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ON INVERSE-CONVEX MEROMORPHIC FUNCTIONS

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Abstract. We introduce a new class K_i of meromorphic functions, called the class of inverse-convex functions, and we study some properties and we prove some theorems for this class. We also study some properties for the integral operator

$$I_\gamma(g)(z) = \frac{\gamma}{z^{\gamma+1}} \int_0^z t^\gamma g(t)dt,\ \gamma \in \mathbb{C},$$

with respect to this class K_i.

1. **Introduction and preliminaries**

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in the complex plane, $\dot{U} = U \setminus \{0\}$ and $H(U) = \{f : U \to \mathbb{C} : f$ is holomorphic in $U\}$.
Let $A_n = \{f \in H(U) : f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots\}$, $n \in \mathbb{N}^*$, and for $n = 1$ we denote A_1 by A and this set is called *the class of analytic functions normalized at the origin*.
Let K be the class of normalized convex functions on the unit disc U, i.e.

$$K = \left\{f \in A : \operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0,\ z \in U\right\}$$

and let S^* be the class of normalized starlike functions on U, i.e.

$$S^* = \left\{f \in A : \operatorname{Re} \frac{zf'(z)}{f(z)} > 0,\ z \in U\right\}.$$

We denote by M_0 the class of meromorphic functions in $\dot{U}$ of the form

$$g(z) = \frac{1}{z} + \alpha_0 + \alpha_1 z + \cdots,\ z \in \dot{U}.$$

Let

$$M_0^* = \left\{g \in M_0 : \text{g is univalent in } \dot{U} \text{ and } \operatorname{Re}\left[-\frac{zg'(z)}{g(z)}\right] > 0,\ z \in \dot{U}\right\}$$

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be called the class of meromorphic starlike functions in $\dot{U}$.

We note that if f is a normalized starlike function on U, then the function $g = \dfrac{1}{f}$ belongs to the class M_0^*.

For $\alpha \in [0,1)$ and $\beta > 1$ let

$$M_0^*(\alpha) = \left\{ g \in M_0 : \text{g is univalent in } \dot{U} \text{ and } \operatorname{Re}\left[-\frac{zg'(z)}{g(z)}\right] > \alpha,\, z \in \dot{U} \right\}$$

and

$$M_0^*(\alpha,\beta) = \left\{ g \in M_0 : \text{g is univalent in } \dot{U} \text{ and } \alpha < \operatorname{Re}\left[-\frac{zg'(z)}{g(z)}\right] < \beta,\, z \in \dot{U} \right\}.$$

Let

$$M_0^c = \left\{ g \in M_0 : \text{g is univalent in } \dot{U} \text{ and } \operatorname{Re}\left[-\left(\frac{zg''(z)}{g'(z)} + 1\right)\right] > 0,\, z \in \dot{U} \right\}$$

be called the class of meromorphic convex functions in $\dot{U}$.

It's easy to see that $M_0^c \subset M_0^*$.

Theorem 1.1. [1, Theorem 2.4f.], [2, p.212] *Let* $p \in H[a,n]$ *with* $\operatorname{Re} a > 0$ *and let* $P : U \to \mathbb{C}$ *be a function with* $\operatorname{Re} P(z) > 0$, $z \in U$. *If*

$$\operatorname{Re}\left[p(z) + P(z)zp'(z)\right] > 0,\, z \in U,$$

then $\operatorname{Re} p(z) > 0$, $z \in U$.

Lemma 1.2. [1, Exemple 2.4e.], [2, p.211] *Let* $p \in H[a,n]$ *with* $\operatorname{Re} a > 0$ *and let* $\alpha : U \to \mathbb{R}$. *If*

$$\operatorname{Re}\left[p(z) + \alpha(z)\frac{zp'(z)}{p(z)}\right] > 0,\, z \in U,$$

then $\operatorname{Re} p(z) > 0$, $z \in U$.

Definition 1.3. Let $g : \dot{U} \to \mathbb{C}$ be a meromorphic function in $\dot{U}$ of the form

$$g(z) = \frac{\alpha_{-1}}{z} + \alpha_0 + \alpha_1 z + \cdots,\, z \in \dot{U}.$$

We say that the function g is inverse-convex in $\dot{U}$ if there exists a convex function f defined on U with $f(0) = 0$ such that $f(z)g(z) = 1$ for each $z \in \dot{U}$.

Remark 1.4. 1. From the above definition we notice that if g is inverse-convex, then $g(z) \neq 0$, $z \in \dot{U}$ and g is univalent in $\dot{U}$.

2. If $\alpha_{-1} = 1$, i.e. $g \in M_0$, we can easily see that the function f from the above definition is also normalized, hence a function $g \in M_0$ is inverse-convex in $\dot{U}$ if there exists a function $f \in K$ such that $f(z)g(z) = 1$ for each $z \in \dot{U}$. We will denote the class of these functions by K_i(the class of normalized inverse-convex functions on $\dot{U}$).

3. If g is inverse-convex in $\dot{U}$ and $\lambda \in \mathbb{C}^*$, then the meromorphic function λg is also inverse-convex in $\dot{U}$.

4. If $g \in K_i$, then $g \in M_0^*\left(\frac{1}{2}\right)$.

Definition 1.5. Let $g : \dot{U} \to \mathbb{C}$ be a meromorphic function in $\dot{U}$ of the form

$$g(z) = \frac{\alpha_{-1}}{z} + \alpha_0 + \alpha_1 z + \cdots .$$

We say that the function g is close-to-inverse-convex in $\dot{U}$ if there exists an inverse-convex function ψ on $\dot{U}$ such that

$$\operatorname{Re} \frac{g'(z)}{\psi'(z)} > 0,\ z \in \dot{U}.$$

We denote by C_i the class of normalized close-to-inverse-convex functions on $\dot{U}$.

For $\beta > 1$ we say that a close-to-inverse-convex function g is in the class $C_{i;\beta}$ if the function $\psi \in K_i \cap M_0^*(0, \beta)$.

2. Main results

Theorem 2.1. (Theorem of analytical characterization of the inverse-convexity for meromorphic functions) *Let $g : \dot{U} \to \mathbb{C}$ be a meromorphic function in $\dot{U}$ of the form*

$$g(z) = \frac{1}{z} + \alpha_0 + \alpha_1 z + \cdots ,$$

such that $g(z) \neq 0$, $z \in \dot{U}$. Then the function g is inverse-convex on $\dot{U}$ if and only if g is univalent on $\dot{U}$ and

$$\operatorname{Re} \left[\frac{zg''(z)}{g'(z)} - 2\frac{zg'(z)}{g(z)} + 1\right] > 0,\ z \in \dot{U}.$$

Proof. Suppose that $g \in K_i$. Then there exists $f \in K$ such that $f(z)g(z) = 1$, $z \in \dot{U}$, so

$$g(z) = \frac{1}{f(z)},\ z \in \dot{U},\ f \in K. \tag{2.1}$$

Because f is univalent also is g, and if we consider the second differential for the equality $f(z)g(z) = 1$, $z \in \dot{U}$ we obtain

$$f''(z)g(z) + 2f'(z)g'(z) + f(z)g''(z) = 0. \tag{2.2}$$

Dividing (2.2) by $f(z)g'(z) \neq 0$, $z \in \dot{U}$ and multiplying the result with z we will have

$$\frac{zf''(z)}{f'(z)}\frac{f'(z)g(z)}{f(z)g'(z)} + 2\frac{zf'(z)}{f(z)} + \frac{zg''(z)}{g'(z)} = 0. \tag{2.3}$$

Using the derivative for $f(z)g(z) = 1$ we obtain

$$\frac{f'(z)g(z)}{f(z)g'(z)} = -1. \tag{2.4}$$

From (2.4) and (2.3) we have

$$\frac{zf''(z)}{f'(z)} + 1 = \frac{zg''(z)}{g'(z)} - 2\frac{zg'(z)}{g(z)} + 1,\ z \in \dot{U},$$

and, since we know that $\mathrm{Re}\,\frac{zf''(z)}{f'(z)} + 1 > 0$, $z \in U$, we obtain

$$\mathrm{Re}\left[\frac{zg''(z)}{g'(z)} - 2\frac{zg'(z)}{g(z)} + 1\right] > 0,\ z \in \dot{U}.$$

To prove the sufficiency we consider the function $f(z) = \frac{1}{g(z)}$, $z \in \dot{U}$, with $f(0) = 0$ and we prove that $f \in K$. □

Remark 2.2. 1. An easy computation shows that the function

$$f(z) = \log(1+z), z \in U \left(\text{with} \quad \log(1+z)\Big|_{z=0} = 0\right)$$

is convex on U and normalized, so the function $g(z) = \frac{1}{f(z)}$, $z \in \dot{U}$ belongs to the class K_i.

On the other hand we have

$$\frac{zg''(z)}{g'(z)} + 1 = \frac{\log(1+z) + 2z}{(1+z)\log(1+z)},$$

and it's easy to see that the inequality

$$\mathrm{Re}\left[-\left(\frac{zg''(z)}{g'(z)} + 1\right)\right] > 0$$

doesn't hold for each $z \in \dot{U}$ $\left(\text{for exemple we can take } z = \frac{1}{2}\right)$, so $g \notin M_0^c$. In other words, $K_i \neq M_0^c$.

2. We know that the function $f(z) = \frac{z}{1 + e^{i\tau}z} \in K$, so

$$g(z) = \frac{1}{f(z)} = \frac{1}{z} + e^{i\tau} \in K_i.$$

But on the other hand, it's easy to show that $g \in M_0^c$, hence $K_i \cap M_0^c \neq \emptyset$.

3. If $g \in K_i$, then $f = \frac{1}{g} \in K \subset S^*$, so $g \in M_0^*$. Therefore, we have $K_i \subset M_0^*$.

Theorem 2.3. (Duality theorem between the classes M_0^* and K_i) *Let $g : \dot{U} \to \mathbb{C}$ be a function in M_0. Then $g \in K_i$ if and only if the function*

$$G(z) = -\frac{g^2(z)}{zg'(z)} \in M_0^*.$$

Proof. Using the definition we have $g \in K_i$ if and only if $f = \frac{1}{g} \in K$.

On the other hand, in view of Alexander's duality theorem (see [2], [3]) we deduce that

$$f \in K \quad \text{is equivalent to} \quad F(z) = zf'(z) = -\frac{zg'(z)}{g^2(z)} \in S^*.$$

But, we know that $F \in S^*$ is equivalent to $G = \frac{1}{F} \in M_0^*$. So, we obtained

$$g \in K_i \quad \text{if and only if} \quad G(z) = -\frac{1}{z}\frac{g^2(z)}{g'(z)} \in M_0^*.$$

□

Theorem 2.4. (Distortion theorem for the class K_i) *If the function g belongs to the class K_i, then we have:*

$$\frac{1}{r} - 1 \le |g(z)| \le \frac{1}{r} + 1,\ |z| = r \in (0,1) \quad \left(\text{equivalent to} \left||g(z)| - \frac{1}{|z|}\right| \le 1,\ z \in \dot{U}\right),$$

$$\left(\frac{1-r}{r+r^2}\right)^2 \le |g'(z)| \le \left(\frac{1+r}{r-r^2}\right)^2,\ |z| = r \in (0,1).$$

For $|g(z)|$ these estimates are sharp and we have equality for $g(z) = \frac{1}{z} + e^{i\tau}$, $\tau \in \mathbb{R}$.

Proof. If $g \in K_i$, then $f = \frac{1}{g} \in K$ and in view of the distortion theorem for the class K we have

$$\frac{r}{1+r} \le |f(z)| \le \frac{r}{1-r} \tag{2.5}$$

$$\frac{1}{(1+r)^2} \le |f'(z)| \le \frac{1}{(1-r)^2},\ |z| = r < 1. \tag{2.6}$$

From (2.5) taking $f = \frac{1}{g}$ we obtain the bounds for $|g(z)|$ and since $r = |z|$ we have

$$\frac{1}{|z|} - 1 \le |g(z)| \le \frac{1}{|z|} + 1 \Leftrightarrow$$

$$\left||g(z)| - \frac{1}{|z|}\right| \le 1.$$

For the bounds of $|g'(z)|$ we use: $g' = -g^2 f'$, the bounds for $|g(z)|$ and (2.6). □

Remark 2.5. If $f : U \to \mathbb{C}$ is a function of the form $f(z) = z + a_1z^2 + a_2z^3 + \cdots$, then the function $g : \dot{U} \to \mathbb{C}$ defined as $g(z) = \dfrac{1}{f(z)}$, $z \in \dot{U}$ has the form

$$g(z) = \frac{1}{z} + \alpha_0 + \alpha_1 z + \cdots + \alpha_n z^n + \cdots ,$$

where

$$\begin{cases} \alpha_0 = & -a_1 \\ \alpha_1 = & -a_2 - \alpha_0 a_1 \\ \vdots & \\ \alpha_n = & -a_{n+1} - \alpha_0 a_n - \alpha_1 a_{n-1} - \cdots - \alpha_{n-1} a_1 \\ \vdots & \end{cases}$$

We know that if a function f belongs to the class K and it is of the form presented above then we have $|a_n| \le 1$ for each $n \in \mathbb{N}^*$ and therefore, after a short computation we obtain that

$$|\alpha_n| \le 2^n, \forall n \in \mathbb{N}.$$

So, if $g \in K_i$, $g(z) = \dfrac{1}{z} + \alpha_0 + \alpha_1 z + \cdots + \alpha_n z^n + \cdots$, then $|\alpha_n| \le 2^n$, $\forall n \in \mathbb{N}$.

Theorem 2.6. *Let be $g \in K_i$, $\lambda \in \mathbb{C} \setminus \{0\}$ with $\operatorname{Re}\lambda > 2|\lambda|^2$, $\beta = \dfrac{\operatorname{Re}\lambda}{2|\lambda|^2}$ and $\operatorname{Re}\left[-\dfrac{zg'(z)}{g(z)}\right] < \beta$ (i.e. $g \in K_i \cap M_0^*(0,\beta)$), then the function*

$$h_\lambda(z) = g(z) + \lambda z g'(z),\ z \in \dot{U},$$

is close-to-inverse-convex.

Proof. From $h_\lambda(z) = g(z) + \lambda z g'(z)$ we obtain $h'_\lambda(z) = g'(z) + \lambda g'(z) + \lambda z g''(z)$ which is equivalent to

$$\frac{h'_\lambda(z)}{\lambda g'(z)} = 1 + \frac{1}{\lambda} + \frac{zg''(z)}{g'(z)} = \frac{1}{\lambda} + 2\frac{zg'(z)}{g(z)} + \left[\frac{zg''(z)}{g'(z)} - 2\frac{zg'(z)}{g(z)} + 1\right]$$

so

$$\operatorname{Re}\frac{h'_\lambda(z)}{\lambda g'(z)} = \operatorname{Re}\frac{1}{\lambda} + 2\operatorname{Re}\frac{zg'(z)}{g(z)} + \operatorname{Re}\left[\frac{zg''(z)}{g'(z)} - 2\frac{zg'(z)}{g(z)} + 1\right] > 0,\ z \in \dot{U}.$$

For the last inequality we have used the fact that $g \in K_i$ implies

$$\operatorname{Re}\left[\frac{zg''(z)}{g'(z)} - 2\frac{zg'(z)}{g(z)} + 1\right] > 0,\ z \in \dot{U},$$

and we have also used the condition

$$\operatorname{Re}\left[-\frac{zg'(z)}{g(z)}\right] < \frac{\operatorname{Re}\lambda}{2|\lambda|^2} \quad \text{equivalent to} \quad \operatorname{Re}\frac{1}{\lambda} + 2\operatorname{Re}\frac{zg'(z)}{g(z)} > 0.$$

Therefore, we have

$$\mathrm{Re}\,\frac{h'_\lambda(z)}{\lambda g'(z)} > 0,\ z \in \dot{U}$$

meaning that the function h_λ is close-to-inverse-convex with respect to the inverse-convex function λg.

We note that we need $\mathrm{Re}\,\lambda > 2|\lambda|^2$ because $\beta > 1$ and that implies $|\lambda| < 1/2$. □

For $\gamma \in \mathbb{C}$ with $\mathrm{Re}\,\gamma > 0$ we consider the integral operator $I_\gamma : M_0 \to M_0$ given by

$$I_\gamma(g)(z) = \frac{\gamma}{z^{\gamma+1}} \int_0^z t^\gamma g(t)dt \tag{2.7}$$

and we have the following result.

Theorem 2.7. *Let be* $\gamma \in \mathbb{C}$ *with* $\mathrm{Re}\,\gamma > 1$ *and* $\beta = \dfrac{\mathrm{Re}\,\gamma + 1}{2}$.
If $I_\gamma[K_i] \subset K_i$*, then* $I_\gamma[C_{i;\beta}] \subset C_i$.

Proof. Let $G = I_\gamma(g)$. If we take the second derivative for the relation

$$G(z) = I_\gamma(g)(z) = \frac{\gamma}{z^{\gamma+1}} \int_0^z t^\gamma g(t)dt$$

we obtain

$$(\gamma + 2)G'(z) + zG''(z) = \gamma g'(z). \tag{2.8}$$

If $g \in C_{i;\beta}$, then there exists a function $\psi \in K_i \cap M_0^*(0, \beta)$ such that

$$\mathrm{Re}\,\frac{g'(z)}{\psi'(z)} > 0,\ z \in U. \tag{2.9}$$

Let's denote $\phi = I_\gamma(\psi)$. From $I_\gamma[K_i] \subset K_i$ we obtain that $\phi \in K_i$.
We also have the relation

$$(\gamma + 2)\phi'(z) + z\phi''(z) = \gamma\psi'(z). \tag{2.10}$$

If we denote

$$p(z) = \frac{G'(z)}{\phi'(z)},$$

then $p(0) = 1$ and the relation (2.8) can be rewritten in the following form

$$(\gamma + 2)p(z)\phi'(z) + z[p'(z)\phi'(z) + p(z)\phi''(z)] = \gamma g'(z). \tag{2.11}$$

Using (2.11) and (2.10) we obtain

$$p(z) + \frac{zp'(z)}{(\gamma + 2) + \dfrac{z\phi''(z)}{\phi'(z)}} = \frac{g'(z)}{\psi'(z)}$$

which is equivalent to

$$p(z) + \frac{zp'(z)}{P(z)} = \frac{g'(z)}{\psi'(z)}, \quad \text{where} \quad P(z) = (\gamma + 2) + \frac{z\phi''(z)}{\phi'(z)}.$$

Using (2.9) we deduce that

$$\operatorname{Re}\left[p(z)+\frac{zp'(z)}{P(z)}\right]>0,\ z\in U. \tag{2.12}$$

The relation (2.10) is equivalent to $\phi'(z)P(z)=\gamma\psi'(z)$ and using the logarithmic derivative for this equality we obtain

$$P(z)+\frac{zP'(z)}{P(z)}=\gamma+2+\frac{z\psi''(z)}{\psi'(z)}=\left[\frac{z\psi''(z)}{\psi'(z)}-2\frac{z\psi'(z)}{\psi(z)}+1\right]+2\frac{z\psi'(z)}{\psi(z)}+\gamma+1.$$

Since we know that

1. $\psi\in K_i$, i.e.
$$\operatorname{Re}\left[\frac{z\psi''(z)}{\psi'(z)}-2\frac{z\psi'(z)}{\psi(z)}+1\right]>0,\ z\in U.$$
2. $\psi\in M_0^*(0,\beta)$, i.e.
$$\operatorname{Re}\left[-\frac{z\psi'(z)}{\psi(z)}\right]<\beta=\frac{\operatorname{Re}\gamma+1}{2}$$

we have

$$\operatorname{Re}\left[P(z)+\frac{zP'(z)}{P(z)}\right]>0,\ z\in U.$$

It is easy to see that $P(0)=\gamma$, so $\operatorname{Re}P(0)>0$ and using Lemma 1.2 we obtain $\operatorname{Re}P(z)>0,\ z\in U$.
Using (2.12), $\operatorname{Re}P(z)>0,\ z\in U$ and Theorem 1.1 we have

$$\operatorname{Re}p(z)>0,\ z\in U$$

which is the same with

$$\operatorname{Re}\frac{G'(z)}{\phi'(z)}>0,\ z\in U,\quad \text{hence}\quad G\in C_i.$$

□

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