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BLENDING SURFACES ON CIRCULAR DOMAINS GENERATED BY HERMITE INTERPOLATION

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Abstract. In this article we use the univariate Hermite interpolation to construct the surfaces on circular domains. The surfaces match given circles. We study the parabolical points of these surfaces. Some examples and graphs are given. These surfaces can be used in civil engineering or in Computer Aided Geometric Design (CAGD).

1. **Introduction**

The blending surfaces have been created by Coons S.A. [7]. These surfaces match a given curve. They can be used in civil engineering (roof-surfaces for large halls) or in Computer Aided Geometric Design (CAGD).

In some previous papers there were constructed the blending surfaces on the rectangular or triangular domains (see [2]-[6]). In [1] we constructed the blending surfaces on circular domain using Lagrange interpolation. In this paper we use Hermite interpolation to get the surfaces which match the given circles. We give the explicit and the parametrical representations for these surfaces. We study the position of the parabolical points because the maximal stress holds in these points (see [3], [5], [6], [9]).

2. **Construction of the surfaces**

Let $0 = y_0 < y_1 < ... < y_{l-1} < y_l = a$, $s_j \in \mathbb{N}$, $j = \overline{0,l}$, $\alpha_{jq} \in \mathbb{R}$, $j = \overline{0,l}$, $q = \overline{1,s_j}$ and $f : [0,a] \to \mathbb{R}$ a function with the properties

$$\begin{aligned} & f(0) = h > 0, f(a) = 0, \\ & f(y_j) = h_j > 0, j = \overline{1,l-1}, \\ & f^{(q)}(y_j) = \alpha_{jq}, j = \overline{0,l}, q = \overline{1,s_j}. \end{aligned} \tag{2.1}$$

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Let the univariate Hermite function

$$(H_n f)(y) = \sum_{j=0}^{l} \sum_{q=0}^{s_j} h_{jq}(y) f^{(q)}(y_j)$$

with $n = l + s_0 + ... + s_l$. The cardinal functions are given by (see [10])

$$h_{jq}(y) = \frac{(y-y_j)^q}{q!} v_j(y) \sum_{\sigma=0}^{s_j - q} \frac{(y-y_j)^\sigma}{\sigma!} \left[\frac{1}{v_j(y)} \right]_{y=y_j}^{(\sigma)}, j = \overline{0,l}, q = \overline{0,s_j}$$

where

$$v_j(y) = v(y)/(y-y_j)^{s_j+1}$$

with

$$v(y) = \prod_{j=0}^{l} (y-y_j)^{s_j+1}.$$

Taking into account (2.1) we obtain

$$(H_n f)(y) = \sum_{j=0}^{l} \sum_{q=1}^{s_j} h_{jq}(y) \alpha_{jq} + \sum_{j=1}^{l-1} h_{j0}(y) h_j + h_{00}(y) h. \tag{2.2}$$

The function (2.2) has the properties

$$\begin{aligned}
&(H_n f)(0) = h, (H_n f)(a) = 0, \\
&(H_n f)(y_j) = h_j, j = \overline{1, l-1}, \\
&(H_n f)^{(q)}(y_j) = \alpha_{jq}, j = \overline{0,l}, q = \overline{1, s_j}.
\end{aligned}$$

Let $D = \{(X,Y) \in \mathbb{R}^2 | X^2 + Y^2 \leq a^2\}$ and $C_j = \{(X,Y) \in \mathbb{R}^2 | X^2 + Y^2 = y_j^2\}$, $j = \overline{1, n-1}$ in the XOY plane.

If we make the substitution

$$y = \sqrt{X^2 + Y^2} \tag{2.3}$$

in (2.2), we obtain the surfaces

$$\widetilde{F}(X,Y) = \sum_{j=0}^{l} \sum_{q=1}^{s_j} h_{jq}(\sqrt{X^2+Y^2}) \alpha_{jq} + \sum_{j=1}^{l-1} h_{j0}(\sqrt{X^2+Y^2}) h_j + \tag{2.4}$$

$$+ h_{00}(\sqrt{X^2+Y^2}) h, \quad X^2 + Y^2 \leq a^2.$$

The surfaces (2.4) have the properties

$$\widetilde{F}|_{\partial D} = 0,$$

$$\widetilde{F}|_{C_j} = h_j, j = \overline{1, n-1},$$

$$\widetilde{F}(0,0) = h.$$

It follows that the surfaces $\widetilde{F}$ match the circle $X^2+Y^2=a^2, Z=0$ (the surfaces are staying on the border of domain D), the circles $X^2+Y^2=y_j^2, Z=h_j$ for $j=\overline{1,n-1}$ and the height of the surfaces in the center of domain D is h.

We can give a parametrical representation for the surfaces

$$\begin{cases} X = u\cos v \\ Y = u\sin v \\ Z = \sum\limits_{j=0}^{l}\sum\limits_{q=1}^{s_j} h_{jq}(u)\alpha_{jq} + \sum\limits_{j=1}^{l-1} h_{j0}(u)h_j + h_{00}(u)h \end{cases} \quad u\in[0,a], v\in[0,2\pi].$$

3. **Parabolical points**

If we take $\alpha_{00}=h, \alpha_{j0}=h_j$, $j=\overline{1,l-1}, \alpha_{l0}=0$ we obtain

$$\widetilde{F}(X,Y)=\sum_{j=0}^{l}\sum_{q=0}^{s_j} h_{jq}(\sqrt{X^2+Y^2})\alpha_{jq}.$$

The parabolical points of the surfaces $\widetilde{F}$ satisfy the condition

$$\widetilde{F}_{XX}(X,Y)\widetilde{F}_{YY}(X,Y)-(\widetilde{F}_{XY}(X,Y))^2=0. \tag{3.1}$$

The second partial derivatives of the function $\widetilde{F}$ can be expressed by

$$\widetilde{F}_{XX}(X,Y)=\frac{X^2}{X^2+Y^2}\sum_{j=0}^{l}\sum_{q=0}^{s_j} h''_{jq}(\sqrt{X^2+Y^2})\alpha_{jq}+ \tag{3.2}$$

$$+\frac{Y^2}{(X^2+Y^2)^{\frac{3}{2}}}\sum_{j=0}^{l}\sum_{q=0}^{s_j} h'_{jq}(\sqrt{X^2+Y^2})\alpha_{jq}$$

$$\widetilde{F}_{XY}(X,Y)=\frac{XY}{X^2+Y^2}\sum_{j=0}^{l}\sum_{q=0}^{s_j} h''_{jq}(\sqrt{X^2+Y^2})\alpha_{jq}- \tag{3.3}$$

$$-\frac{XY}{(X^2+Y^2)^{\frac{3}{2}}}\sum_{j=0}^{l}\sum_{q=0}^{s_j} h'_{jq}(\sqrt{X^2+Y^2})\alpha_{jq}$$

$$\widetilde{F}_{YY}(X,Y)=\frac{Y^2}{X^2+Y^2}\sum_{j=0}^{l}\sum_{q=0}^{s_j} h''_{jq}(\sqrt{X^2+Y^2})\alpha_{jq}+ \tag{3.4}$$

$$+\frac{X^2}{(X^2+Y^2)^{\frac{3}{2}}}\sum_{j=0}^{l}\sum_{q=0}^{s_j} h'_{jq}(\sqrt{X^2+Y^2})\alpha_{jq}$$

if $(X,Y)\neq(0,0)$. From (3.1) and (3.2)-(3.4) we obtain

$$\frac{1}{\sqrt{X^2+Y^2}}\left(\sum_{j=0}^{l}\sum_{q=0}^{s_j} h'_{jq}(\sqrt{X^2+Y^2})\alpha_{jq}\right)\left(\sum_{j=0}^{l}\sum_{q=0}^{s_j} h''_{jq}(\sqrt{X^2+Y^2})\alpha_{jq}\right)=0.$$

We consider the following two polynomial equations with unknown y

$$\sum_{j=0}^{l}\sum_{q=0}^{s_j} h'_{jq}(y)\alpha_{jq} = 0, \tag{3.5}$$

$$\sum_{j=0}^{l}\sum_{q=0}^{s_j} h''_{jq}(y)\alpha_{jq} = 0. \tag{3.6}$$

If the equations have no roots in $[0, a]$ then the surfaces generated by function $\widetilde{F}$ have no parabolical points. If $y = \overline{y} \in (0, a]$ is a solution for one of the equations (3.5) or (3.6), the surfaces have parabolical points of which projections on XOY plane are the circle $X^2 + Y^2 = \overline{y}^2$. If $\overline{y} = 0$ then the point $(0, 0, h)$ is the parabolical point of the surfaces if

$$\lim_{X,Y\to 0} \frac{1}{\sqrt{X^2+Y^2}} \left(\sum_{j=0}^{l}\sum_{q=0}^{s_j} h'_{jq}(\sqrt{X^2+Y^2})\alpha_{jq}\right)\left(\sum_{j=0}^{l}\sum_{q=0}^{s_j} h''_{jq}(\sqrt{X^2+Y^2})\alpha_{jq}\right) \tag{3.7}$$

is zero. It is important to study the position of the parabolical points. The maximal stress holds in these points. The surfaces are more resistent if the projection of parabolical points are situated as closed as possible to the border, if possible just on the border or outside of domain (see [6], [9]). We can give conditions on parameters α_{ij} to control the position of the parabolical points.

4. **Examples**

We study the parabolical points in two particular cases.

First, we take $l = 1$, $s_0 = 1$, $s_1 = 0$. We have

$$h_{00}(y) = -\frac{y^2}{a^2} + 1, h_{01}(y) = -\frac{y^2}{a} + y.$$

From (2.4), it follows that the surface is given by

$$\widetilde{F}_1(X, Y) = \left(-\frac{X^2+Y^2}{a} + \sqrt{X^2+Y^2}\right)\alpha_{01} + \left(-\frac{X^2+Y^2}{a^2} + 1\right)h, X^2 + Y^2 \le a^2$$

It has the properties

$$\widetilde{F}_1|_{\partial D} = 0, \widetilde{F}_1(0, 0) = h.$$

The equations (3.5) and (3.6) become

$$\left(-\frac{2}{a^2}h - \frac{2}{a}\alpha_{01}\right)y + \alpha_{01} = 0 \tag{4.1}$$

$$-\frac{2}{a^2}h - \frac{2}{a}\alpha_{01} = 0 \tag{4.2}$$

Let $\overline{y} = \frac{a^2\alpha_{01}}{2(h+a\alpha_{01})}$. We have the following cases:

i) If $\alpha_{01} = -\frac{h}{a}$ then the relation (4.2) holds. It follows that all the points of the surface are parabolical points.

ii) If $\alpha_{01} < -\frac{2h}{a}$ or $\alpha_{01} > 0$ the equation (4.1) has the solution $y = \overline{y}$ in interval $(0, a)$. It follows that the surface has parabolical points of which projections are situated on a circle, inside of domain D.

iii) If $\alpha_{01} = -\frac{2h}{a}$ then $y = a$ is the solution of the equation (4.1). The parabolical points are on the border of domain D.

iv) If $\alpha_{01} \in (-2\frac{h}{a}, -\frac{h}{a})$ then the equation (4.1) has the solution $y = \overline{y} > a$. The surface has no parabolical points.

v) If $\alpha_{01} \in (-\frac{h}{a}, 0)$ the equation (4.1) has the solution $y = \overline{y} < 0$. The surface has no parabolical points.

vi) If $\alpha_{01} = 0$ then $y = 0$ is the solution of the equation (4.1). The limit (3.7) is different to zero. It follows that the surface has no parabolical points.

In Figure 1 we plot the surface $\widetilde{F}_1$ for $a = 3$, $h = 5$ and some values of α_{01}.

In second case, we take $l = 1$, $s_0 = 0$, $s_1 = 1$. We have

$$h_{00}(y) = \frac{y^2}{a^2} - 2\frac{y}{a} + 1, h_{11}(y) = \frac{y^2}{a} - y.$$

From (2.4) we get the surface

$$\widetilde{F}_2(X, Y) = \left(\frac{X^2 + Y^2}{a} - \sqrt{X^2 + Y^2}\right)\alpha_{11} + \left(\frac{X^2 + Y^2}{a^2} - 2\frac{\sqrt{X^2 + Y^2}}{a} + 1\right)h,$$

$$X^2 + Y^2 \leq a^2$$

The surface has the properties

$$\widetilde{F}_2|_{\partial D} = 0, \widetilde{F}_2(0, 0) = h.$$

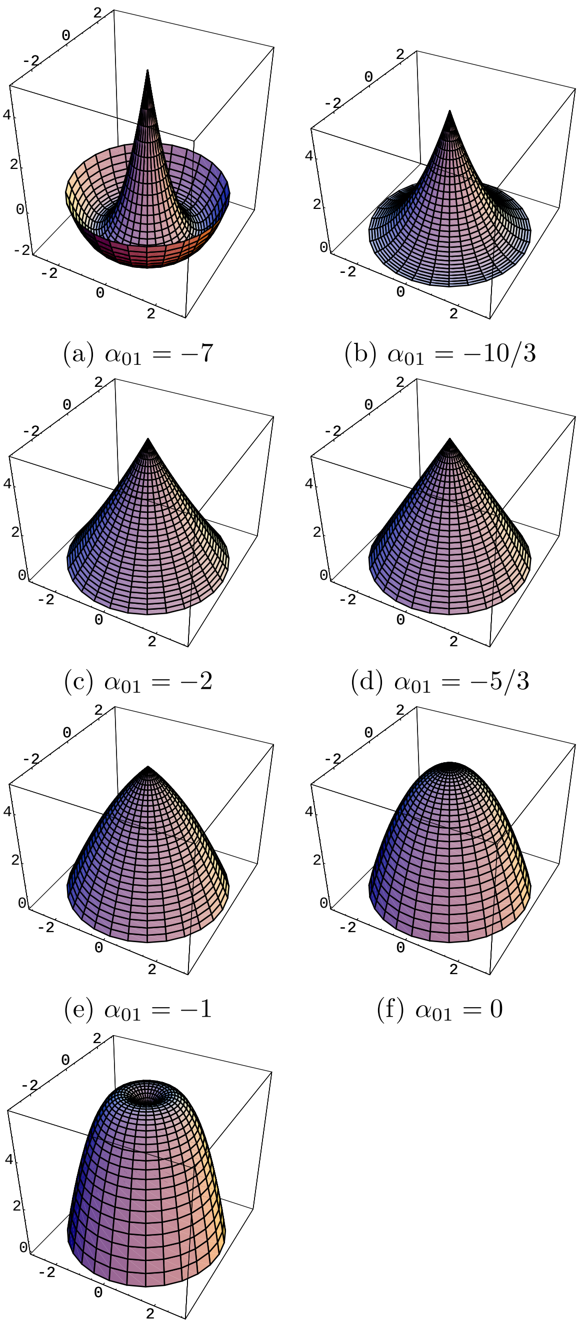
The equations (3.5) and (3.6) become

$$\left(\frac{2}{a^2}h + \frac{2}{a}\alpha_{11}\right)y - \frac{2}{a}h - \alpha_{11} = 0 \tag{4.3}$$

$$\frac{2}{a^2}h + \frac{2}{a}\alpha_{11} = 0 \tag{4.4}$$

Let $\overline{y} = \frac{a(2h+a\alpha_{11})}{2(h+a\alpha_{11})}$. We have the following cases:

i) If $\alpha_{11} = -\frac{h}{a}$ then the relation (4.4) holds. It follows that all the points of the surface are parabolical points.



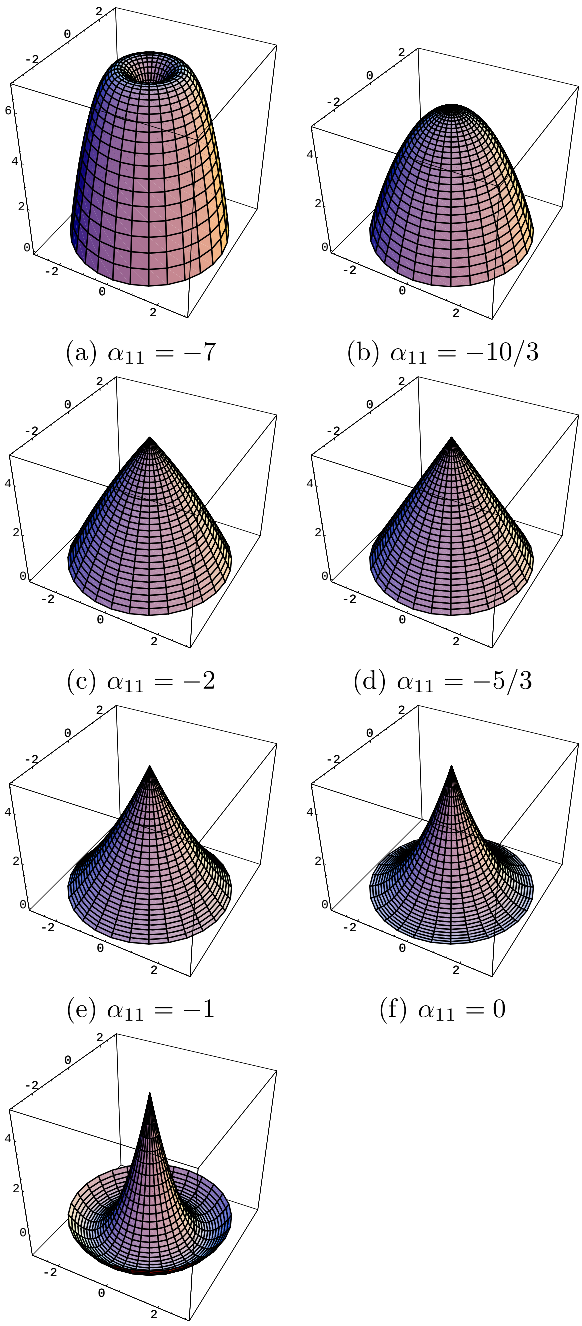
(a) $\alpha_{01} = -7$ (b) $\alpha_{01} = -10/3$

(c) $\alpha_{01} = -2$ (d) $\alpha_{01} = -5/3$

(e) $\alpha_{01} = -1$ (f) $\alpha_{01} = 0$

(g) $\alpha_{01} = 2$

FIGURE 1. The surface $\widetilde{F}_1$ for $a = 3$ and $h = 5$.



(a) $\alpha_{11} = -7$

(b) $\alpha_{11} = -10/3$

(c) $\alpha_{11} = -2$

(d) $\alpha_{11} = -5/3$

(e) $\alpha_{11} = -1$

(f) $\alpha_{11} = 0$

(g) $\alpha_{11} = 2$

FIGURE 2. The surface $\widetilde{F}_2$ for $a = 3$ and $h = 5$.

ii) If $\alpha_{11} < -\frac{2h}{a}$ or $\alpha_{01} > 0$ the equation (4.3) has the solution $y = \overline{y}$ in interval $(0, a)$. The surface has parabolical points of which projections are situated on a circle, inside of domain D.

iii) If $\alpha_{11} = -\frac{2h}{a}$ then $y = 0$ is solution of the equation (4.3). The limit (3.7) is different to zero. It follows that the surface has no parabolical points.

iv) If $\alpha_{11} \in (-2\frac{h}{a}, -\frac{h}{a})$ then the equation (4.3) has the solution $y = \overline{y} < 0$. It follows that the surface has no parabolical points.

v) If $\alpha_{11} \in (-\frac{h}{a}, 0)$ the equation (4.3) has the solution $y = \overline{y} > a$. The surface has no parabolical points.

vi) If $\alpha_{11} = 0$ then $y = a$ is the solution of the equation (4.3). The parabolical points of the surface are on the border of domain D.

In Figure 2 we plot the surface $\widetilde{F}_2$ for $a = 3$, $h = 5$ and some values of α_{11}.

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