THE SOLVABILITY AND PROPERTIES OF SOLUTIONS OF ONE WIENER-HOPF TYPE EQUATION IN THE SINGULAR CASE

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Abstract. The work defines the conditions of solvability of one integral convolutional equation with degreely difference kernels in a singular case. This type of integral equations was not studied earlier, and it turned out that all methods used for the investigation of such equations with the help of Riemann boundary problem at the real axis are not applied there. The investigation of such type equations is based on the investigation of the equivalent singular integral equation with the Cauchy type kernel at the real axis in a singular case. It is determined that the equation is not a Noetherian one. Besides, there are shown the number of the linear independent solutions of the homogeneous equation and the number of conditions of solvability for the heterogeneous equation in the singular case. The general form of these conditions is also shown and there are determined the spaces of solutions of the equation. Thus the convolutional equation that wasn't studied earlier is presented in this work and the theory of its solvability in the singular case is built here. So some new and interesting theoretical results are got in this paper.

The present work is devoted to studying the next Wiener-Hopf type integral equation such as

$$P_m(x)\varphi(x) + \frac{1}{\sqrt{2\pi}} \int\limits_{0}^{+\infty} k(t, x-t)\varphi(t)\,dt = h(x), x \in \mathbf{R},\qquad(1.1)$$

where $\mathbf{R}$ is the real axis;

$$k(t, x-t) = \sum_{j=0}^{n} k_j(x-t)t^j,$$

and

$$P_m(x) = \sum_{k=0}^{m} A_k x^k$$

is the known polynomial of degree m and $k_j(x) \in \mathbf{L}$, $j = \overline{1, n}$, $h(x) \in \mathbf{L}_2$ are known functions.

The theory of solvability of Wiener-Hopf equations with difference kernels was constructed in [2] and there were made quite wide assumptions concerning their kernels and right parts. This theory was based on the investigation of the Riemann boundary-value problem on the real axis, that was obtained with the help of the Fourier transformation. But we can't use methods from [2] to study the equation (1.1), because the investigation of it with the help of the Fourier transformation goes to the investigation of the Riemann differential boundary-value problem on the real axis. The ordinary Wiener-Hopf equation was studied in details in [6], where the conditions of solvability and some properties of solutions in the normal and singular cases were determined. The number of the linear independent solutions and the number of conditions of solvability for the both cases were also established there. Now we are studying the Wiener-Hopf type equation with more complicated kernel.

Let $D^+ = \{z \in \mathbf{C} : \mathbf{Im\,z} > 0\}$ be an upper half plane and $D^- = \{z \in \mathbf{C} : \mathbf{Im\,z} < 0\}$ be a lower half plane of the complex plane $\mathbf{C}$. According to the properties of the Fourier transformation [3], [2] the investigation of the equation (1.1) reduces to the investigation of the following Riemann differential boundary-value problem

$$\left[\sum_{k=0}^{m} A_k (-1)^k \Phi^{+(k)}(x) + \sum_{j=0}^{n} (-1)^j K_j(x) \Phi^{+(j)}(x) \right] - \Phi^-(x) = H(x), x \in \mathbf{R}, \quad (1.2)$$

where $K_j(x), H(x)$ are accordingly the Fourier transforms of functions $k_j(x), h(x)$, $j = \overline{1, n}$. $\Phi^{+(p)}(x)$ and $\Phi^-(x)$ are the boundary values at $\mathbf{R}$ of functions $\Phi^{+(p)}(z)$ and $\Phi^-(z)$ accordingly, where $\Phi^+(z), \Phi^-(z)$ are unknown functions, which are analytical in the domains D^+ and D^- accordingly. As all the transformations of the Riemann differential boundary-value problem (1.2) and the equation (1.1) are identical, then the problem (1.2) and the equation (1.1) are equivalent in such a sense that they are simultaneously solvable or are not, and there is one and only one solution $\Phi^{\pm}(x)$ of the Riemann differential boundary-value problem (1.2) that corresponds to one and only one solution $\varphi(x)$ of the equation (1.1) and vice versa. The solutions of the equation (1.1) are expressed over the solutions of the problem (1.2) by the formula

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} \Phi^+(t) e^{-ixt}\, dt, x > 0. \tag{1.3}$$

We consider the functions $K_j(x) \in \mathbf{H}_\alpha^{(r)}$, $r \geq 0, 0 < \alpha \leq 1$, $\mathbf{H}_\alpha^{(0)} = \mathbf{H}_\alpha$, $j = \overline{1,n}$ and the function $H(x) \in \mathbf{L}_2^{(r)}$, $r \geq 0$, $\mathbf{L}_2^{(0)} = \mathbf{L}_2$. As functions $k_j(x) \in \mathbf{L}$, then according to Riemann-Lebesgue theorem [3] $\lim\limits_{x \to \infty} K_j(x) = 0$, $j = \overline{1,n}$. The investigation of the equation (1.1) we will do basing on the investigation of the Riemann differential boundary-value problem (1.2). The investigation of the Riemann differential boundary-value problem (1.2) reduces to the investigation of the singular integral equation with Cauchy kernel at the real axis with the help of integral representations for functions and derivatives of them built in [8]. Let construct functions $\Phi^+(z)$ and $\Phi^-(z)$ such that they are analytic in the domains D^+, D^- respectively and decay at infinity. Besides, the boundary values on $\mathbf{R}$ of functions $\Phi^{+(p)}(z)$ and $\Phi^-(z)$ satisfy the following condition $\Phi^{+(p)}(x), \Phi^-(x) \in \mathbf{L}_2^{(r)}$, $r \geq 0, p \geq 0$. These conditions satisfy such functions as:

$$\Phi^\pm(z) = (2\pi\imath)^{-1} \int\limits_{\mathbf{R}} P^\pm(x,z)\rho(x)\,dx, z \in D^\pm, \tag{1.4}$$

where

$$P^+(x,z) = \frac{(-1)^p(x+\imath)^{-p}}{(p-1)!}\left[(x-z)^{p-1}\ln\left(1 - \frac{x+\imath}{z+\imath}\right) - \sum_{k=0}^{p-2} d_{p-k-2}(x+\imath)^{k+1}(z+\imath)^{p-k-2}\right],$$

$$x \in \mathbf{R}, \mathbf{z} \in \mathbf{D}^+;$$

$$P^-(x,z) = \frac{1}{x-z}$$

$$x \in \mathbf{R}, \mathbf{z} \in \mathbf{D}^-;$$

$$d_{p-k-2} = (-1)^{k+1}\sum_{j=0}^{k} C_{p-1}^{p-1-j}(k-j+1)^{-1},$$

where C_n^m are binomial coefficients and the function $\ln\left[1 - \frac{x+\imath}{z+\imath}\right]$ is the main branch ($\ln 1 = 0$) of the logarithmic function in the complex plane with the cut that connects such points as $z = -\imath$ and $z = \infty$, following the negative direction of the axis of ordinate. It's easy to verify, that defined by (1.4) functions $\Phi^+(z)$ and $\Phi^-(z)$ are the unique analytic functions in domains D^+, D^- respectively. It is easy to verify that the function $\rho(x) \in \mathbf{L}_2$ is defined uniquely by the functions $\Phi^+(z)$ and $\Phi^-(z)$ and vice versa, so with the help of the given function $\rho(x) \in \mathbf{L}_2$ both functions $\Phi^+(z)$ and $\Phi^-(z)$ are constructed uniquely. The following representations take place:

$$\Phi^{+(p)}(z) = (2\pi\imath)^{-1} \int\limits_{\mathbf{R}} (z+\imath)^{-p}(x-z)^{-1}\rho(x)\,dx, z \in D^+,$$

$$\Phi^-(z) = (2\pi\imath)^{-1} \int_{\mathbf{R}} (x-z)^{-1} \rho(x)\, dx, z \in D^-. \tag{1.5}$$

We consider the case, when $m = n$. Using the properties [8] of partial derivatives of functions $P^+(x,z)$ with respect to z and Sohotski formulas for derivatives from [1], with the help of the representations (1.4), (1.5), we transform the Riemann differential boundary-value problem (1.2) into the following singular integral equation

$$A(x)\rho(x) + B(x)(\pi\imath)^{-1} \int_{\mathbf{R}} (t-x)^{-1}\rho(t)\, dt + (T\rho)(x) = H(x), x \in \mathbf{R}, \tag{1.6}$$

where

$$A(x) = 0,5(-1)^m \left\{ [A_m + K_m(x)](x+\imath)^{-m} + 1 \right\},$$

$$B(x) = 0,5(-1)^m \left\{ [A_m + K_m(x)](x+\imath)^{-m} - 1 \right\}, \tag{1.7}$$

$$(T\rho)(x) = \int_{\mathbf{R}} K(x,t)\rho(t)\, dt, x \in \mathbf{R}, \tag{1.8}$$

$$K(x,t) = \frac{1}{2\pi\imath} \sum_{j=0}^{m-1} (-1)^j [A_j + K_j(x)] \frac{\partial^j P^+(t,x)}{\partial x^j}, \tag{1.9}$$

and $\frac{\partial^j P^+(t,x)}{\partial x^j}$ is a limiting value at $\mathbf{R}$ of the function $\frac{\partial^j P^+(t,z)}{\partial z^j}$, $j = \overline{0, m-1}$.

Lemma 1.1. *If functions $K_j(x) \in \mathbf{H}_\alpha^{(r)}$, $j = \overline{1,n}$, then the operator $T: \mathbf{L}_2^{(r)} \to \mathbf{L}_2^{(r)}$, $r \geq 0$, defined by the formula (1.8) is a compact operator.*

The proof of lemma follows from Frechet-Kolmogorov-Riesz criterion of compactness of integral operators on the real axis in the space $\mathbf{L_p}$, $p > 1$, the properties of functions $P^\pm(x,z)$ follow from the results of the papers [8] and [9].

According to the work [4], the problem (1.2) and the singular integral equation (1.6) are equivalent in such a sense that they are simultaneously solvable or are not, and for the every solution $\rho(x)$ of the equation (1.6) there exists may be an ununique solution $\Phi^\pm(x)$ of the problem (1.2) and vice versa. In order to make this solution to be the unique one, it is necessary to set initial conditions for the problem (1.2). As its solutions $\Phi^\pm(x)$ are found in spaces of decaying at infinity functions, then according to the properties of Cauchy type integral, solutions of the problem (1.2) are such that $\Phi^{\pm(j)}(\infty) = 0$, $j = \overline{0, m-1}$, thus we obtain trivial initial conditions of (1.2) and they are set automatically. So it follows that the Riemann differential boundary-value problem (1.2) and the singular integral equation (1.6) are equivalent in such a sense that they are simultaneously solvable or are not, and there is one and only one solution $\rho(x)$ of the equation (1.6) for the every solution $\Phi^\pm(x)$ of the problem (1.2)

and vice versa. By the force of formula (1.4), the solutions of the problem (1.2) are expressed over solutions of the equation (1.6) according to the formula

$$\Phi^+(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} P^+(t, x)\rho(t)\,dt, x \in \mathbf{R}, \tag{1.10}$$

where $p = m$; $P^+(t, x)$ is the boundary value at $x \in \mathbf{R}$ of function $P^+(t, z)$, and $\rho(x)$ is the solution of the equation (1.6). As the equation (1.1) and the problem (1.2) are equivalent, the problem (1.2) and the singular integral equation (1.6) are equivalent, too, it follows that the equation (1.1) and the equation (1.6) are equivalent in such a sense that they are simultaneously solvable or are not, and there is one and only one solution $\varphi(x)$ of the equation (1.1) for the every solution $\rho(x)$ of the equation (1.6) and vice versa. Thus the solutions of the equation (1.1) are expressed over solutions of the equation (1.6) according to the formulas (1.10), (1.3). That is why the equation (1.1) we will call Noetherian if the equation (1.6) is Noetherian.

Theorem 1.2. *The equation (1.1) is not Noetherian.*

Proof. According to the work [4] the equation (1.6) is Noetherian if and only if when $A(x) + B(x) \neq 0$, $A(x) - B(x) \neq 0$ on $x \in \mathbf{R}$. From the formula (1.7) it follows that

$$A(x) + B(x) = (-1)^m[A_m + K_m(x)](x + \imath)^{-m},$$

$$A(x) - B(x) = 1.$$

So we have got that the function $A(x)+B(x)$ possesses a zero at infinity of at least the order m. It means that the equation (1.6) is not Noetherian. Then as the equations (1.1) and (1.6) are equivalent, the equation (1.1) is not Noetherian, too.

The theorem is proved.

Let determine $\chi = -ind\,[A_m + K_m(x)]$.

Here we don't study the case when $A(x) + B(x) \neq 0$ on $\mathbf{R}$ as this is the normal case and the results of [6] for it remain correct if in Theorems 1.3 and 1.4 in [6] we will study the function $A_m + K_m(x)$ instead of $A_m + B_m K(x)$.

Let's study the singular case when the condition $A_m + K_m(x) \neq 0$ at $\mathbf{R}$ is not executed. Then we suppose that the function $A_m + K_m(x)$ goes to zero on the real axis in such points as $a_1, a_2, \ldots, a_s$ with accordingly integer orders $\gamma_1, \gamma_2, \ldots, \gamma_s$. Then in virtue of [5] the following representation takes place

$$A(x) + B(x) = (x + \imath)^{-m} M(x) \prod_{k=1}^{s} \left(\frac{x - a_k}{x + \imath} \right)^{\gamma_k}, \tag{1.11}$$

where the function $M(x) \neq 0$ on $\mathbf{R}$, $M(x) \in \mathbf{H}_\alpha^{(r)}$.

Let

$$r_0 = \max\{\gamma_1, \gamma_2, \ldots, \gamma_s, m\}, \gamma = \sum_{k=1}^{s} \gamma_k, \chi = -indM(x). \qquad (1.12)$$

Theorem 1.3. *Let the functions $k_j(x) \in \mathbf{L}$, $h(x) \in \mathbf{L}_2$; the functions $K_j(x) \in \mathbf{H}_\alpha^{(r)}$, $j = \overline{1, m}$, $H(x) \in \mathbf{L}_2^{(r)}$, $r \geq r_0$, where the number r_0 is defined by the formula (1.12) and the representations (1.11) take place, where $M(x) \neq 0$ on $\mathbf{R}$.*

If $\chi + m + \gamma \leq 0$, where the numbers χ, γ are defined by formulas (1.12), then the homogeneous equation (1.1) has not less than $|\chi + m + \gamma|$ linear independent solutions; the heterogeneous equation (1.1) is unconditionally solvable and its general solution depends upon not less than $|\chi + m + \gamma|$ arbitrary constants.

If $\chi + m + \gamma > 0$, then generally speaking the heterogeneous equation (1.1) is unsolvable. It will be solvable when not less than $m + \gamma + \chi$ conditions of solvability

$$\int\limits_R H(x)\psi_j(x)dx = 0, \qquad (1.13)$$

will be executed. Here $H(x)$ is a right part of the equation (1.6), and $\psi_j(x)$ are the linear independent solutions of the homogeneous equation

$$A(x)\psi(x) - (\pi i)^{-1} \int\limits_R \frac{B(t)\psi(t)}{t-x}dt + \int\limits_R K(t, x)\psi(t)dt = 0,$$

allied to the equation (1.6), where $A(x), B(x), K(x, t)$ are the coefficients and the regular kernel of the singular integral equation (1.6).

Proof. According to [5], the index of the equation (1.6) is equal to $-(\chi + m + \gamma)$. Then due to [4], if $\chi + m + \gamma \leq 0$, then the homogeneous equation (1.6) has not less than $|\chi + m + \gamma|$ linear independent solutions; the heterogeneous equation (1.6) is unconditionally solvable and its general solution depends upon not less than $|\chi + m + \gamma|$ arbitrary constants. If $\chi + m + \gamma > 0$, then due to [4], the heterogeneous equation (1.6) is unsolvable. It will be solvable, when not less than $m + \gamma + \chi$ conditions of solvability (1.13) will be executed. As the equations (1.1) and (1.6) are equivalent, then the theorem is proved.

Theorem 1.4. *Let the functions $k_j(x) \in \mathbf{L}$, $h(x) \in \mathbf{L}_2$; the functions $K_j(x) \in \mathbf{H}_\alpha^{(r)}$, $j = \overline{1, m}$, $H(x) \in \mathbf{L}_2^{(r)}$, $r \geq r_0$, where the number r_0 is defined by the formula (1.12), the representations (1.11) take place, where $M(x) \neq 0$ on $\mathbf{R}$, and the equation (1.1) is solvable. Then its solutions belong to the space $\mathbf{L}_2[-r - m + r_0; 0]$, $r \geq r_0$.*

Proof. According to [5], the solutions of the equation (1.6) $\rho(x) \in \mathbf{L}_2^{(r-r_0)}$, $r \geq r_0$. Then in virtue of the representations (1.5) and the properties of the Cauchy type integral the limit values $\Phi^+(x)$ on $\mathbf{R}$ of the function $\Phi^+(z)$ belong to the space $\mathbf{L}_2^{(r-r_0)}$.

194

From the properties of Fourier transformation [2] we obtain that the solutions of the equation (1.1) belong to the space $\mathbf{L}_2[-r - m + r_0; 0]$, $r \geq r_0$, and the theorem is proved.

If the function $A_m + K_m(x)$ goes to zero on the real axis $\mathbf{R}$ in the points $a_1, a_2, \ldots, a_s$ with accordingly fractional orders $\gamma_1, \gamma_2, \ldots, \gamma_s$, then the representation (1.11) where $M(x) \neq 0$ at $\mathbf{R}$ and $M(x) \in H_\alpha^{(r)}$ is also fulfilled. But for the numbers γ_k, $k = \overline{1, s}$ the following representation takes place:

$$\gamma_k = [\gamma_k] + \{\gamma_k\}, \quad k = \overline{0, s}.$$

There $[a]$ means the integer part, and $\{a\}$ - the fractional part of the number a.
So here we will note

$$r_0 = max\{\alpha_1', \ldots, \alpha_s', m\}, \quad \alpha = \sum_{k=1}^{s} \alpha_k', \quad \varkappa = -indM(x), \qquad (1.14)$$

$$\alpha_k' = \begin{cases} [\gamma_k], 0 < \{\gamma_k\} < \frac{1}{2} \\ [\gamma_k] + 1, \frac{1}{2} < \{\gamma_k\} < 1, \quad k = \overline{0, s}. \end{cases} \qquad (1.15)$$

With such assumption the following theorem takes place.

Theorem 1.5. *Let the functions $k_j(x) \in L$, $j = \overline{1, m}$, $h(x) \in L_2$ and the functions $K_j(x) \in H_\alpha^{(r)}$, $j = \overline{1, m}$, $0 < \alpha \leqslant 1$, $H(x) \in L_2^{(r)}$, $r \geq r_0$, where the number r_0 is defined by (1.14), (1.15) and the representation (1.11) takes place, where $M(x) \neq 0$ on $\mathbf{R}$.*

If $\chi + m + \alpha \leq 0$, where the numbers χ, α are defined by formulas (1.14), then the homogeneous equation (1.1) has not less than $|\chi + m + \alpha|$ linear independent solutions; the heterogeneous equation (1.1) is unconditionally solvable and its general solution depends upon not less than $|\chi + m + \alpha|$ arbitrary constants.

If $\chi + m + \alpha > 0$, then generally speaking the heterogeneous equation (1.1) is unsolvable. It will be a solvable one when not less than $m + \alpha + \chi$ conditions of solvability (1.13) will be executed.

Proof. According to [5], [7] the index of the equation (1.6) is equal to $-(\chi + m + \alpha)$. Then due to [4], if $\chi + m + \alpha \leq 0$, then the homogeneous equation (1.6) has not less than $|\chi + m + \alpha|$ linear independent solutions; the heterogeneous equation (1.6) is unconditionally solvable and its general solution depends upon not less than $|\chi + m + \alpha|$ arbitrary constants. If $\chi + m + \alpha > 0$, then due to [4], the heterogeneous equation (1.6) is unsolvable. It will be solvable, when not less than $m + \alpha + \chi$ conditions of solvability (1.13) will be executed. As the equations (1.1) and (1.6) are equivalent, then the theorem is proved.

Theorem 1.6. *Let the functions $k_j(x) \in \mathbf{L}$, $h(x) \in \mathbf{L}_2$; the functions $K_j(x) \in \mathbf{H}_\alpha^{(r)}$, $j = \overline{1,m}$, $H(x) \in \mathbf{L}_2^{(r)}$, $r \geq r_0$, where the number r_0 is defined by the formula (1.14), the representations (1.11) take place, where $M(x) \neq 0$ on $\mathbf{R}$, and the equation (1.1) is solvable. Then its solutions belong to the space $\mathbf{L}_2[-r - m + r_0; 0]$, $r \geq r_0$.*

The proof of the Theorem 1.6 coincides with the proof of the theorem 3.

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