STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LV**, Number 4, December 2010

COMBINED VARIATIONAL AND SUB-SUPERSOLUTION APPROACH FOR MULTI-VALUED ELLIPTIC VARIATIONAL INEQUALITIES

SIEGFRIED CARL

Abstract. This paper provides a variational approach for a class of multi-valued elliptic variational inequalities governed by the p-Laplacian and Clarke's generalized gradient of some locally Lipschitz function including a number of (multi-valued) elliptic boundary value problems as special cases. Since only local growth conditions are imposed on the multi-valued term, the problem under consideration is neither coercive nor of variational structure beforehand meaning that it cannot be related to the derivative of some associated (nonsmooth) potential. By combining a recently developed sub-supersolution method for multi-valued elliptic variational inequalities and a suitable modification of the given locally Lipschitz function the main goal of this paper is to construct a (nonsmooth) functional whose critical points turn out to be solutions of the problem under consideration lying in an ordered interval of sub-supersolution.

1. **Introduction**

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$, and let $V = W^{1,p}(\Omega)$ and $V_0 = W_0^{1,p}(\Omega)$, $1 < p < +\infty$, denote the usual Sobolev spaces with their dual spaces V^* and V_0^*, respectively. Let K be a closed, convex subset of V, and let $j : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function $(x, s) \mapsto j(x, s)$ that is only supposed to be measurable in $x \in \Omega$ and locally Lipschitz continuous in $s \in \mathbb{R}$. Let q denote the Hölder conjugate to p, i.e., q satisfies $1/p + 1/q = 1$. In this paper we are dealing with the following multi-valued variational inequality: Find $u \in K$, $\eta \in L^q(\Omega)$ such that

$$\langle -\Delta_p u, v - u\rangle + \int_\Omega \eta\,(v - u)\,dx \ge 0, \quad \forall\, v \in K, \tag{1.1}$$

$$\eta(x) \in \partial j(x, u(x)) \ \text{ for a.a. } x \in \Omega, \tag{1.2}$$

Received by the editors: 20.09.2010.
2000 *Mathematics Subject Classification.* 35B51, 35J87, 47J22, 49J52.
Key words and phrases. Multi-valued variational inequality, p-Laplacian, Clarke's generalized gradient, sub-supersolution, critical point, locally Lipschitz functional, comparison principle.

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $s \mapsto \partial j(x,s)$ denotes Clarke's generalized gradient of some locally Lipschitz function $s \mapsto j(x,s)$, and $\langle \cdot, \cdot \rangle$ denotes the duality pairing. The operator $-\Delta_p$ is defined by

$$\langle -\Delta_p u, \varphi \rangle = \int_\Omega |\nabla u|^{p-2} \nabla u \, \nabla \varphi \, dx, \quad \forall \, \varphi \in V,$$

which implies that $-\Delta_p : V \to V^*$ is continuous, bounded, monotone, and thus pseudomonotone, see [1, Theorem 2.109, Lemma 2.111].

Only for the sake of simplifying our presentation and in order to emphasize the key ideas we have confined our consideration to problem (1.1)-(1.2). Making use of the arguments developed in this paper, more general multi-valued problems can be considered as well such as, for example, the following one: Find $u \in K$, $\eta \in L^q(\Omega)$, and $\xi \in L^q(\partial\Omega)$ such that

$$\begin{cases} \eta(x) \in \partial j_1(x, u(x)), \text{ a.e. } x \in \Omega, \quad \xi(x) \in \partial j_2(x, \gamma u(x)), \text{ a.e. } x \in \partial\Omega, \\ \langle -\Delta_p u - h, v - u \rangle + \displaystyle\int_\Omega \eta \, (v-u) \, dx + \int_{\partial\Omega} \xi \, (\gamma v - \gamma u) \, d\sigma \geq 0, \ \forall \, v \in K, \end{cases} \tag{1.3}$$

where $\gamma : V \to L^p(\partial\Omega)$ denotes the trace operator, and $h \in V^*$.

The main goal of this paper is to develop a variational approach to the multi-valued variational inequality (1.1)-(1.2). Since only a local growth condition is imposed on the multi-valued term, the problem under consideration is neither coercive nor of variational structure beforehand meaning that it cannot be related to the derivative of some associated (nonsmooth) potential. Therefore, the main difficulty one is faced with is to associate to (1.1)-(1.2) a corresponding potential that can be studied by (nonsmooth) variational methods. By combining a recently developed sub-supersolution method for elliptic variational inequalities (see [1]) with a suitable modification of the given locally Lipschitz function, the aim of this paper is to construct a (nonsmooth) functional whose critical points turn out to be solutions of problem (1.1)-(1.2).

2. **Special Cases**

Let us consider a few special cases that are included in (1.1)-(1.2).

Example 2.1. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. Consider its primitive given by

$$j(x,s) := \int_0^s f(x,t) \, dt.$$

Then the function $s \mapsto j(x,s)$ is continuously differentiable, and thus Clarke's gradient reduces to a singleton, i.e.,

$$\partial j(x,s) = \{\partial j(x,s)/\partial s\} = \{f(x,s)\}.$$

If $K = V$, then (1.1)-(1.2) becomes the following quasilinear elliptic boundary value problem (BVP)

$$\langle -\Delta_p u, v\rangle + \int_\Omega f(x,u)\, v\, dx = 0, \quad \forall\, v \in V, \tag{2.1}$$

which is the formulation for the weak solution of the quasilinear Neumann BVP

$$-\Delta_p u + f(x,u) = 0 \ \text{ in } \ \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \ \ \partial\Omega, \tag{2.2}$$

where $\partial/\partial\nu$ denotes the outward pointing conormal derivative associated with $-\Delta_p$.

Example 2.2. If $K = V_0$, and j as in Example 2.1, then (1.1)-(1.2) is equivalent to

$$u \in V_0 : \quad \langle -\Delta_p u, v\rangle + \int_\Omega f(x,u)\, v\, dx = 0, \quad \forall\, v \in V_0, \tag{2.3}$$

which is nothing but the weak formulation of the homogeneous Dirichlet problem

$$-\Delta_p u + f(x,u) = 0 \ \text{ in } \ \Omega, \ \ u = 0 \quad \text{on} \ \ \partial\Omega. \tag{2.4}$$

Example 2.3. If $K = V_0$ or $K = V$, then (1.1)-(1.2) reduces to elliptic inclusion problems, which for $K = V_0$ yields the following multi-valued Dirichlet problem

$$-\Delta_p u + \partial j_1(x,u) \ni 0 \ \text{ in } \ \Omega, \ \ u = 0 \quad \text{on} \ \ \partial\Omega, \tag{2.5}$$

and for $K = V$ the multi-valued Neumann BVP

$$-\Delta_p u + \partial j(x,u) \ni 0 \ \text{ in } \ \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \ \ \partial\Omega. \tag{2.6}$$

Example 2.4. Let Γ_1 and Γ_2 be relatively open subsets of $\partial\Omega$ satisfying $\overline{\Gamma_1} \cup \overline{\Gamma_2} = \partial\Omega$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$. If $K \subseteq V$ is the closed subspace given by

$$K = \{v \in V : \gamma v = 0 \ \text{ on } \ \Gamma_1\},$$

then we obtain the following special case of (1.1)-(1.2):

$$-\Delta_p u + \partial j(x,u) \ni 0 \ \text{ in } \ \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \ \ \Gamma_2, \ \ u = 0 \ \text{ on } \ \Gamma_1. \tag{2.7}$$

Example 2.5. If $K \subseteq V$, and $j = 0$, then (1.1)-(1.2) is equivalent to the usual variational inequality of the form

$$u \in K : \quad \langle -\Delta_p u, v - u\rangle \ge 0, \quad \forall\, v \in K.$$

3. **Definitions, Assumptions and Preliminaries**

Based on comparison principles for nonsmooth variational problems developed in [1] we first provide a natural extension of the notion of sub-supersolution to the multi-valued variational problem (1.1)-(1.2). To this end we introduce the following notations for functions $w,\ z$ and sets W and Z of functions defined on Ω: $w \wedge z = \min\{w, z\},\ w \vee z = \max\{w, z\},\ W \wedge Z = \{w \wedge z : w \in W,\ z \in Z\}$, $W \vee Z = \{w \vee z : w \in W,\ z \in Z\}$, and $w \wedge Z = \{w\} \wedge Z$, $w \vee Z = \{w\} \vee Z$.

Definition 3.1. A function $\underline{u} \in V$ is called a **subsolution** of (1.1)-(1.2) if there is an $\underline{\eta} \in L^q(\Omega)$ satisfying

$$(i)\ \underline{u} \vee K \subseteq K,$$

$$(ii)\ \underline{\eta}(x) \in \partial j(x, \underline{u}(x)), \text{for a.e. } x \in \Omega,$$

$$(iii)\ \langle -\Delta_p \underline{u}, v - \underline{u}\rangle + \int_\Omega \underline{\eta}\,(v - \underline{u})\, dx \geq 0, \quad \text{for all} \quad v \in \underline{u} \wedge K.$$

Definition 3.2. A function $\overline{u} \in V$ is called a **supersolution** of (1.1)-(1.2) if there is an $\overline{\eta} \in L^q(\Omega)$ satisfying

$$(i)\ \overline{u} \wedge K \subseteq K,$$

$$(ii)\ \overline{\eta}(x) \in \partial j(x, \overline{u}(x)),\ \text{for a.e. } x \in \Omega,$$

$$(iii)\ \langle -\Delta_p \overline{u}, v - \overline{u}\rangle + \int_\Omega \overline{\eta}\,(v - \overline{u})\, dx \geq 0, \quad \text{for all } v \in \overline{u} \vee K.$$

Remark 3.3. Note that the notions for sub- and supersolution defined in Definition 3.1 and Definition 3.2 have a symmetric structure, i.e., one obtains the definition for the supersolution $\overline{u}$ from the definition of the subsolution by replacing $\underline{u}$ in Definition 3.1 by $\overline{u}$, and interchanging $\vee$ by $\wedge$.

To see that Definitions 3.1 and 3.2 are in fact natural extensions of the usual notions of sub-supersolutions for elliptic BVP let us consider the following special cases.

Example 3.4. Consider Example 2.1, i.e., $K = V$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, and j is the primitive of f as given above. Then Clarke's generalized gradient ∂j reduces to a singleton, i.e.,

$$\partial j(x, s) = \{f(x, s)\},$$

and (1.1)-(1.2) becomes the quasilinear elliptic BVP (2.1). If $\underline{u} \in V$ is a subsolution according to Definition 3.1, then the first condition (i) is trivially satisfied. The second condition (ii) of Definition 3.1 means that

$$\underline{\eta}(x) = f(x, \underline{u}(x)), \text{ for a.e. } x \in \Omega.$$

Since $K = V$, any $v \in \underline{u} \wedge V$ has the form $v = \underline{u} \wedge \varphi = \underline{u} - (\underline{u} - \varphi)^+$ with $\varphi \in V$, where $w^+ = \max\{w, 0\}$, condition (iii) becomes

$$\langle -\Delta_p \underline{u}, -(\underline{u} - \varphi)^+ \rangle + \int_\Omega f(\cdot, \underline{u}) \, (-(\underline{u} - \varphi)^+) \, dx \geq 0, \quad \forall \, \varphi \in V. \tag{3.1}$$

Since $\underline{u} \in V$, we have

$$M = \{(\underline{u} - \varphi)^+ : \varphi \in V\} = V \cap L^p_+(\Omega),$$

where $L^p_+(\Omega)$ is the positive cone of $L^p(\Omega)$, and thus we obtain from inequality (3.1)

$$\langle -\Delta_p \underline{u}, \chi \rangle + \int_\Omega f(x, \underline{u}) \, \chi \, dx \leq 0, \quad \forall \, \chi \in V \cap L^p_+(\Omega, \tag{3.2}$$

which is nothing but the usual notion of a (weak) subsolution for the BVP (2.1). Similarly, one verifies that $\overline{u} \in V$ which is a supersolution according to Definition 3.2 is equivalent with the usual supersolution of the BVP (2.1).

Example 3.5. In case that $K = V_0$, and j as in Example 3.4, then (1.1)-(1.2) is equivalent to the BVP (2.3) (resp. (2.4)). Let us consider the notion of subsolution in this case given via Definition 3.1. For $\underline{u} \in V$ condition (i) means $\underline{u} \vee V_0 \subseteq V_0$. This last condition is satisfied if and only if

$$\gamma \underline{u} \leq 0 \quad \text{i.e.,} \quad \underline{u} \leq 0 \;\; \text{on} \;\; \partial\Omega, \tag{3.3}$$

and condition (ii) means, as above,

$$\underline{\eta}(x) = f(x, \underline{u}(x)), \text{ a.e. } x \in \Omega.$$

Since any $v \in \underline{u} \wedge V_0$ can be represented in the form $v = \underline{u} - (\underline{u} - \varphi)^+$ with $\varphi \in V_0$, from (iii) of Definition 3.1 we obtain

$$\langle -\Delta_p \underline{u}, -(\underline{u} - \varphi)^+ \rangle + \int_\Omega f(\cdot, \underline{u}) \, (-(\underline{u} - \varphi)^+) \, dx \geq 0, \; \forall \, \varphi \in V_0. \tag{3.4}$$

Set $\chi = (\underline{u} - \varphi)^+$, then (3.4) results in

$$\langle -\Delta_p \underline{u}, \chi \rangle + \int_\Omega f(\cdot, \underline{u}) \, \chi \, dx \leq 0, \; \forall \, \chi \in M_0, \tag{3.5}$$

where $M_0 := \{\chi \in V : \chi = (\underline{u} - \varphi)^+, \; \varphi \in V_0\} \subseteq V_0 \cap L^p_+(\Omega)$. In [1] it has been proved that the set M_0 is a dense subset of $V_0 \cap L^p_+(\Omega)$, which shows that (3.5) together with (3.3) is nothing but the weak formulation for the subsolution of the Dirichlet problem (2.3). Similarly, $\overline{u} \in V$ given by Definition 3.6 is shown to be a supersolution of the Dirichlet problem (2.3).

Assumption on j.

(H1) The function $j : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies: $x \mapsto j(x, s)$ is measurable in Ω for all $s \in \mathbb{R}$, and $s \mapsto j(x, s)$ is locally Lipschitz continuous in $\mathbb{R}$ for a.e. $x \in \Omega$.

We next introduce a certain local L^q-boundedness condition for Clarke's generalized gradient $s \mapsto \partial j(x, s)$.

Definition 3.6. Let $[v, w] \subset L^p(\Omega)$ be an ordered interval. Clarke's gradient $\partial j : \Omega \times \mathbb{R} \to 2^{\mathbb{R}}$ is called **$\mathbf{L^q}$-bounded with respect to the ordered interval $[\mathbf{v}, \mathbf{w}]$** provided that there exists $k_\Omega \in L^q_+(\Omega)$ such that for a.e. $x \in \Omega$ and for all $s \in [v(x), w(x)]$ the inequality

$$|\eta| \leq k_\Omega(x), \quad \forall\, \eta \in \partial j(x, s),$$

is fulfilled.

Remark 3.7. (i) We note that $\partial j : \Omega \times \mathbb{R} \to 2^{\mathbb{R}}$ is trivially L^q-bounded with respect to any ordered interval $[v, w] \subset L^p(\Omega)$ if we suppose the following natural growth condition on ∂j: There exist $c > 0$, $k_\Omega \in L^q_+(\Omega)$ such that

$$|\eta| \leq k_\Omega(x) + c|s|^{p-1}, \quad \forall\, \eta \in \partial j(x, s),$$

for a.a. $x \in \Omega$ and for all $s \in \mathbb{R}$.

(ii) If ∂j is a singleton, i.e., $\partial j(x, s) = \{f(x, s)\}$ then accordingly we call the function $(x, s) \mapsto f(x, s)$ L^q-bounded with respect to the ordered interval $[v, w] \subset L^p(\Omega)$ provided that there exists $k_\Omega \in L^q_+(\Omega)$ such that for a.e. $x \in \Omega$ and for all $s \in [v(x), w(x)]$ the inequalty

$$|f(x, s)| \leq k_\Omega(x)$$

is fulfilled.

The construction of an appropriate functional related to (1.1)-(1.2) relies amongst others on a suitable modification of the function j outside the interval $[\underline{u}, \overline{u}]$ formed by a given pair of sub- and supersolutions. Let $(\underline{u}, \underline{\eta}) \in V \times L^q(\Omega)$ and $(\overline{u}, \overline{\eta}) \in V \times L^q(\Omega)$ satisfy the conditions of Definition 3.1 and Definition 3.2, respectively, with $\underline{u} \leq \overline{u}$. Then we define the following modification $\tilde{j}$ of the given j:

$$\tilde{j}(x, s) = \begin{cases} j(x, \underline{u}(x)) + \underline{\eta}(x)(s - \underline{u}(x)) & \text{if} \quad s < \underline{u}(x), \\ j(x, s) & \text{if} \quad \underline{u}(x) \leq s \leq \overline{u}(x), \\ j(x, \overline{u}(x)) + \overline{\eta}(x)(s - \overline{u}(x)) & \text{if} \quad s > \overline{u}(x). \end{cases} \tag{3.6}$$

Assumption on j.

(H2) Let $\underline{u}$ and $\overline{u}$ be sub-and supersolution of (1.1)-(1.2) such that $\underline{u} \le \overline{u}$. We assume that $\partial j : \Omega \times \mathbb{R} \to 2^{\mathbb{R}}$ is L^q-bounded with respect to the ordered interval $[\underline{u}, \overline{u}]$.

Lemma 3.8. *Let hypotheses (H1)–(H2) be satisfied. Then the function $\tilde{j} : \Omega \times \mathbb{R} \to \mathbb{R}$ has the following properties:*

(i) *$x \mapsto \tilde{j}(x,s)$ is measurable in Ω for all $s \in \mathbb{R}$, and $s \mapsto \tilde{j}(x,s)$ is Lipschitz continuous in $\mathbb{R}$ for a.e. $x \in \Omega$.*

(ii) *Let $\partial\tilde{j}$ denote Clarke's generalized gradient of $s \mapsto \tilde{j}(x,s)$, then for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$ the growth*

$$|\eta| \le k_\Omega(x), \quad \forall\, \eta \in \partial\tilde{j}(x,s)$$

is fulfilled.

(iii) *Clarke's generalized gradient of $s \mapsto \tilde{j}(x,s)$ is given by*

$$\partial\tilde{j}(x,s) = \begin{cases} \underline{\eta}(x) & \text{if} \quad s < \underline{u}(x), \\ \partial\tilde{j}(x,\underline{u}(x)) & \text{if} \quad s = \underline{u}(x), \\ \partial j(x,s) & \text{if} \quad \underline{u}(x) < s < \overline{u}(x), \\ \partial\tilde{j}(x,\overline{u}(x)) & \text{if} \quad s = \overline{u}(x), \\ \overline{\eta}(x) & \text{if} \quad s > \overline{u}(x), \end{cases} \tag{3.7}$$

and the inclusions $\partial\tilde{j}(x,\underline{u}(x)) \subseteq \partial j(x,\underline{u}(x))$ and $\partial\tilde{j}(x,\overline{u}(x)) \subseteq \partial j(x,\overline{u}(x))$ hold true.

Proof. The proof follows immediately from the definition (3.6) of $\tilde{j}$, and using the assumptions (H1)–(H2) on j as well as from the fact that Clarke's generalized gradient $\partial j_1(x,s)$ is a convex set. ☐

Using $\tilde{j}$ we define an integral functional $\tilde{J}$ on $L^p(\Omega)$ given by

$$\tilde{J}(u) = \int_\Omega \tilde{j}(x,u(x))\,dx, \quad u \in L^p(\Omega). \tag{3.8}$$

Due to (ii) of Lemma 3.8, and applying Lebourg's mean value theorem (see [1, Theorem 2.177]) the functional $\tilde{J} : L^p(\Omega) \to \mathbb{R}$ is well-defined and Lipschitz continuous, so that Clarke's generalized gradients $\partial\tilde{J} : L^p(\Omega) \to 2^{(L^p(\Omega))^*}$ is well-defined too. Moreover, Aubin–Clarke theorem (cf. [8, p. 83]) provides the following characterization of the generalized gradient. For $u \in L^p(\Omega)$ we have

$$\tilde{\eta} \in \partial\tilde{J}(u) \Longrightarrow \tilde{\eta} \in L^q(\Omega) \text{ with } \tilde{\eta}(x) \in \partial\tilde{j}(x,u(x)) \text{ for a.e. } x \in \Omega. \tag{3.9}$$

Lemma 3.9. *Let $i : V \hookrightarrow L^p(\Omega)$ denote the embedding operator and let $i^* : L^q(\Omega) \hookrightarrow V^*$ be its adjoint operator. Then Clarke's generalized gradient of $\tilde{J}$ at $u \in V$ is given* *by*

$$\partial \tilde{J}(u) = \partial(\tilde{J} \circ i)(u) = (i^* \circ \partial \tilde{J} \circ i)(u) = i^* \partial \tilde{J}(u), \quad \forall\, u \in V.$$

Proof. Apply the chain rule, cf. [1, Corollary 2.180]. □

Finally, let b be the cut-off function related to an ordered pair $(\underline{u}, \overline{u})$ of sub-supersolution and defined as follows:

$$b(x,s) = \begin{cases} (s - \overline{u}(x))^{p-1} & \text{if} \quad s > \overline{u}(x) \\ 0 & \text{if} \quad \underline{u}(x) \le s \le \overline{u}(x) \\ -(\underline{u}(x) - s)^{p-1} & \text{if} \quad s < \underline{u}(x). \end{cases}$$

Apparently, $b : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the growth condition

$$|b(x,s)| \le k(x) + c_1 |s|^{p-1} \tag{3.10}$$

for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$, where $c_1 > 0$ is a constant and $k \in L^q_+(\Omega)$. Moreover, one has the following estimate

$$\int_\Omega b(x, u(x))\, u(x)\, dx \ge c_2 \|u\|^p_{L^p(\Omega)} - c_3, \quad \forall u \in L^p(\Omega), \tag{3.11}$$

for some constants $c_2 > 0$ and $c_3 > 0$. Due to (3.10) the functional $\mathbb{B}$ given by

$$\mathbb{B}(u) = \int_\Omega \int_0^{u(x)} b(x,s)\, ds dx, \quad \forall\, u \in L^p(\Omega) \tag{3.12}$$

is well defined, and $\mathbb{B} \in C^1(V, \mathbb{R})$ with

$$\langle \mathbb{B}'(u), \varphi \rangle = \int_\Omega b(x, u(x))\, \varphi(x)\, dx, \quad \forall u \in V. \tag{3.13}$$

Lemma 3.10. *There exist constants $c_4 > 0$, $c_5 > 0$ such that*

$$\mathbb{B}(u) \ge c_4 \|u\|^p_{L^p(\Omega)} - c_5, \quad \forall u \in L^p(\Omega). \tag{3.14}$$

Proof. From the definition of the cut-off function b we readily see that β given by

$$\beta(x,s) = \begin{cases} \frac{1}{p}(s - \overline{u}(x))^p & \text{if} \quad s > \overline{u}(x) \\ 0 & \text{if} \quad \underline{u}(x) \le s \le \overline{u}(x) \\ \frac{1}{p}(\underline{u}(x) - s)^p & \text{if} \quad s < \underline{u}(x). \end{cases} \tag{3.15}$$

is a primitive of $s \mapsto b(x,s)$, i.e., $\partial\beta(x,s)/\partial s = b(x,s)$, which yields

$$\int_0^{u(x)} b(x,s)\, ds = \beta(x, u(x)) - \beta(x, 0). \tag{3.16}$$

By using (3.15) we get the estimate

$$|\beta(x,0)| \le \frac{1}{p}(|\underline{u}|^p + |\overline{u}|^p). \tag{3.17}$$

For functions v, $w \in L^p(\Omega)$ we denote

$$\{v < (\leq) w\} = \{x \in \Omega : v(x) < (\leq) w(x)\}.$$

We next estimate the first term on the right-hand side of (3.16). To this end we make use of the following inequality:

$$|u(x)|^p \leq c\left(|u(x) - \overline{u}(x)|^p + |\overline{u}(x)^p\right)$$

for some generic positive constant c, which yields

$$\frac{1}{p}|u(x) - \overline{u}(x)|^p \geq \frac{1}{pc}|u(x)|^p - \frac{1}{p}|\overline{u}(x)|^p, \tag{3.18}$$

and similarly

$$\frac{1}{p}|\underline{u}(x) - u(x)|^p \geq \frac{1}{pc}|u(x)|^p - \frac{1}{p}|\underline{u}(x)|^p. \tag{3.19}$$

With the help of (3.15) and (3.18)–(3.19) we obtain

$$\begin{aligned}
\int_\Omega \beta(x, u(x))\, dx &= \int_{\{u>\overline{u}\}} \beta(x, u(x))\, dx + \int_{\{u<\underline{u}\}} \beta(x, u(x))\, dx \\
&\geq \frac{1}{pc}\int_\Omega |u(x)|^p\, dx - \frac{1}{pc}\int_{\{\underline{u}\leq u\leq\overline{u}\}} |u(x)|^p\, dx \\
&\quad -\frac{1}{p}\int_\Omega \left(|\underline{u}(x)|^p + |\overline{u}(x)|^p\right) dx \\
&\geq \frac{1}{pc}\|u\|^p_{L^p(\Omega)} - \frac{2}{p}\left(\|\underline{u}\|^p_{L^p(\Omega)} + \|\overline{u}\|^p_{L^p(\Omega)}\right). \qquad (3.20)
\end{aligned}$$

Finally, (3.16), (3.17) and (3.20) imply the assertion of the lemma, i.e.,

$$\mathbb{B}(u) \geq \frac{1}{pc}\|u\|^p_{L^p(\Omega)} - \frac{3}{p}\left(\|\underline{u}\|^p_{L^p(\Omega)} + \|\overline{u}\|^p_{L^p(\Omega)}\right)$$

with

$$c_4 = \frac{1}{pc},\ c_5 = \frac{3}{p}\left(\|\underline{u}\|^p_{L^p(\Omega)} + \|\overline{u}\|^p_{L^p(\Omega)}\right).$$

□

4. Combined Variational and Sub-Supersolution Approach

In this section we formulate and prove our main result. A crucial role in our approach plays the following functional

$$\Phi(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\, dx + \mathbb{B}(u) + \tilde{J}(u), \quad u \in V, \tag{4.1}$$

where $\mathbb{B}$ and $\tilde{J}(u)$ are defined by (3.8) and (3.12), respectively.

Lemma 4.1. *Let hypotheses (H1) and (H2) be satisfied. Then the functional* $\Phi : V \to \mathbb{R}$ *is locally Lipschitz continuous, bounded below, coercive, and weakly lower semicontinuous.*

Proof. By the definition of $\tilde{J}$ and due to Lemma 3.8 (ii) we readily see that $\tilde{J} : L^p(\Omega) \to \mathbb{R}$ is Lipschitz continuous, which in view of the compact embedding $V \hookrightarrow L^p(\Omega)$ shows that $\tilde{J} : V \to \mathbb{R}$ is weakly lower semicontinuous. The functionals

$$u \mapsto P(u) := \frac{1}{p}\int_\Omega |\nabla u|^p \, dx \text{ and } u \mapsto \mathbb{B}(u)$$

are $C^1(V, \mathbb{R})$, and thus, in particular, locally Lipschitz continuous as well. The derivative $P' + \mathbb{B}' : V \to V^*$ results in

$$\langle P'(u) + \mathbb{B}'(u), \varphi\rangle = \langle -\Delta_p u + \mathbb{B}'(u), \varphi\rangle = \int_\Omega \Big(|\nabla u|^{p-2}\nabla u \nabla\varphi + b(\cdot, u)\varphi\Big)\, dx, \quad \forall\, \varphi \in V.$$

Taking into account (3.10) and applying [1, Theorem 2.109, Lemma 2.111] we see that the operator $P' + \mathbb{B}' : V \to V^*$ is bounded, and pseudomonotone, which in view of [10, Proposition 41.8] implies that the functional $P + \mathbb{B} : V \to \mathbb{R}$ is weakly lower semicontinuous. Due to Lemma 3.8 (ii), (iii) the functional $\tilde{J} : L^p(\Omega) \to \mathbb{R}$ is (globally) Lipschitz continuous with Lipschitz constant L. Thus by means of Lemma 3.10 we obtain the following estimate (for some constant $c_6 > 0$)

$$\Phi(u) = P(u) + \mathbb{B}(u) + \tilde{J}(u) \geq \frac{1}{p}\|\nabla u\|^p_{L^p(\Omega)} + c_4\|u\|^p_{L^p(\Omega)} - L\|u\|_{L^p(\Omega)} - c_6, \quad (4.2)$$

which shows that $\Phi : V \to \mathbb{R}$ is bounded below and coercive. □

Let $I_K : V \to \mathbb{R} \cup \{+\infty\}$ be the indicator function related to the given closed convex set $K \neq \emptyset$, i.e.,

$$I_K(u) = \begin{cases} 0 & \text{if} \quad u \in K, \\ +\infty & \text{if} \quad u \notin K, \end{cases}$$

which is known to be proper, convex, and lower semicontinuous, and thus weakly lower semicontinuous as well (cf. [10, Proposition 38.7]). The following functional $\mathbb{E} : V \to \mathbb{R} \cup \{+\infty\}$ will allow us to study the multi-valued variational inequality (1.1)–(1.2) via variational methods for nonsmooth and nonconvex functionals:

$$\mathbb{E}(u) = \Phi(u) + I_K(u), \quad u \in V, \quad (4.3)$$

i.e., $\mathbb{E}$ is the sum of a locally Lipschitz functional and a convex, proper and lower semicontinuous functional. This type of functional has been studied, e.g., in [9].

Definition 4.2. The function $u \in V$ is called a **critical point** of $\mathbb{E} : V \to \mathbb{R} \cup \{+\infty\}$ if the following holds:

$$\Phi^o(u; v-u) + I_K(v) - I_K(u) \geq 0, \quad \forall\, v \in V,$$

where $\Phi^o(u; v)$ denotes Clarke's generalized directional derivative of Φ at u in the direction v.

The following definition is equivalent to Definition 4.2, see [9, p.46].

Definition 4.3. The function $u \in V$ is called a **critical point** of $\mathbb{E} : V \to \mathbb{R} \cup \{+\infty\}$ if and only if

$$0 \in \partial\Phi(u) + \partial I_K(u),$$

where $\partial\Phi(u)$ denotes Clarke's generalized gradient of Φ at u, and $\partial I_K(u)$ is the subdifferential of I_K at u in the sense of convex analysis.

Our main result is given by the following theorem.

Theorem 4.4. *Let hypotheses (H1)–(H2) be satisfied. Then the functional* $\mathbb{E} = \Phi + I_K : V \to \mathbb{R} \cup \{+\infty\}$ *possesses critical points. Moreover, any critical point* u *of* $\mathbb{E}$ *is a solution of the multi-valued variational inequality (1.1)–(1.2) which belongs to the ordered interval* $[\underline{u}, \overline{u}]$ *formed by the given ordered sub- and supersolution.*

Proof. (a) *Existence of critical points*

By Lemma 4.1 in conjunction with the properties of the indicator function I_K, the functional $\mathbb{E} : V \to \mathbb{R} \cup \{+\infty\}$ defined in (4.3) is weakly lower semicontinuous, bounded below and coercive. Applying the basic minimization principle (cf. e.g. [10, Proposition 38.15]) there exists a global minimizer u of $\mathbb{E}$ which necessarily is a critical point of $\mathbb{E}$ (see [9]), i.e., $u \in K$ and $0 \in \partial\Phi(u) + \partial I_K(u)$.

(b) *Critical points are solutions of (1.1)–(1.2) in* $[\underline{u}, \overline{u}]$

Let $u \in K$ be a critical point of $\mathbb{E}$, which implies the existence of an $\xi \in \partial\Phi(u)$ satisfying $-\xi \in \partial I_K(u)$. The latter is equivalent to

$$\langle \xi, v-u \rangle \geq 0, \quad \forall\, v \in K. \tag{4.4}$$

Since Φ is the sum of a differentiable functional and Lipschitz continuous functional, we have

$$\partial\Phi(u) = P'(u) + \mathbb{B}'(u) + \partial\tilde{J}(u),$$

and $\xi \in \partial\Phi(u)$ leads to

$$\xi = P'(u) + \mathbb{B}'(u) + i^*\tilde{\eta}, \tag{4.5}$$

where $\tilde{\eta} \in \partial\tilde{J}(u)$, which in turn implies (see (3.9)) that $\tilde{\eta} \in L^q(\Omega) \hookrightarrow V^*$ and $\tilde{\eta}(x) \in \partial\tilde{j}(x, u(x))$. Hence by (4.4), (4.5) it follows that to any critical point u of

$\mathbb{E}$ there is an $\tilde{\eta} \in L^q(\Omega)$ such that the following multi-valued variational inequality holds:

$$u \in K,\ \tilde{\eta}(x) \in \partial\tilde{j}(x, u(x)) : \langle -\Delta_p u + \mathbb{B}'(u) + i^*\tilde{\eta}, v - u\rangle \geq 0, \quad \forall\, v \in K. \tag{4.6}$$

By comparison we are going to prove next that any solution of (4.6) belongs to the interval $[\underline{u}, \overline{u}]$. We first note that (4.6) is equivalent to

$$u \in K,\ \tilde{\eta}(x) \in \partial\tilde{j}(x, u(x)) : \langle -\Delta_p u, v - u\rangle + \int_\Omega \big(b(\cdot, u) + \tilde{\eta}\big)\,(v - u)\,dx \geq 0, \quad \forall\, v \in K. \tag{4.7}$$

Let us show first that for any solution u of (4.7) the inequality $u \leq \overline{u}$ holds, where $\overline{u}$ is the given supersolution of (1.1)–(1.2). To this end we recall the definition of $\overline{u}$ according to Definition 3.2: $\overline{u} \in V$ satisfies

$$\begin{aligned}
&(i)\ \overline{u} \wedge K \subseteq K,\\
&(ii)\ \overline{\eta}(x) \in \partial j(x, \overline{u}(x)),\ \text{ for a.e. } x \in \Omega,\\
&(iii)\ \langle -\Delta_p \overline{u}, v - \overline{u}\rangle + \int_\Omega \overline{\eta}\,(v - \overline{u})\,dx \geq 0, \quad \text{for all } v \in \overline{u} \vee K.
\end{aligned}$$

We apply the special test function $v = \overline{u} \vee u = \overline{u} + (u - \overline{u})^+$ in (iii), and $v = \overline{u} \wedge u = u - (u - \overline{u})^+ \in K$ in (4.7), and get by adding the resulting inequalities (with $A := -\Delta_p$ for short)

$$\langle A\overline{u} - Au, (u - \overline{u})^+\rangle - \int_\Omega b(\cdot, u)\,(u - \overline{u})^+\,dx + \int_\Omega (\overline{\eta} - \tilde{\eta})\,(u - \overline{u})^+\,dx \geq 0. \tag{4.8}$$

Applying Lemma 3.8 (iii) we have

$$\int_\Omega (\overline{\eta} - \tilde{\eta})\,(u - \overline{u})^+\,dx = \int_{\{u > \overline{u}\}} (\overline{\eta} - \tilde{\eta})\,(u - \overline{u})\,dx = 0, \tag{4.9}$$

because $\tilde{\eta}(x) = \overline{\eta}(x)$ for $x \in \{u > \overline{u}\}$. Taking the definition of the cut-off function b into account we get

$$\int_\Omega b(\cdot, u)\,(u - \overline{u})^+\,dx = \int_\Omega \Big((u - \overline{u})^+\Big)^p\,dx. \tag{4.10}$$

The first trem on the left-hand side of (4.8)yields the estimate

$$\langle A\overline{u} - Au, (u - \overline{u})^+\rangle = -\langle Au - A\overline{u}, (u - \overline{u})^+\rangle \leq 0. \tag{4.11}$$

Applying the results (4.9)–(4.11) to (4.8) we finally obtain

$$\int_\Omega \Big((u - \overline{u})^+\Big)^p\,dx = 0,$$

which implies $(u - \overline{u})^+ = 0$, and thus $u \leq \overline{u}$. The proof for $\underline{u} \leq u$ can be done in a similar way.

So far we have shown that any solution u of the multi-valued variational inequality (4.7) belongs to the interval $[\underline{u}, \overline{u}]$, and thus satisfies: $u \in K$, $b(x, u(x)) = 0$, $\tilde{\eta} \in L^q(\Omega)$ and

$$\tilde{\eta}(x) \in \partial\tilde{j}(x, u(x)), \text{ a.e. } x \in \Omega, \tag{4.12}$$

$$\langle Au, v - u\rangle + \int_\Omega \tilde{\eta}\,(v - u)\, dx \geq 0, \ \forall\, v \in K. \tag{4.13}$$

From Lemma 3.8 (iii) we see that $\partial\tilde{j}(x, u(x)) \subseteq \partial j(x, u(x))$ for any $u \in [\underline{u}, \overline{u}]$, and therefore we also have

$$\tilde{\eta}(x) \in \partial j(x, u(x)), \text{ a.e. } x \in \Omega,$$

which shows that the solution $u \in [\underline{u}, \overline{u}]$ of the problem (4.7) is in fact a solution of the original multi-valued variational inequality (1.1)–(1.2). This completes the proof. $\square$

Remark 4.5. (i) Theorem 4.4 yields the desired variational tool in form of the nonsmooth functional $\mathbb{E}$ given by (4.1) and (4.3), which not only allows to get existence results for the multi-valued variational inequality (1.1)–(1.2), but also to localize the critical points of $\mathbb{E}$, i.e., any critical point of $\mathbb{E}$ belongs automatically to the ordered interval $[\underline{u}, \overline{u}]$. Under the assumptions (H1)–(H2) we were able to verify the existence of critical points by showing that $\mathbb{E}$ has a global minimizer. Under more specific assumptions on j other types of critical points may occur which allows the study of multiple solutions for (1.1)–(1.2).

(ii) By inspection of the notion of sub- and supersolution according to Definition 3.1 and Definition 3.2, respectively, one readily observes that any solution of the multi-valued variational inequality (1.1)–(1.2) is both a subsolution and supersolution provided the closed, convex set $K \subseteq V$ satisfies the following lattice condition:

$$K \wedge K \subseteq K, \quad K \vee K \subseteq K.$$

(iii) In specific cases for K the general potential $\mathbb{E}$ may be replaced by a more simpler potential having the same critical points which according to Theorem 4.4 are solutions of (1.1)–(1.2) within the order interval $[\underline{u}, \overline{u}]$ of sub-supersolution. In regard with the latter two examples will be considered in the next section.

5. **Applications**

In this section we apply Theorem 4.4 to the special cases given in Example 2.1 and Example 2.2. In particular, the corresponding functional $\mathbb{E}$ is studied in more detail. In the result we may replace $\mathbb{E}$ by functionals $\hat{\mathbb{E}}$ and $\hat{\mathbb{E}}_0$ that are simpler to handle and that have the same critical points as $\mathbb{E}$.

Example 5.1. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, and let j be its primitive given by

$$j(x,s) := \int_0^s f(x,t)\,dt. \tag{5.1}$$

Then the function $s \mapsto j(x,s)$ is continuously differentiable, and thus Clarke's gradient reduces to a singleton, i.e.,

$$\partial j(x,s) = \{\partial j(x,s)/\partial s\} = \{f(x,s)\}.$$

If $K = V$, then (1.1)-(1.2) reduces to the following quasilinear elliptic BVP

$$\langle -\Delta_p u, v\rangle + \int_\Omega f(x,u)\,v\,dx = 0, \quad \forall\ v \in V, \tag{5.2}$$

which is Example 2.1 and which is equivalent to a homogeneous Neumann problem. Let $\underline{u}$ and $\overline{u}$ be sub- and supersolution with $\underline{u} \le \overline{u}$. The hypothesis (H1) on j given by (5.1) is trivially satisfied. To fulfill hypothesis (H2) we need to impose an L^q-boundedness with respect to $[\underline{u}, \overline{u}]$ on f, i.e., for some $k_\Omega \in L^q_+(\Omega)$ the following inequality is assumed to be satisfied:

$$|f(x,s)| \le k_\Omega(x), \quad \text{for a.e. } x \in \Omega,\ \forall\ s \in [\underline{u}(x), \overline{u}(x)]. \tag{5.3}$$

The associated potential $\mathbb{E}$ whose critical points are solutions of (5.2) within $[\underline{u}, \overline{u}]$ is now given by (note that $I_V = 0$)

$$\mathbb{E}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \mathbb{B}(u) + \tilde{J}(u),$$

where $\mathbb{B}$ and $\tilde{J}$ are given by (3.12) and (3.8), respectively. Our aim is to replace the functional $\tilde{J}$ by a functional $\hat{J}$ that can easier be handled and that satisfies

$$\tilde{J}(u) - \hat{J}(u) = C, \quad \forall\ u \in V,$$

where C is a constant (not depending on u). Let $\hat{\mathbb{E}}$ be defined by

$$\hat{\mathbb{E}}(u) := \frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \mathbb{B}(u) + \hat{J}(u), \quad u \in V. \tag{5.4}$$

Since $\mathbb{E}$ and $\hat{\mathbb{E}}$ differ only by some constant C, we have that u is a critical point of $\mathbb{E}$ if and only if u is a critical point of $\hat{\mathbb{E}}$. Therefore, Theorem 4.4 holds true if $\mathbb{E}$ is replaced by $\hat{\mathbb{E}}$.

For the construction of the new functional $\hat{J}$ let us first recall $\tilde{j}$ where j is given by (5.1):

$$\tilde{j}(x,s) = \begin{cases} j(x,\underline{u}(x)) + f(x,\underline{u}(x))(s - \underline{u}(x)) & \text{if} \quad s < \underline{u}(x), \\ j(x,s) & \text{if} \quad \underline{u}(x) \le s \le \overline{u}(x), \\ j(x,\overline{u}(x)) + f(x,\overline{u}(x))(s - \overline{u}(x)) & \text{if} \quad s > \overline{u}(x). \end{cases} \tag{5.5}$$

Since $s \mapsto \tilde{j}(x,s)$ given by (5.5) is differentiable, Clarke's gradient $\partial\tilde{j}(x,s)$ is single-valued, i.e., $\partial\tilde{j}(x,s) = \{\frac{\partial}{\partial s}\tilde{j}(x,s)\ \}$ which apparently is given by

$$\frac{\partial}{\partial s}\tilde{j}(x,s) = \begin{cases} f(x,\underline{u}(x)) & \text{if} \quad s < \underline{u}(x), \\ f(x,s) & \text{if} \quad \underline{u}(x) \le s \le \overline{u}(x), \\ f(x,\overline{u}(x) & \text{if} \quad s > \overline{u}(x). \end{cases} \tag{5.6}$$

By means of the truncation τ related to the given sub- and supersolution and defined by

$$\tau(x,s) = \begin{cases} \underline{u}(x) & \text{if } s < \underline{u}(x), \\ s & \text{if } \underline{u}(x) \le s \le \overline{u}(x), \\ \overline{u}(x) & \text{if } s > \overline{u}(x), \end{cases} \tag{5.7}$$

we may rewrite (5.6) in the following compact form

$$\frac{\partial}{\partial s}\tilde{j}(x,s) = (f \circ \tau)(x,s) := f(x,\tau(x,s)).$$

Define the function $(x,s) \mapsto \hat{j}(x,s)$ by

$$\hat{j}(x,s) := \int_0^s (f \circ \tau)(x,t)\,dt. \tag{5.8}$$

Then we have

$$\frac{\partial}{\partial s}\tilde{j}(x,s) = \frac{\partial}{\partial s}\hat{j}(x,s),$$

and thus

$$\hat{j}(x,s) = \tilde{j}(x,s) - \tilde{j}(x,0),$$

which yields

$$\tilde{J}(u) - \hat{J}(u) = \tilde{J}(0) =: C \quad \forall\ u \in V,$$

where

$$\hat{J}(u) := \int_\Omega \hat{j}(x,u(x))\,dx = \int_\Omega \int_0^{u(x)} (f \circ \tau)(x,t)\,dt\,dx, \quad \forall\ u \in V. \tag{5.9}$$

Applying Theorem 4.4 to the elliptic problem (5.2) we get the following result.

Corollary 5.2. *Let $\underline{u}$, $\overline{u}$ be sub-and supersolution of (5.2) with $\underline{u} \le \overline{u}$, and assume the local L^q-boundedness with respect to $[\underline{u},\overline{u}]$ for f. Then the (smooth) functional*

$$\hat{\mathbb{E}}(u) := \frac{1}{p}\int_\Omega |\nabla u|^p\,dx + \mathbb{B}(u) + \hat{J}(u), \quad u \in V,$$

with $\hat{J}$ given by (5.9) possesses critical points. Any critical point u of $\hat{\mathbb{E}}$ is a solution of the elliptic problem (5.2) which belongs to the interval $[\underline{u},\overline{u}]$.

Example 5.3. Let j and f satisfy the same condition as in Example 2.1, and assume $K = V_0$. Then (1.1)–(1.2) is equivalent to the homogeneous Dirichlet boundary value problem (2.3) which is:

$$u \in V_0 : \quad \langle -\Delta_p u, v\rangle + \int_\Omega f(x,u)\, v\, dx = 0, \quad \forall\, v \in V_0. \tag{5.10}$$

Similarly as in the previous subsection the potential $\mathbb{E} : V \to \mathbb{R} \cup \{+\infty\}$ provided by Theorem 4.4 and whose critical points are the solutions of (5.10)in $[\underline{u}, \overline{u}]$ may be replaced first by the following simpler functional $\hat{\mathbb{E}} : V_0 \to \mathbb{R}$ (note V_0 is a closed subspace of V):

$$\hat{\mathbb{E}}(u) := \frac{1}{p}\int_\Omega |\nabla u|^p\, dx + \mathbb{B}(u) + \int_\Omega \int_0^{u(x)} (f \circ \tau)(x,t)\, dt\, dx, \quad u \in V_0. \tag{5.11}$$

The specific properties of problem (5.10) allow even further to simplify the associated potential $\hat{\mathbb{E}}$ in that the term $\mathbb{B}(u)$ which is required in the general situation and in the previous subsection may now be dropped, i.e., we have the following result.

Corollary 5.4. *Let $\hat{\mathbb{E}}_0 : V_0 \to \mathbb{R}$ be defined by*

$$\hat{\mathbb{E}}_0(u) := \frac{1}{p}\int_\Omega |\nabla u|^p\, dx + \int_\Omega \int_0^{u(x)} (f \circ \tau)(x,t)\, dt\, dx, \quad u \in V_0.$$

Then $\hat{\mathbb{E}}_0 \in C^1(V_0, \mathbb{R})$ possesses critical points in V_0, and any critical point $u \in V_0$ of $\hat{\mathbb{E}}_0$ is a solution of (5.10) satisfying $\underline{u} \le u \le \overline{u}$. Moreover, $\hat{\mathbb{E}}$ and $\hat{\mathbb{E}}_0$ have the same critical points.

Proof. As $\|u\|_{V_0}^p := \int_\Omega |\nabla u|^p\, dx$ defines an equivalent norm in V_0, and since

$$\Big| \int_\Omega \int_0^{u(x)} (f \circ \tau)(x,t)\, dt\, dx \Big| \le \|k_\Omega\|_{L^q(\Omega)}\, \|u\|_{L^p(\Omega)},$$

we readily see that $\hat{\mathbb{E}}_0 : V_0 \to \mathbb{R}$ is bounded below, coercive, and weakly lower semicontinous. Thus there is a global minimizer $u \in V_0$ of $\hat{\mathbb{E}}_0$ which is a critical point, i.e, we have

$$0 = \langle \hat{\mathbb{E}}_0'(u), \varphi\rangle = \int_\Omega \Big(|\nabla u|^{p-2} \nabla u \nabla \varphi + (f \circ \tau)(\cdot, u)\, \varphi \Big) dx, \quad \forall\, \varphi \in V_0. \tag{5.12}$$

The supersolution $\overline{u} \in V$ of (5.10) satisfies: $\overline{u}|_{\partial\Omega} \ge 0$ and the inequality

$$\int_\Omega \Big(|\nabla \overline{u}|^{p-2} \nabla \overline{u} \nabla \varphi + f(\cdot, \overline{u})\, \varphi \Big) dx \ge 0, \quad \forall\, \varphi \in V_0 \cap L^p_+(\Omega). \tag{5.13}$$

Subtracting (5.13) from (5.12) and using $\varphi = (u - \overline{u})^+ \in V_0$ we get

$$\int_\Omega \Big(|\nabla u|^{p-2}\nabla u - |\nabla \overline{u}|^{p-2} \nabla \overline{u} \Big) \nabla (u - \overline{u})^+\, dx + \int_\Omega \big((f \circ \tau)(\cdot, u) - f(\cdot, \overline{u}) \big) (u - \overline{u})^+\, dx \le 0. \tag{5.14}$$

Applying the definition of τ we readily see that

$$\int_\Omega \left((f\circ\tau)(\cdot,u) - f(\cdot,\overline{u})\right)(u-\overline{u})^+\,dx = 0,$$

which by means of (5.14) implies

$$\begin{aligned} 0 &\geq \int_\Omega \Big(|\nabla u|^{p-2}\nabla u - |\nabla \overline{u}|^{p-2}\nabla\overline{u}\Big)\nabla(u-\overline{u})^+\,dx \\ &= \int_{\{u>\overline{u}\}} \Big(|\nabla u|^{p-2}\nabla u - |\nabla \overline{u}|^{p-2}\nabla\overline{u}\Big)\nabla(u-\overline{u})\,dx \geq 0. \end{aligned}$$

Hence it follows that $\nabla(u-\overline{u}) = 0$ a.e. in $\{u>\overline{u}\}$, which means $\nabla(u-\overline{u})^+ = 0$ a.e. in Ω, and thus $\|(u-\overline{u})^+\|_{V_0} = 0$, i.e., $(u-\overline{u})^+ = 0$ a.e. in Ω, that is $u\leq\overline{u}$. In a similar way one shows that $\underline{u}\leq u$ holds true which proves that any critical point u of $\hat{\mathbb{E}}_0$ is a solution of (5.10) because $\underline{u}\leq u\leq\overline{u}$ and therefore $(f\circ\tau)(x,u(x)) = f(x,u(x))$. So far we know that critical points of both $\hat{\mathbb{E}}$ and $\hat{\mathbb{E}}_0$ are necessarily solutions of (5.10) in $[\underline{u},\overline{u}]$. By (3.12) and (3.16) we see that $\mathbb{B}(u) = c$ for $u\in[\underline{u},\overline{u}]$, and therefore

$$\hat{\mathbb{E}}(u) = \hat{\mathbb{E}}_0 + c, \quad \forall\, u\in[\underline{u},\overline{u}],$$

which shows that $\hat{\mathbb{E}}$ and $\hat{\mathbb{E}}_0$ have the same critical points. This completes the proof. $\square$

Example 5.5. For illustration let us consider the Dirichlet problem depending on a parameter $\lambda\in\mathbb{R}$:

$$-\Delta_p u = \lambda\,|u|^{p-2}u - g(u), \quad \text{in} \quad V_0^*, \tag{5.15}$$

where we assume the following assumptions on g:

(g1) $g:\mathbb{R}\to\mathbb{R}$ is continuous;
(g2) $\lim_{|s|\to\infty}\frac{g(s)}{|s|^{p-2}s} = +\infty$;
(g3) $\lim_{s\to 0}\frac{g(s)}{|s|^{p-2}s} = 0$.

The following specific g which is of exponential growth satisfies (g1)–(g3):

$$g(s) = \begin{cases} |s|^{p-2}se^{-s-1} & \text{if } s<-1 \\ \frac{|s|^p}{2}((s-1)\cos(s+1)+s+1) & \text{if } -1\leq s\leq 1 \\ (1+(s-1))s^{p-1}e^{s-1} & \text{if } s>1, \end{cases}$$

see [4]. By means of (g2) one readily verifies that $\overline{u} = M > 0$ with M sufficiently large is a supersolution of (5.15), and $\underline{u} = -M$ with $M>0$ sufficiently large is a subsolution of (5.15). Since $s\mapsto\lambda\,|s|^{p-2}s - g(s)$ is continuous, and thus bounded in

$[\underline{u}, \overline{u}] = [-M, M]$ we may apply Corollary 5.4 with τ given by

$$\tau(s) = \begin{cases} -M & \text{if } s < -M, \\ s & \text{if } -M \le s \le M, \\ M & \text{if } s > M, \end{cases}$$

and $\hat{\mathbb{E}}_0 : V_0 \to \mathbb{R}$ given by

$$\hat{\mathbb{E}}_0(u) = \frac{1}{p}\,\|\nabla u\|_p^p - \int_\Omega \int_0^{u(x)} (\lambda|\tau(s)|^{p-2}\tau(s) - g(\tau(s)))\,ds\,dx.$$

According to Corollary 5.4 the functional $\hat{\mathbb{E}}_0$ has critical points, and any critical point $u \in V_0$ is a solution of (5.15) satisfying $-M \le u \le M$ for $M > 0$ sufficiently large. Due to (g3) problem (5.15) always has the trivial solution. How to decide the existence of nontrivial, and moreover, multiple nontrivial solutions? In a first step we can show that the global minimizer of $\hat{\mathbb{E}}_0$ is a nontrivial solution of (5.15) in $[-M, M]$ provided $\lambda > \lambda_1$, where λ_1 is the first eigenvalue of $(-\Delta_p, V_0)$. Let φ_1 be the (normalized, positive) eigenfunction corresponding to λ_1 ($\|\varphi_1\|_p = 1$), then it is known that $\varphi_1 \in \operatorname{int}(C_0^1(\overline{\Omega})_+)$. By using (g3) we have for $\varepsilon > 0$ small the estimate:

$$\begin{aligned}
\hat{\mathbb{E}}_0(\varepsilon\varphi_1) &= \frac{\lambda_1}{p}\,\varepsilon^p - \int_\Omega \int_0^{\varepsilon\varphi_1(x)} (\lambda s^{p-1} - g(s))\,ds\,dx \\
&= \frac{\lambda_1 - \lambda}{p}\,\varepsilon^p + \int_\Omega \int_0^{\varepsilon\varphi_1(x)} g(s)\,ds\,dx \\
& \qquad (g3) \Longrightarrow \frac{|g(s)|}{|s|^{p-1}} < \lambda - \lambda_1,\ \forall\, s:\ |s| < \delta_\lambda \\
&\le \frac{\lambda_1 - \lambda}{p}\,\varepsilon^p + \int_\Omega \int_0^{\varepsilon\varphi_1(x)} \frac{|g(s)|}{s^{p-1}}\,s^{p-1}\,ds\,dx \\
& \qquad \text{choose } \varepsilon : \varepsilon\|\varphi_1\|_\infty < \delta_\lambda \\
&< \frac{\lambda_1 - \lambda}{p}\,\varepsilon^p + \frac{\lambda - \lambda_1}{p}\,\varepsilon^p = 0.
\end{aligned}$$

Therefore, the global minimizer u of $\hat{\mathbb{E}}_0$ satisfies $\hat{\mathbb{E}}_0(u) \le \hat{\mathbb{E}}_0(\varepsilon\varphi_1) < 0$, and thus $u \ne 0$ is a nontrivial solution.

Remark 5.6. Multiple solution results for (1.1)–(1.2) in case that $K = V_0$ which refers to the Dirichlet problem for elliptic equations with smooth or nonsmooth functions j have been obtained by the author jointly with D. Motreanu and K. Perera in [2, 3, 4, 5, 6, 7]. In particular, in [4] multiple solutions have been obtained for (5.15) by applying a combined approach of variational and comparison principles in the smooth case. The approach developed in this paper allows to extend the study of multiple solutions to a wide range of (nonsmooth) multi-valued variational inequality in the form (1.1)–(1.2) or, more general, in the form (1.3).

References

[1] Carl, S., Le, Vy K., Motreanu, D., *Nonsmooth Variational Problems and Their Inequalities. Comparison Principles and Applications*, Springer Monogr. Math., Springer, New York, 2007.

[2] Carl, S., Motreanu, D., *Multiple and sign-changing solutions for the multivalued p-Laplacian equation*, Math. Nachr., **283** (2010), 965–981.

[3] Carl, S., Motreanu, D., *Sign-changing solutions for nonlinear elliptic problems depending on parameters*, Int. J. Differential Equations, **2010**, Article ID 536236, DOI: 10.1155/2010/ (2010)/536236.

[4] Carl, S., Motreanu, D., *Constant-sign and sign-changing solutions for nonlinear eigenvalue problems*, Nonlinear Anal., **68** (2008), 2668–2676.

[5] Carl, S., Motreanu, D., *Constant-sign and sign-changing solutions of a nonlinear eigenvalue problem involving the p-Laplacian*, Differential Integral Equations, **20** (3) (2007), 309–324.

[6] Carl, S., Motreanu, D., *Sign-changing and extremal constant-sign solutions of nonlinear elliptic problems with supercritical nonlinearities*, Commun. Appl. Nonlinear Anal., **14** (2007), 85–100.

[7] Carl, S., Perera, K., *Sign-changing and multiple solutions for the p-Laplacian*, Abstract Appl. Anal., **7** (2002), 613–625.

[8] Clarke, F. H., *Optimization and Nonsmooth Analysis*, Classics Appl. Math. **5**, SIAM, Philadelphia, 1990.

[9] Motreanu, D., Radulescu, V., *Variational and Non-variational Methods in Nonlinear Analysis and Boundary Value Problems*, Kluwer Academic Publishers, Boston, Dordrecht, London (2003).

[10] Zeidler, E., *Nonlinear Functional Analysis and Its Applications*, Vol. III. Springer-Verlag, New York (1985).

Institut für Mathematik
Martin-Luther-Universität Halle-Wittenberg
06099 Halle, Germany
E-mail address: siegfried.carl@mathematik.uni-halle.de