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SUBCLASSES OF HARMONIC FUNCTIONS BASED ON GENERALIZED DERIVATIVE OPERATOR

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Abstract. Making use of Salagean and Ruscheweyh derivative operator we introduced a new class of complex-valued harmonic functions which are orientation preserving, univalent and starlike functions. We investigate the coefficient bounds, distortion inequalities, extreme points and inclusion results for the generalized class of functions.

1. **Introduction**

A continuous function $f = u + iv$ is a complex-valued harmonic function in a complex domain Ω if both u and v are real and harmonic in Ω. In any simply connected domain $\mathcal{D} \subset \Omega$ we can write $f = h + \overline{g}$ where h and g are analytic in $\mathcal{D}$. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in $\mathcal{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathcal{D}$ (see [1]).

Denote by $\mathcal{H}$ the family of functions

$$f = h + \overline{g} \tag{1.1}$$

which are harmonic univalent and orientation preserving in the open unit disc $U = \{z : |z| < 1\}$ so that f is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Thus, for $f = h + \overline{g} \in \mathcal{H}$, we may express the analytic functions h and g in the forms $h(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ and $g(z) = \sum\limits_{n=1}^{\infty} b_n z^n, (0 \leq b_1 < 1)$. Then

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}, \ \ |b_1| < 1. \tag{1.2}$$

We note that the family $\mathcal{H}$ of orientation preserving, normalized harmonic univalent functions reduces to the well known class S of normalized univalent functions if the co-analytic part of $f = h + \overline{g}$ is identically zero that is $g \equiv 0$. Due to Silverman

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[6] we denote $\overline{\mathcal{H}}$ the subclass of $\mathcal{H}$ consisting of functions of the form $f = h + \overline{g}$ given by

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1, a_n, b_n \geq 0. \tag{1.3}$$

In 1999 Jahangiri [2] introduced a subclass of $\mathcal{H}$ called the class of harmonic starlike functions of order α denoted by $S_H(\alpha)$ which consist of functions of the form (1.1) and satisfying the inequality:

$$\frac{\partial}{\partial \theta}\left(arg(f(z)\right) > \alpha \tag{1.4}$$

Equivalently

$$\text{Re}\left\{\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}}\right\} \geq \alpha \tag{1.5}$$

where $z \in \mathcal{U}$.

Given two functions $\phi(z) = z + \sum\limits_{n=2}^{\infty} \phi_n z^n$ and $\psi(z) = z + \sum\limits_{n=2}^{\infty} \psi_n z^n$ in $\mathcal{S}$ their Hadamard product or convolution is defined by $(\phi * \psi)(z) = \phi(z) * \psi(z) = z + \sum\limits_{n=2}^{\infty} \phi_n \psi_n z^n$. Using the convolution, Ruscheweyh [5] introduced the derivative operator

$$D^m \phi(z) := \frac{z}{(1-z)^{m-1}} = z + \sum_{n=2}^{\infty} \begin{pmatrix} m+n-1 \\ n-1 \end{pmatrix} \phi_n z^n, \quad (z \in U,\, m > -1). \tag{1.6}$$

Recently in [4] Jahangiri and etal. defined the Ruscheweyh derivative for harmonic functions, as given below

$$D^m f(z) := z + \sum_{n=2}^{\infty} \begin{pmatrix} m+n-1 \\ n-1 \end{pmatrix} a_n z^n + \sum_{n=1}^{\infty} \begin{pmatrix} m+n-1 \\ n-1 \end{pmatrix} \overline{b_n z^n}, \tag{1.7}$$

which was initially studied for the class of harmonic starlike functions $S_H(\alpha)$ in [4]. Further motivated by the works of Jahangiri et. al. [3] we define a new generalized derivative operator on harmonic function $f = h + \overline{g}$ in $\mathcal{H}$ as

$$D_k^m f(z) = D_k^m h(z) + (-1)^k \overline{D_k^m g(z)}, \quad m > -1, \text{ and } k \geq 0 \tag{1.8}$$

where

$$D_k^m h(z) = z + \sum_{n=2}^{\infty} n^k C(n,m) a_n z^n, D_k^m g(z) = \sum_{n=1}^{\infty} n^k C(n,m) b_n z^n,$$

and

$$C(n,m) = \begin{pmatrix} n+m-1 \\ n-1 \end{pmatrix}.$$

For $0 \leq \alpha < 1$, we let $\mathcal{HR}_k^m(\lambda,\alpha)$ a subclass of $\mathcal{H}$ of the form $f = h + \overline{g}$ given by (1.2) and satisfying the analytic criteria

$$\text{Re}\,\left\{\frac{z(D_k^m f(z))'}{(1-\lambda)D_k^m f(z) + \lambda z(D_k^m f(z))'}\right\} \geq \alpha \tag{1.9}$$

where $0 \leq \lambda < 1$, $D_k^m f$ is given by (1.8) and $z \in U$. We also let $\overline{\mathcal{H}\mathcal{R}}_k^m(\lambda,\alpha) = \mathcal{HR}_k^m(\lambda,\alpha) \cap \overline{\mathcal{H}}$.

We investigate the coefficient bounds,distortion inequalities, extreme points and inclusion results for the generalized class $\overline{\mathcal{H}\mathcal{R}}_k^m(\lambda,\alpha)$

The Class $\mathcal{HR}_k^m(\lambda,\alpha)$

In our first theorem, we obtain a sufficient coefficient condition for harmonic functions in $\mathcal{HR}_k^m(\lambda,\alpha)$.

Theorem 1.1. *Let $f = h + \overline{g}$ be given by (1.2). If*

$$\sum_{n=1}^{\infty} n^k C(n,m)\left[(n-\alpha-\alpha\lambda(n-1))|a_n| + (n+\alpha-\alpha\lambda(n+1))|b_n|\right] \leq 2(1-\alpha), \tag{1.10}$$

where $a_1 = 1$ and $0 \leq \alpha < 1$, then $f \in \mathcal{HR}_k^m(\lambda,\alpha)$.

Proof. We first show that if (1.10) holds for the coefficients of $f = h + \overline{g}$, the required condition (1.9) is satisfied. From (1.9) we can write

$$\begin{aligned} &\text{Re}\left\{\frac{z(D_k^m h(z))' - \overline{z(D_k^m g(z))'}}{(1-\lambda)(D_k^m h(z) + \overline{D_k^m g(z)}) + \lambda(z(D_k^m h(z))' - \overline{z(D_k^m g(z))'})}\right\} \geq \alpha \\ = \quad &\text{Re}\,\frac{A(z)}{B(z)} \geq \alpha, \end{aligned}$$

where

$$\begin{aligned} A(z) &= z(D_k^m h(z))' - \overline{z(D_k^m g(z))'} \\ &= z + \sum_{n=2}^{\infty} n^k C(n,m) a_n z^n - \sum_{n=1}^{\infty} n^k C(n,m)\overline{b_n}\overline{z}^n \\ \text{and}\;\; B(z) &= (1-\lambda)(D_k^m h(z) + \overline{D_k^m g(z)}) + \lambda(z(D_k^m h(z))' - \overline{z(D_k^m g(z))'}\ ) \\ &= z + \sum_{n=2}^{\infty} n^k C(n,m)(1-\lambda+n\lambda) a_n z^n + \sum_{n=1}^{\infty} n^k C(n,m)(1-\lambda-n\lambda)\overline{b_n}\overline{z}^n. \end{aligned}$$

Using the fact that Re $\{w\} \geq \alpha$ if and only if $|1-\alpha+w| \geq |1+\alpha-w|$, it suffices to show that

$$|A(z) + (1-\alpha)B(z)| - |A(z) - (1+\alpha)B(z)| \geq 0. \tag{1.11}$$

Substituting for $A(z)$ and $B(z)$ in (1.11), we get

$$|A(z)+(1-\alpha)B(z)|-|A(z)-(1+\alpha)B(z)|$$

$$=|\ (2-\alpha)z+\sum_{n=2}^{\infty}n^kC(n,m)[(n+1-\alpha)(1-\lambda+n\lambda)]a_nz^n$$

$$-\sum_{n=1}^{\infty}n^kC(n,m)[n-(1-\alpha)(1-\lambda+n\lambda)]\overline{b}_n\ \overline{z}^n\ |$$

$$-\mid-\alpha z+\sum_{n=2}^{\infty}n^kC(n,m)[n-(1+\alpha)(1-\lambda+n\lambda)a_nz^n$$

$$-\sum_{n=1}^{\infty}n^kC(n,m)[n+(1+\alpha)(1-\lambda+n\lambda)]\overline{b}_n\overline{z}^n\ |$$

$$\geq(2-\alpha)|z|-\sum_{n=2}^{\infty}n^kC(n,m)[n+(1-\alpha)(1-\lambda+n\lambda)|a_n||z|^n$$

$$-\sum_{n=1}^{\infty}n^kC(n,m)[n-(1-\alpha)(1-\lambda-n\lambda)]|b_n|\ |z|^n$$

$$-\alpha|z|-\sum_{n=2}^{\infty}n^kC(n,m)[n-(1+\alpha)(1-\lambda+n\lambda)]|a_n|\ |z|^n$$

$$-\sum_{n=1}^{\infty}n^kC(n,m)[n+(1+\alpha)(1-\lambda-n\lambda)]|b_n|\ |z|^n$$

$$\geq 2(1-\alpha)|z|\left\{2-\sum_{n=1}^{\infty}n^kC(n,m)\left[\frac{n-\alpha-\alpha\lambda(n-1)}{1-\alpha}|a_n|\right.\right.$$

$$\left.\left.+\frac{n+\alpha-\alpha\lambda(n+1)}{1-\alpha}|b_n|\right]|z|^{n-1}\right\}$$

$$\geq 2(1-\alpha)\left\{2-\sum_{n=1}^{\infty}n^kC(n,m)\left[\frac{n-\alpha-\alpha\lambda(n-1)}{1-\alpha}|a_n|+\frac{n+\alpha-\alpha\lambda(n+1)}{1-\alpha}|b_n|\right]\right\}.$$

The above expression is non negative by (1.10), and so $f(z)\in\mathcal{HR}_k^m(\lambda,\alpha)$. $\square$

Corollary 1.2. *Let $f=h+\overline{g}$ be of the form (1.2) and satisfy the condition (1.10). Then each $D^i(z)$, $-1<i\leq m$, is orientation preserving, harmonic univalent and starlike of order α in U.*

Proof. Observe that $n^kC(n,m)$ is an increasing function of n. Therefore, by (1.10) for each i, $-1 < i \leq m$, we can write

$$\begin{aligned}
&\sum_{n=1}^{\infty}[(n-\alpha-\alpha\lambda(n-1)]|a_n| + \sum_{n=1}^{\infty}[n+\alpha-\alpha\lambda(n+1)]|b_n|ht] \\
\leq\ & \sum_{n=1}^{\infty} C(n,i)[n-\alpha-\alpha\lambda(n-1)]|a_n| + \sum_{n=1}^{\infty}[n+\alpha-\alpha\lambda(n+1)]|b_n| \\
\leq\ & \sum_{n=1}^{\infty} n^kC(n,m)[n-\alpha-\alpha\lambda(n-1)]|a_n| + [n+\alpha-\alpha\lambda(n+1)]|b_n| \\
\leq\ & 2(1-\alpha).
\end{aligned}$$

Thus, by (1.10) each $D^i(z)$, $-1 < i \leq m$, is orientation preserving, harmonic univalent and starlike of order α in U.
The harmonic function

$$\begin{aligned}
f(z) &= z + \sum_{n=2}^{\infty} \frac{1-\alpha}{n^kC(n,m)[n-\alpha-\alpha\lambda(n-1)]} x_n z^n \\
&+ \sum_{n=1}^{\infty} \frac{1-\alpha}{n^kC(n,m)[n+\alpha-\alpha\lambda(n+1)]} \overline{y_n}(\overline{z})^n \qquad (1.7)
\end{aligned}$$

where $\sum\limits_{n=2}^{\infty} |x_n| + \sum\limits_{n=1}^{\infty} |y_n| = 1$ shows that the coefficient bound given by (1.10) is sharp.
The functions of the form (1.7) are in $\mathcal{HR}_k^m(\lambda,\alpha)$ because

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left(\frac{n^kC(n,m)[n-\alpha-\alpha\lambda(n-1)]}{1-\alpha}|a_n| + \frac{n^kC(n,m)[n+\alpha-\alpha\lambda(n+1)]}{1-\alpha}|b_n|\right) \\
&= 1 + \sum_{n=2}^{\infty}|x_n| + \sum_{n=1}^{\infty}|y_n| = 2.
\end{aligned}$$

□

Next theorem establishes that such coefficient bounds cannot be improved further.

Theorem 1.3. *For $a_1 = 1$ and $0 \leq \alpha < 1$, $f = h + \overline{g} \in \overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha)$ if and only if*

$$\sum_{n=1}^{\infty} n^kC(n,m)\left\{[n-\alpha-\alpha\lambda(n-1)]|a_n| + [n+\alpha-\alpha\lambda(n+1)]|b_n|\right\} \leq 2(1-\alpha). \quad (1.8)$$

Proof. Since $\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha) \subset \mathcal{HR}_k^m(\lambda,\alpha)$, we only need to prove the "only if" part of the theorem. To this end, for functions f of the form (1.3), we notice that the condition

$$\text{Re}\left\{\frac{z(D_k^m f(z))'}{(1-\lambda)D_k^m f(z) + \lambda z(D_k^m f(z))'}\right\} \geq \alpha.$$

Equivalently,

$$\operatorname{Re}\left\{\frac{(1-\alpha)z-\sum\limits_{n=2}^{\infty}[n-\alpha-\alpha\lambda(n-1)]n^kC(n,m)a_nz^n-\sum\limits_{n=1}^{\infty}[n+\alpha-\alpha\lambda(n+1)]n^kC(n,m)\overline{b}_n\overline{z}^n}{z-\sum\limits_{n=2}^{\infty}n^kC(n,m)(1-\lambda+n\lambda)a_nz^n+\sum\limits_{n=1}^{\infty}n^kC(n,m)(1-\lambda-n\lambda)\overline{b}_n\overline{z}^n}\right\}\geq 0.$$

The above required condition must hold for all values of z in U. Upon choosing the values of z on the positive real axis where $0\leq z=r<1$, we must have

$$\frac{(1-\alpha)-\sum\limits_{n=2}^{\infty}[n-\alpha-\alpha\lambda(n-1)]n^kC(n,m)a_nr^{n-1}-\sum\limits_{n=1}^{\infty}[n+\alpha-\alpha\lambda(n+1)]n^kC(n,m)b_nr^{n-1}}{1-\sum\limits_{n=2}^{\infty}n^kC(n,m)(1-\lambda+n\lambda)a_nr^{n-1}+\sum\limits_{n=1}^{\infty}n^kC(n,m)(1-\lambda-n\lambda)b_nr^{n-1}}\geq 0. \quad (1.9)$$

If the condition (1.8) does not hold, then the numerator in (1.9) is negative for r sufficiently close to 1. Hence, there exists $z_0=r_0$ in (0,1) for which the quotient of (1.9) is negative. This contradicts the required condition for $f(z)\in\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha)$. This completes the proof of the theorem. $\square$

Corollary 1.4. *Let $f=h+\overline{g}$ be given by (1.3). Then $D^if(z)$, $-1<i\leq m$ is orientation preserving, harmonic and starlike of order α, $0\leq\alpha<1$, if and only if the coefficient condition (1.8) holds.*

Next we determine the extreme points of closed convex hulls of $\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha)$ denoted by $clco\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha)$.

Theorem 1.5. *A function $f(z)\in\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha)$ if and only if*

$$f(z)=\sum_{n=1}^{\infty}\left(X_nh_n(z)+Y_ng_n(z)\right),$$

where

$$h_1(z)=z, h_n(z)=z-\frac{1-\alpha}{n^kC(n,m)[n-\alpha-\alpha\lambda(n-1)]}z^n;\quad (n\geq 2),$$
$$g_n(z)=z+\frac{1-\alpha}{n^kC(n,m)[+\alpha-\alpha\lambda(n+1)]}\overline{z}^n;\quad (n\geq 2),$$

$$\sum_{n=1}^{\infty}(X_n+Y_n)=1,\quad X_n\geq 0 \textit{ and } \; Y_n\geq 0.$$

In particular, the extreme points of $\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha)$ are $\{h_n\}$ and $\{g_n\}$.

Proof. First, we note that for f as in the theorem above, we may write

$$f(z)=\sum_{n=1}^{\infty}\left(X_nh_n(z)+Y_ng_n(z)\right)$$

$$=\sum_{n=1}^{\infty}(X_n+Y_n)z-\sum_{n=2}^{\infty}\frac{1-\alpha}{n^kC(n,m)[n-\alpha-\alpha\lambda(n-1)]}X_nz^n$$

$$+\sum_{n=1}^{\infty}\frac{1-\alpha}{n^kC(n,m)[n+\alpha-\alpha\lambda(n+1)]}Y_n\overline{z}^n = z-\sum_{n=2}^{\infty}A_nz^n+\sum_{n=1}^{\infty}B_n\overline{z}^n,$$

where

$$A_n = \frac{1-\alpha}{n^kC(n,m)[n-\alpha-\alpha\lambda(n-1)])}X_n,$$

and

$$B_n = \frac{1-\alpha}{n^kC(n,m)[n+\alpha-\alpha\lambda(n+1)]}Y_n.$$

Therefore

$$\sum_{n=2}^{\infty}\frac{n^kC(n,m)[n-\alpha-\alpha\lambda(n-1)]}{1-\alpha}A_n+\sum_{n=1}^{\infty}\frac{n^kC(n,m)[n+\alpha-\alpha\lambda(n+1)]}{1-\alpha}B_n$$

$$=\sum_{n=2}^{\infty}X_n+\sum_{n=1}^{\infty}Y_n = 1-X_1 \leq 1,$$

and hence $f(z)\in clco\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha)$.
Conversely, suppose that $f(z)\in clco\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha)$. Setting

$$X_n = \frac{n^kC(n,m)[n-\alpha-\alpha\lambda(n-1)]}{1-\alpha}A_n, \quad (n\geq 2)$$

and

$$Y_n = \frac{n^kC(n,m)[n+\alpha-\alpha\lambda(n-1)]}{1-\alpha}B_n, \quad (n\geq 1)$$

where $\sum\limits_{n=1}^{\infty}(X_n+Y_n)=1$. Then

$$\begin{aligned}f(z) &= z-\sum_{n=2}^{\infty}a_nz^n+\sum_{n=1}^{\infty}\overline{b_n}\overline{z}^n, \quad a_n,\ b_n\geq 0.\\
&= z-\sum_{n=2}^{\infty}\frac{1-\alpha}{n^kC(n,m)[n-\alpha-\alpha\lambda(n-1)]}X_nz^n\\
&+\sum_{n=1}^{\infty}\frac{1-\alpha}{n^kC(n,m)[n+\alpha-\alpha\lambda(n-1)]}Y_n\overline{z}^n\\
&= z-\sum_{n=2}^{\infty}(h_n(z)-z)X_n+\sum_{n=1}^{\infty}(g_n(z)-z)Y_n\\
&= \sum_{n=1}^{\infty}(X_nh_n(z)+Y_ng_n(z))\end{aligned}$$

as required. □

The following theorem gives the distortion bounds for functions in $R_{\overline{H}}(m,\alpha)$ which yields a covering result for the class $\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha)$.

Theorem 1.6. *Let $f \in \overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha)$. Then for $|z| = r < 1$, we have*

$$(1-b_1)r - \frac{1}{2^kC(2,m)}\left(\frac{1-\alpha}{2-\alpha-\alpha\lambda} - \frac{1+\alpha}{2-\alpha-\alpha\lambda}b_1\right)r^2 \le |f(z)|$$
$$\le (1+b_1)r + \frac{1}{2^kC(2,m)}\left(\frac{1-\alpha}{2-\alpha-\alpha\lambda} - \frac{1+\alpha}{2-\alpha-\alpha\lambda}b_1\right)r^2.$$

Proof. We only prove the right hand inequality. Taking the absolute value of $f(z)$, we obtain

$$\begin{aligned}
|f(z)| &= \left|z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{z}^n\right| \\
&= \left|z + b_1\ \overline{z} + \sum_{n=2}^{\infty}(a_n z^n + \overline{b}_n \overline{z}^n)\right| \\
&\le (1+b_1)|z| + \sum_{n=2}^{\infty}(a_n + b_n)|z|^n \\
&\le (1+|b_1|)r + \sum_{n=2}^{\infty}(a_n + b_n)r^n \\
&\le (1+b_1)r + \sum_{n=2}^{\infty}(a_n + b_n)r^2 \\
&\le (1+b_1)r + \frac{1-\alpha}{2^kC(2,m)(2-\alpha-\alpha\lambda)} \\
&\quad \sum_{n=2}^{\infty}\left(\frac{2^kC(2,m)(2-\alpha-\alpha\lambda)}{1-\alpha}a_n + \frac{2^kC(2,m)(2-\alpha-\alpha\lambda)}{1-\alpha}b_n\right)r^2 \\
&\le (1+b_1)r + \frac{1-\alpha}{2^kC(2,m)(2-\alpha-\alpha\lambda)}\left(1 - \frac{1+\alpha}{1-\alpha}b_1\right)r^2 \\
&\le (1+b_1)r + \frac{1}{2^kC(2,m)}\left(\frac{1-\alpha}{2-\alpha-\alpha\lambda} - \frac{1+\alpha}{2-\alpha-\alpha\lambda}b_1\right)r^2.
\end{aligned}$$

□

The proof of the left hand inequality follows on lines similar to that of the right hand side inequality.
The covering result follows from the left hand inequality given in Theorem 1.6.

Corollary 1.7. *If $f(z) \in \overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha)$. Then*

$$\left\{w : |w| < \frac{2^{k+1}C(2,m) - 1 - ((1+\lambda)2^kC(2,m)-1)\alpha}{2^kC(2,m)(2-\alpha-\alpha\lambda)}\right.$$
$$\left. - \frac{2^{k+1}C(2,m) - 1 - ((1+\lambda)2^kC(2,m)-1)\alpha}{2^kC(2,m)(2-\alpha-\alpha\lambda)}b_1\right\} \subset f(U).$$

Proof. Using the left hand inequality of Theorem 1.6 and letting $r \to 1$, we prove that

$$\begin{aligned}
&(1-b_1) - \frac{1}{2^k C(2,m)}\left(\frac{1-\alpha}{2-\alpha-\alpha\lambda} - \frac{1+\alpha}{2-\alpha-\alpha\lambda}b_1\right)\\
&= (1-b_1) - \frac{1}{2^k C(2,m)(2-\alpha-\alpha\lambda)}[1-\alpha-(1+\alpha)b_1]\\
&= \frac{(1-b_1)2^k C(2,m)(2-\alpha-\alpha\lambda) - (1-\alpha) + (1+\alpha)b_1}{2^k C(2,m)(2-\alpha-\alpha\lambda)}\\
&= \frac{2^k C(2,m)(2-\alpha-\alpha\lambda) - 2^k C(2,m)(2-\alpha-\alpha\lambda)b_1 - (1-\alpha) + (1+\alpha)b_1}{2^k C(2,m)(2-\alpha-\alpha\lambda)}\\
&= \frac{2^k C(2,m)(2-\alpha-\alpha\lambda) - 1 + \alpha - [2^k C(2,m)(2-\alpha-\alpha\lambda) - (1+\alpha)]b_1}{2^k C(2,m)(2-\alpha-\alpha\lambda)}\\
&= \frac{2^{k+1} C(2,m) - 1 - \alpha[(1+\lambda)2^k C(2,m) - 1]}{2^k C(2,m)(2-\alpha-\alpha\lambda)}\\
&- \frac{[2C(2,m) - 1 - \alpha((1+\lambda)C(2,m) - 1)]}{2^k C(2,m)(2-\alpha-\alpha\lambda)} b_1 \subset f(U).
\end{aligned}$$

□

Now we show that $\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha)$ is closed under convex combinations of its member and also closed under the convolution product.

Theorem 1.8. *The family $\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha)$ is closed under convex combinations.*

Proof. For $i = 1, 2, \ldots$, suppose that $f_i \in \overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha)$ where

$$f_i(z) = z - \sum_{n=2}^{\infty} a_{i,n} z^n + \sum_{n=2}^{\infty} \overline{b}_{i,n} \overline{z}^n.$$

Then, by Theorem 1.3

$$\sum_{n=2}^{\infty} \frac{n^k C(n,m)[n-\alpha-\alpha\lambda(n-1)]}{(1-\alpha)} a_{i,n} + \sum_{n=1}^{\infty} \frac{n^k C(n,m)[n+\alpha-\alpha\lambda(n+1)]}{(1-\alpha)} b_{i,n} \le 1. \tag{1.10}$$

For $\sum_{i=1}^{\infty} t_i$, $0 \le t_i \le 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{n=2}^{\infty}\left(\sum_{i=1}^{\infty} t_i a_{i,n}\right) z^n + \sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty} t_i \overline{b}_{i,n}\right) \overline{z}^n.$$

Using the inequality (1.8), we obtain

$$\sum_{n=2}^{\infty} \frac{n^k C(n,m)[n-\alpha-\alpha\lambda(n-1)]}{(1-\alpha)} \left(\sum_{i=1}^{\infty} t_i a_{i,n}\right) +$$

$$+\sum_{n=1}^{\infty}\frac{n^kC(n,m)[n+\alpha-\alpha\lambda(n+1)]}{(1-\alpha)}\left(\sum_{i=1}^{\infty}t_ib_{i,n}\right)$$

$$=\sum_{i=1}^{\infty}t_i\left(\sum_{n=2}^{\infty}\frac{n^kC(n,m)[n-\alpha-\alpha\lambda(n-1)]}{(1-\alpha)}a_{i,n}+\sum_{n=1}^{\infty}\frac{n^kC(n,m)[n+\alpha-\alpha\lambda(n+1)]}{(1-\alpha)}b_{i,n}\right)$$

$$\leq\sum_{i=1}^{\infty}t_i=1,$$

and therefore $\sum\limits_{i=1}^{\infty}t_if_i\in\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha)$. □

Theorem 1.9. *For $0\leq\beta\leq\alpha<1$, let $f(z)\in\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha)$ and $F(z)\in\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\beta)$. Then $f(z)*F(z)\in\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha))\subset\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\beta)$.*

Proof. Let

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n+\sum_{n=1}^{\infty}\overline{b}_n\overline{z}^n\in\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha)$$

and

$$F(z)=z-\sum_{n=2}^{\infty}A_nz^n+\sum_{n=1}^{\infty}\overline{B}_n\overline{z}^n\in\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\beta).$$

Then $f(z)*F(z)$ is

$$f(z)*F(z)=z-\sum_{n=2}^{\infty}a_nA_nz^n+\sum_{n=1}^{\infty}\overline{b}_n\overline{B}_n\overline{z}^n.$$

For $f(z)*F(z)\in\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\beta)$ we note that $|A_n|\leq1$ and $|B_n|\leq1$.
Now by Theorem 1.3 we have

$$\sum_{n=2}^{\infty}\frac{n^kC(n,m)[n-\beta-\beta\lambda(n-1)]}{1-\beta}|a_n|\;|A_n|$$

$$+\sum_{n=1}^{\infty}\frac{n^kC(n,m)[n+\beta-\beta\lambda(n+1)]}{1-\beta}|b_n|\;|B_n|$$

$$\leq\sum_{n=2}^{\infty}\frac{n^kC(n,m)[n-\beta-\beta\lambda(n-1)]}{1-\beta}|a_n|+\sum_{n=1}^{\infty}\frac{n^kC(n,m)[n+\beta-\beta\lambda(n+1)]}{1-\beta}|b_n|$$

and since $0\leq\beta\leq\alpha<1$

$$\sum_{n=2}^{\infty}\frac{n^kC(n,m)[n-\alpha-\alpha\lambda(n-1)]}{1-\alpha}|a_n|+\sum_{n=1}^{\infty}\frac{n^kC(n,m)[n+\alpha-\alpha\lambda(n+1)]}{1-\alpha}|b_n|\leq1,$$

by Theorem 1.3 $f(z)\in\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha)$. Therefore

$$f(z)*F(z)\in\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\alpha)\subset\overline{\mathcal{H}}\mathcal{R}_k^m(\lambda,\beta).$$ □

Concluding remarks. We observe that, if we specialize the parameter $\lambda = 0$, for suitable choice of $k = 0$ and $m = 0$; $m = 0$ and $k = 0$ we obtain the analogous results for the classes studied in [2, 3] and [4] respectively.

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