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ON A DEGENERATE AND SINGULAR ELLIPTIC EQUATION WITH CRITICAL EXPONENT AND NON-STANDARD GROWTH CONDITIONS

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Abstract. In this paper we study a class of degenerate and singular elliptic equations involving critical exponents and non-standard growth conditions in the whole space $\mathbb{R}^N$. We show the existence of at least one nontrivial solution using as main argument Ekeland's variational principle.

1. **Introduction**

In this paper we are concerned with the study of the following problem

$$-\mathrm{div}(|x|^\alpha \nabla u) = \lambda g(x)|u|^{q(x)-2}u + |u|^{2^\star_\alpha - 2}u \qquad \text{in } \mathbb{R}^N \tag{1.1}$$

where $N \geq 2$, $0 < \alpha < 2$, $2^\star_\alpha = 2N/(N-2+\alpha)$ is the critical exponent, $q : \mathbb{R}^N \to (1, 2^\star_\alpha)$ is a function satisfying $q \in L^\infty(\mathbb{R}^N)$, $g : \mathbb{R}^N \to (0,\infty)$ is a measurable function satisfying certain properties that will be described later in the paper and $\lambda > 0$ is a constant.

The main interest in studying problem (1.1) is due to the presence of the degenerate and singular potential $|x|^\alpha$ in the divergence operator. This potential leads to a differential operator

$$\mathrm{div}(|x|^\alpha \nabla u(x))$$

which is degenerate and singular in the sense that

$$\lim_{|x|\to 0} |x|^\alpha = 0 \text{ and } \lim_{|x|\to\infty} |x|^\alpha = \infty,$$

provided that $\alpha \in (0, 2)$. Consequently, we will analyze equation (1.1) in the case when the operator $\mathrm{div}(|x|^\alpha \nabla u(x))$ is not strictly elliptic in the sense pointed out in D. Gilbarg & N. S. Trudinger [6] (see, page 31 in [6] for the definition of strictly

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elliptic operators). It follows that some of the techniques that can be applied in solving equations involving strictly elliptic operators fail in this new context. For instance some concentration phenomena may occur in the degenerate and singular case which lead to a lack of compactness. On the other hand, such kind of problems are exacerbated by the presence of the critical exponent $2_\alpha^\star$ in the right-hand side of equation (1.1).

2. **Preliminary results**

In this paper the convenient (and natural) functional space where we are seeking solutions for problem (1.1) is $\mathcal{D}_\alpha^{1,2}(\mathbb{R}^N)$, which is defined as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the inner product

$$\langle u, v\rangle_\alpha := \int_{\mathbb{R}^N} |x|^\alpha \nabla u \nabla v dx.$$

Recall that $\mathcal{D}_\alpha^{1,2}(\mathbb{R}^N)$ is a Hilbert space with respect to the norm

$$\|u\|_\alpha^2 := \int_{\mathbb{R}^N} |x|^\alpha |\nabla u|^2 dx$$

We say that $u \in \mathcal{D}_\alpha^{1,2}(\mathbb{R}^N)$ is a *weak solution* of (1.1) if

$$\int_{\mathbb{R}^N} |x|^\alpha \nabla u \nabla v dx - \lambda \int_{\mathbb{R}^N} g(x)|u|^{q(x)-2} uv\, dx - \int_{\mathbb{R}^N} |u|^{2_\alpha^\star - 2} uv\, dx = 0$$

for all $v \in C_0^\infty(\mathbb{R}^N)$.

Remark 2.1. Actually, it can be proved that $\mathcal{D}_\alpha^{1,2}(\mathbb{R}^N) = \overline{C_0^\infty(\mathbb{R}^N \setminus \{0\})}^{\|\cdot\|_\alpha}$ (see [2]).

The starting point of the variational approach to problems of this type is the following inequality which can be obtained essentially "interpolating" between Sobolev's and Hardy's inequalities [1] (see also [3] and [4]).

Lemma 2.2. *(Caffarelli-Kohn-Nirenberg) Let* $N \geq 2$, $\alpha \in (0,2)$ *and denote* $2_\alpha^\star = \frac{2N}{N-2+\alpha}$. *Then there exists* $C_\alpha > 0$ *such that*

$$\left(\int_{\mathbb{R}^N} |\varphi|^{2_\alpha^\star} dx\right)^{2/2_\alpha^\star} \leq C_\alpha \int_{\mathbb{R}^N} |x|^\alpha |\nabla \varphi|^2 dx \tag{2.1}$$

for every $\varphi \in C_0^\infty(\mathbb{R}^N)$.

Remark 2.3. By Lemma 2.2 we deduce that $\mathcal{D}_\alpha^{1,2}(\mathbb{R}^N)$ is continuously embedded in $L^{2_\alpha^\star}(\mathbb{R}^N)$.

On the other hand, in order to study problem (1.1), we will appeal to the variable exponent Lebesgue spaces $L^{q(\cdot)}(\mathbb{R}^N)$. We point out certain properties of that spaces according to the papers of Kováčik and Rákosník [7] and Mihăilescu and Rădulescu [8, 9].

For any function $p : \mathbb{R}^N \to (1, \infty)$ with $p \in L^\infty(\mathbb{R}^N)$ define

$$p^- := \operatorname{ess}\inf_{x \in \mathbb{R}^N} p(x) \quad \text{and} \quad p^+ := \operatorname{ess}\sup_{x \in \mathbb{R}^N} p(x)\,.$$

It is usually assumed that $p^+ < +\infty$, since this condition implies many useful properties for the associated variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^N)$. This function space is defined by

$$L^{p(\cdot)}(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ measurable } : \int_{\mathbb{R}^N} |u(x)|^{p(x)}\, dx < \infty \right\}.$$

$L^{p(\cdot)}(\mathbb{R}^N)$ is a Banach space when endowed with the so-called *Luxemburg norm*, defined by

$$|u|_{p(\cdot)} := \inf \left\{ \mu > 0 \ : \int_{\mathbb{R}^N} \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \le 1 \right\}.$$

The variable exponent Lebesgue space is a special case of an Orlicz-Musielak space. For constant functions p the space $L^{p(\cdot)}(\mathbb{R}^N)$ reduces to the classical Lebesgue space $L^p(\mathbb{R}^N)$, endowed with the standard norm

$$\|u\|_{L^p(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |u(x)|^p dx \right)^{1/p}.$$

We recall that if $1 < p^- \le p^+ < +\infty$ the variable exponent Lebesgue spaces are separable and reflexive.

We denote by $L^{p'(\cdot)}(\mathbb{R}^N)$ the conjugate space of $L^{p(\cdot)}(\mathbb{R}^N)$, where $1/p(x) + 1/p'(x) = 1$. For any $u \in L^{p(\cdot)}(\mathbb{R}^N)$ and $v \in L^{p'(\cdot)}(\mathbb{R}^N)$ the Hölder type inequality

$$\left| \int_{\mathbb{R}^N} uv\, dx \right| \le \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) |u|_{p(\cdot)} |v|_{p'(\cdot)} \tag{2.2}$$

holds.

A key role in the theory of variable exponent Lebesgue and Sobolev (defined below) spaces is played by the *modular* of the space $L^{p(\cdot)}(\mathbb{R}^N)$, which is the mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$\rho_{p(\cdot)}(u) := \int_{\mathbb{R}^N} |u(x)|^{p(x)}\, dx.$$

If $u \in L^{p(\cdot)}(\mathbb{R}^N)$ then the following relations hold:

$$|u|_{p(\cdot)} > 1 \quad \Rightarrow \quad |u|_{p(\cdot)}^{p^-} \le \rho_{p(\cdot)}(u) \le |u|_{p(\cdot)}^{p^+}; \tag{2.3}$$

$$|u|_{p(\cdot)} < 1 \quad \Rightarrow \quad |u|_{p(\cdot)}^{p^+} \le \rho_{p(\cdot)}(u) \le |u|_{p(\cdot)}^{p^-}; \tag{2.4}$$

$$|u|_{p(\cdot)} = 1 \Leftrightarrow \rho_{p(\cdot)}(u) = 1. \tag{2.5}$$

3. **The main result**

In this paper we study the existence of nontrivial weak solutions for problem (1.1) in the case when $q : \mathbb{R}^N \to (1, 2_\alpha^\star)$, $q \in L^\infty(\mathbb{R}^N)$ satisfies the property that there exists $x_0 \in \overline{\Omega}$ and $s > 0$ such that q is continuous on the ball centered in x_0 of radius s, that is $B_s(x_0)$, and

$$1 < q(x_0) < 2\,. \tag{3.1}$$

Our main result is given by the following theorem.

Theorem 3.1. *Assume $q : \mathbb{R}^N \to (1, 2_\alpha^\star)$, $q \in L^\infty(\mathbb{R}^N)$ satisfies the property that there exists $x_0 \in \mathbb{R}^N$ and $s > 0$ such that q is continuous in $B_s(x_0)$ and relation (3.1) is fulfilled. Assume that $g : \mathbb{R}^N \to (0, \infty)$ satisfies $g \in L^\infty(\mathbb{R}^N) \cap L^{r(\cdot)}(\mathbb{R}^N)$, where $r(x) = \frac{2_\alpha^\star}{2_\alpha^\star - q(x)}$ for each $x \in \mathbb{R}^N$. Then, there exists $\lambda^\star > 0$ such that problem (1.1) has a nontrivial weak solution for any $\lambda \in (0, \lambda^\star)$.*

4. **Proof of the main result**

In order to prove Theorem 3.1 we define the functional $J : \mathcal{D}_\alpha^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ by

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^N} |x|^\alpha |\nabla u|^2 \, dx - \lambda \int_{\mathbb{R}^N} \frac{g(x)}{q(x)} |u|^{q(x)} \, dx - \frac{1}{2_\alpha^\star} \int_{\mathbb{R}^N} |u|^{2_\alpha^\star} \, dx\,.$$

Standard arguments show that $J \in C^1(\mathcal{D}_\alpha^{1,2}(\mathbb{R}^N), \mathbb{R})$ and

$$\langle J'(u), v\rangle = \int_{\mathbb{R}^N} |x|^\alpha \nabla u \nabla v dx - \lambda \int_{\mathbb{R}^N} g(x) |u|^{q(x)-2} uv \, dx - \int_{\mathbb{R}^N} |u|^{2_\alpha^\star - 2} uv \, dx\,,$$

for all u, $v \in \mathcal{D}_\alpha^{1,2}(\mathbb{R}^N)$. Thus, we remark that in order to find weak solutions of equation (1.1) it is enough to find critical points for the functional J.

Lemma 4.1. *There exists $\lambda^\star > 0$ such that for any $\lambda \in (0, \lambda^\star)$ there exist $\xi > 0$ and $\theta > 0$ such that*

$$J(u) \ge \theta, \quad \forall\, u \in \mathcal{D}_\alpha^{1,2}(\mathbb{R}^N) \text{ with } \|u\|_\alpha = \xi\,.$$

Proof. By Lemma 2.2 and Remark 2.3 it follows taht

$$|u|_{2_\alpha^\star} \le C_\alpha^{1/2} \|u\|_\alpha, \quad \forall\, u \in \mathcal{D}_\alpha^{1,2}(\mathbb{R}^N)\,. \tag{4.1}$$

Consider $\xi \in (0, 1)$ with $\xi < 1/\sqrt{C_\alpha}$. Then the above relation implies

$$|u|_{2_\alpha^\star} < 1, \quad \forall\, u \in \mathcal{D}_\alpha^{1,2}(\mathbb{R}^N), \text{ with } \|u\|_\alpha = \xi\,. \tag{4.2}$$

On the other hand, by relation (2.4) we have

$$| \, |u|^{q(\cdot)} \, |_{\frac{2^\star_\alpha}{q(\cdot)}}^{2^\star_\alpha/q^-} \leq |u|_{2^\star_\alpha}^{2^\star_\alpha}, \quad \forall\, u \in \mathcal{D}^{1,2}_\alpha(\mathbb{R}^N), \text{ with } \|u\|_\alpha = \xi\,. \tag{4.3}$$

Since $g \in L^{r(\cdot)}(\mathbb{R}^N)$, with $r(x) = \frac{2^\star_\alpha}{2^\star_\alpha - q(x)}$ we deduce by Hölder's inequality that there exists a constant $c_1 > 0$ such that

$$\int_{\mathbb{R}^N} g(x)|u|^{q(x)}\,dx \leq c_1 |g|_{r(\cdot)} | \, |u|^{q(\cdot)} \, |_{\frac{2^\star_\alpha}{q(\cdot)}}, \quad \forall\, u \in \mathcal{D}^{1,2}_\alpha(\mathbb{R}^N) \tag{4.4}$$

Relations (4.1), (4.2), (4.3) and (4.4) imply that

$$\int_{\mathbb{R}^N} g(x)|u|^{q(x)}\,dx \leq c_1 |g|_{r(\cdot)} C_\alpha^{q^-/2} \|u\|_\alpha^{q^-}, \quad \forall\, u \in \mathcal{D}^{1,2}_\alpha(\mathbb{R}^N), \text{ with } \|u\|_\alpha = \xi\,. \tag{4.5}$$

Relations (2.1) and (4.5) yield that for any $u \in \mathcal{D}^{1,2}_\alpha(\mathbb{R}^N)$ with $\|u\|_\alpha = \xi$ the following inequalities hold true

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|_\alpha^2 - \lambda \int_{\mathbb{R}^N} \frac{g(x)}{q(x)}|u|^{q(x)}\,dx - \frac{1}{2^\star_\alpha} \cdot |u|_{2^\star_\alpha}^{2^\star_\alpha} \\ &\geq \frac{1}{2}\|u\|_\alpha^2 - \lambda \int_{\mathbb{R}^N} \frac{g(x)}{q(x)}|u|^{q(x)}\,dx - \frac{C_\alpha^{2^\star_\alpha/2}}{2^\star_\alpha} \cdot \|u\|_\alpha^{2^\star_\alpha} \\ &\geq \tfrac{1}{2}\|u\|_\alpha^2 - \tfrac{\lambda}{q^-} \cdot c_2^{q^-} \|u\|_\alpha^{q^-} - c_3 \|u\|_\alpha^{2^\star_\alpha}, \end{aligned} \tag{4.6}$$

where c_2 and c_3 are two positive constants. In other words, for any $u \in \mathcal{D}^{1,2}_\alpha(\mathbb{R}^N)$ with $\|u\|_\alpha = \xi$ we have

$$J(u) \geq \|u\|_\alpha^{q^-} \cdot \left[\frac{1}{2}\|u\|_\alpha^{2-q^-} - \frac{\lambda}{q^-}\, c_2^{q^-} - c_3 \|u\|_\alpha^{2^\star_\alpha - q^-} \right].$$

Define $Q : [0,\infty) \to \mathbb{R}$ by

$$Q(t) = \frac{1}{2}\, t^{2-q^-} - c_3\, t^{2^\star_\alpha - q^-}\,.$$

Since relation (3.1) holds true we deduce that $q^- < 2 < 2^\star_\alpha$ and thus, it is clear that there exists $\beta > 0$ such that $\max_{t \geq 0} Q(t) = Q(\beta) > 0$. We take $\lambda^\star = \frac{q^-}{c_2^{q^-}}\, Q(\beta)$ and we remark that there exists $\theta > 0$ such that for any $\lambda \in (0, \lambda^\star)$ we have

$$J(u) \geq \theta, \quad \forall\, u \in \mathcal{D}^{1,2}_\alpha(\mathbb{R}^N) \text{ with } \|u\|_\alpha = \xi\,.$$

Lemma 4.1 is verified. □

Lemma 4.2. *There exists $\phi \in \mathcal{D}^{1,2}_\alpha(\mathbb{R}^N)$ such that $\phi \geq 0$, $\varphi \neq 0$ and $J(t\phi) < 0$, for $t > 0$ small enough.*

Proof. Since there exists $x_0 \in \mathbb{R}^N$ and $s > 0$ such that q is continuous in $B_s(x_0)$ and relation (3.1) is satisfied we deduce that there exists $\theta \in (1,2)$ such that the open set $\Omega_0 := \{x \in \Omega;\, q(x) < \theta\}$ is nonempty and bounded.

Let $\phi \in C_0^\infty(\mathbb{R}^N)$ be such that $\mathrm{supp}(\phi) \supset \overline{\Omega}_0$, $\phi(x) = 1$ for all $x \in \overline{\Omega}_0$ and $0 \leq \phi \leq 1$ in $\mathbb{R}^N$. For any $t \in (0,1)$ we have

$$\begin{array}{rcl} J(t\phi) & = & \dfrac{t^2}{2}\displaystyle\int_{\mathbb{R}^N} |x|^\alpha |\nabla\phi|^2\,dx - \lambda\int_{\mathbb{R}^N} \frac{t^{q(x)}g(x)}{q(x)}|\phi|^{q(x)}\,dx - \frac{t^{2^\star_\alpha}}{2^\star_\alpha}\int_{\mathbb{R}^N} |\phi|^{2^\star_\alpha}\,dx \\ & \leq & \dfrac{t^2}{2}\displaystyle\int_{\mathbb{R}^N} |x|^\alpha |\nabla\phi|^2\,dx - \frac{\lambda}{q^+}\int_{\Omega_0} g(x)t^{q(x)}|\phi|^{q(x)}\,dx \\ & \leq & \dfrac{t^2}{2}\displaystyle\int_{\mathbb{R}^N} |x|^\alpha |\nabla\phi|^2\,dx - \frac{\lambda\cdot t^\theta}{q^+}\int_{\Omega_0} g(x)|\phi|^{q(x)}\,dx\,. \end{array}$$

It is clear that

$$J(t\phi) < 0\,,$$

providing that

$$0 < t < \min\left\{1, \frac{\lambda\cdot 2}{q^+}\cdot\frac{\int_{\Omega_0} g(x)|\phi|^{q(x)}\,dx}{\int_{\mathbb{R}^N}|x|^\alpha|\nabla\phi|^2\,dx}\right\}.$$

Lemma 4.2 is verified. □

Proof of Theorem 3.1. By inequality (4.6) we obtain that J is bounded from below on $\overline{B_\xi(0)}$. Thus, usinging Ekeland's variational principle (see [5] or [10]) to the functional $J : \overline{B_\xi(0)} \to \mathbb{R}$, it follows that there exists $u_\epsilon \in \overline{B_\xi(0)}$ such that

$$\begin{array}{rcl} J(u_\epsilon) & < & \displaystyle\inf_{\overline{B_\xi(0)}} J + \epsilon \\ J(u_\epsilon) & < & J(u) + \epsilon\cdot\|u - u_\epsilon\|_\alpha, \quad u \neq u_\epsilon. \end{array}$$

Using Lemmas 4.1 and 4.2 we find

$$\inf_{\partial B_\xi(0)} J \geq \theta > 0 \quad \text{and} \quad \inf_{\overline{B_\xi(0)}} J < 0\,.$$

We choose $\epsilon > 0$ such that

$$0 < \epsilon \leq \inf_{\partial B_\xi(0)} J - \inf_{\overline{B_\xi(0)}} J\,.$$

Therefore, $J(u_\epsilon) < \inf_{\partial B_\xi(0)} J$ and thus, $u_\epsilon \in B_\xi(0)$.

We define $I : \overline{B_\xi(0)} \to \mathbb{R}$ by $I(u) = J(u) + \epsilon\cdot\|u - u_\epsilon\|_\alpha$. It is clear that u_ϵ is a minimum point of I and thus

$$\frac{I(u_\epsilon + \delta\cdot v) - I(u_\epsilon)}{\delta} \geq 0$$

for small $\delta > 0$ and any $v \in B_1(0)$. The above relation yields

$$\frac{J(u_\epsilon + \delta\cdot v) - J(u_\epsilon)}{\delta} + \epsilon\cdot\|v\|_\alpha \geq 0.$$

Letting $\delta \to 0$ it follows that $\langle J^{'}(u_\epsilon), v\rangle + \epsilon\cdot\|v\|_\alpha > 0$ and we infer that $\|J^{'}(u_\epsilon)\| \leq \epsilon$.

We deduce that there exists a sequence $\{u_n\} \subset B_\xi(0)$ such that

$$J(u_n) \to c = \inf_{B_\xi(0)} J < 0 \quad \text{and} \quad J^{'}(u_n) \to 0. \tag{4.7}$$

It is clear that $\{u_n\}$ is bounded in $\mathcal{D}^{1,2}_\alpha(\mathbb{R}^N)$. Thus, there exists $w \in \mathcal{D}^{1,2}_\alpha(\mathbb{R}^N)$ such that, up to a subsequence, $\{u_n\}$ converges weakly to u in $\mathcal{D}^{1,2}_\alpha(\mathbb{R}^N)$. Then Lemma 2.2 (actually, Remark 2.3) implies that $\{u_n\}$ converges weakly to u in $L^{2^\star_\alpha}(\Omega)$. Using these information and the fact that $g \in L^{2^\star_\alpha/q(\cdot)}(\mathbb{R}^N)$, we get that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} g(x)|u_n|^{q(x)-2}u_n v \, dx = \int_{\mathbb{R}^N} g(x)|u|^{q(x)-2}uv \, dx \,,$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{2^\star_\alpha-2}u_n v \, dx = \int_{\mathbb{R}^N} |u|^{2^\star_\alpha-2}uv \, dx \,,$$

for any $v \in C_0^\infty(\mathbb{R}^N)$.

On the other hand, relation (4.7) implies

$$\lim_{n\to\infty} \langle J'(u_n), v\rangle = 0 \,,$$

for all $v \in C_0^\infty(\mathbb{R}^N)$ and actually, (by density) for all $v \in \mathcal{D}^{1,2}_\alpha(\mathbb{R}^N)$.

The above information implies

$$J'(u) = 0 \,,$$

and thus, u is a weak solution of equation (1.1).

We prove now that $u \neq 0$. Assume by contradiction that $u \equiv 0$ and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |x|^\alpha |\nabla u_n|^2 \, dx = l \geq 0 \,.$$

Since by relation (4.7) we have $\lim_{n\to\infty} \langle J'(u_n), u_n\rangle = 0$ and $\{u_n\}$ converges weakly to 0 in $L^{2^\star_\alpha}(\mathbb{R}^N)$ and $g \in L^{2^\star_\alpha/q(\cdot)}(\mathbb{R}^N)$ we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} g(x)|u_n|^{q(x)} \, dx = 0 \,,$$

or

$$\int_{\mathbb{R}^N} |x|^\alpha |\nabla u_n|^2 \, dx - \int_{\mathbb{R}^N} |u_n|^{2^\star_\alpha} \, dx = o(1)$$

or

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{2^\star_\alpha} \, dx = l \,.$$

Using again (4.7) we deduce

$$\begin{aligned} 0 > c + o(1) &= \frac{1}{2} \int_{\mathbb{R}^N} |x|^\alpha |\nabla u_n|^2 \, dx - \lambda \int_{\mathbb{R}^N} \frac{g(x)}{q(x)} |u_n|^{q(x)} \, dx - \frac{1}{2^\star_\alpha} \int_{\mathbb{R}^N} |u_n|^{2^\star_\alpha} \, dx \\ &\to \left(\frac{1}{2} - \frac{1}{2^\star_\alpha}\right) l \geq 0 \end{aligned}$$

and that is a contradiction. We conclude that $u \neq 0$.

Thus, Theorem 3.1 is proved. $\square$

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References

[1] Caffarelli, L., Kohn, R., Nirenberg, L., *First order interpolation inequalities with weights*, Compositio Math., **53** (1984), 259-275.

[2] Catrina, F., Wang, Z.-Q., *On the Caffarelli-Kohn-Nirenberg inequalities: sharp constants, existence (and nonexistence) and symmetry of extremal function*, Commun. Pure Appl. Math., **54** (2001), 229-258.

[3] Caldiroli, P., Musina, R., *On the existence of extremal functions for a weighted Sobolev embedding with critical exponent*, Calc. Var. Partial Differential Equations, **8** (1999), 365-387.

[4] aldiroli, P., Musina, R., *On a variational degenerate elliptic problem*, NoDEA Nonlinear Differential Equations Appl., **7** (2000), 187-199.

[5] Ekeland, I., *On the variational principle*, J. Math. Anal. App., **47** (1974), 324-353.

[6] Gilbarg, D., Trudinger, N. S., *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 1998.

[7] Kováčik, O., Rákosník, J., *On spaces $L^{p(x)}$ and $W^{1,p(x)}$*, Czechoslovak Math. J., **41** (1991), 592-618.

[8] Mihăilescu, M., Rădulescu, V., *A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids*, Proc. Roy. Soc. London Ser. A, **462** (2006), 2625-2641.

[9] Mihăilescu, M., Rădulescu, V., *On a nonhomogeneous quasilinear eigenvalue problem in Sobolev spaces with variable exponent*, Proceedings Amer. Math. Soc., **135** (2007), 2929-2937.

[10] Willem, M., *Minimax Theorems*, Birkhäuser, Boston, 1996.

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