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DATA DEPENDENCE FOR SOME INTEGRAL EQUATIONS

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Abstract. In the paper *Integral equations, periodicity and fixed points*, published in Fixed Point Theory, 9(2008), No 1, 47-65 the author T.A. Burton considered the equation

$$x(t) = g(t) + \int_{-\infty}^{t} K(t, s, x(s))ds.$$

In this paper we shall study the data dependence for this integral equations.

1. **Introduction**

Let $(P_T, \|\cdot\|)$ denote the Banach space of continuous scalar $T-$periodic functions with the supremum norm.

We consider the equation

$$x(t) = g(t) + \int_{-\infty}^{t} K(t, s, x(s))ds,\ t \in \mathbb{R} \tag{1.1}$$

under the conditions:

(C_1) there exists $T > 0$ such that

$$g(t + T) = g(t),\ K(t + T, s + T, u) = K(t, s, u)$$

for all $t, s, u \in \mathbb{R}$;

(C_2) for all $x \in P_T$ we have that $\int_{-\infty}^{(\cdot)} K((\cdot), s, x(s))ds \in P_T$

Now we define the operator

$$A : P_T \to P_T,$$

$$A(x)(t) = g(t) + \int_{-\infty}^{t} K(t, s, x(s))ds.$$

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In [1], T.A. Burton was considered the following conditions

(C_3) there exists a function $B(t,s)$ with $\int\limits_{-\infty}^{t} B(t,s)$ defined such that

$$|K(t,s,u)-K(t,s,v)| \leq B(t,s)|u-v|,$$

for all $-\infty < s \leq t < \infty$, $u,v \in \mathbb{R}$;

(C_4) there exists $0<\alpha<1$ such that $\int\limits_{-\infty}^{t} B(t,s) \leq \alpha$.

Under conditions $(C_1)-(C_4)$ we have that the operator A has a unique fixed point $x_A^\star$, and $A^n(x) \to x_A^\star$ for $n \to \infty$ and for all $x \in P_T$, so the operator A is Picard(I.A. Rus [3])

The purpose of this article is to establish a Gronwall type lemma corresponding to the equation (1.1) and also data dependence theorems, comparison theorems for the solutions of the equation (1.1). More results about nonlinear integral equations we find in [2].

2. **A Gronwall type inequalities**

We consider the following integral inequalities:

$$x(t) \leq g(t) + \int\limits_{-\infty}^{t} K(t,s,x(s))ds, \ t \in \mathbb{R} \tag{2.1}$$

$$x(t) \geq g(t) + \int\limits_{-\infty}^{t} K(t,s,x(s))ds \ t \in \mathbb{R}. \tag{2.2}$$

Throughout this section we use the following

Lemma 2.1. *I.A. Rus* [5] *Let (X,d) be an ordered metric space and $A : X \to X$ be such that:*

(i) *the operator A is Picard, with the set of fixed points $F_A = \{x_A^\star\}$;*
(ii) *the operator A is monotone increasing.*

Then

(a) *$x \leq A(x)$ implies $x \leq x_A^\star$;*
(b) *$x \geq A(x)$ implies $x \geq x_A^\star$;*

We have

Theorem 2.2. *We suppose that:*

(i) *the conditions $(C_1)-(C_4)$ hold;*
(ii) *the operator $K(t,s,\cdot)$ is monotone increasing, for all $-\infty < s \leq t < \infty$.*

Then

(a) *the equation (1.1) has a unique solution* $x^\star$;

(b) *for all solutions* $x \in P_T$ *of the inequality (2.1) we have that* $x \leq x^\star$*;*

(c) *for all solutions* $x \in P_T$ *of the inequality (2.2) we have that* $x \geq x^\star$,

Proof. (a) We consider the operator

$$A : P_T \to P_T,$$

$$A(x)(t) = g(t) + \int_{-\infty}^{t} K(t, s, x(s))ds.$$

T.A Burton [1] proves that the operator A is Picard operator, $F_A = \{x^\star\}$.

(b)+(c) From the condition (ii) we obtain that A is an increasing operator. Then, by Lemma 2.1 we have the conclusions.

3. **A comparison result**

Now we shall give a comparison result for the solution of the equation (1.1). For this study we need the following abstract result ([5]).

Lemma 3.1. *Let* $(X, d, \leq)$ *be an ordered metric space and* $A, B, C : X \to X$ *be such that:*

(i) $A \leq B \leq C$

(ii) A, B, C *are Picard operators,* $F_A = \{x_A^\star\}$, $F_B = \{x_B^\star\}$, $F_C = \{x_C^\star\}$*;*

(iii) *the operator* B *is increasing.*

Then

$$x_A^\star \leq x_B^\star \leq x_C^\star.$$

We consider the equations

$$(4)_i \quad x(t) = g_i(t) + \int_{-\infty}^{t} K(t, s, x(s))ds, \ t \in \mathbb{R}, \ i = \overline{1,3},$$

We have

Theorem 3.2. *We consider the equation* $(4)_i$*. We suppose that:*

(i) g_i *and* K_i, $i = \overline{1,3}$, *satisfy the condition* (i) *in Theorem 2.2;*

(ii) $g_1(t) \leq g_2(t) \leq g_3(t)$ *and* $K_1(t, s, \cdot) \leq K_2(t, s, \cdot) \leq K_3(t, s, \cdot)$ *for all* $-\infty < s \leq t < \infty$;

(iii) $K_2(t, s, \cdot)$ *is monotone increasing for all* $-\infty < s \leq t < \infty$.

Then

(a) *the equations* $(4)_i$ *have a unique solution* $x_i^\star \in P_T$, $i = \overline{1,3}$
(b) $x_1^\star \leq x_2^\star \leq x_3^\star$.

Proof. (a) We consider the operator

$$A_i : P_T \to P_T,$$
$$A_i(x)(t) = g_i(t) + \int_{-\infty}^{t} K_i(t,s,x(s))ds,\ i = \overline{1,3}.$$

The condition (i) from Theorem 2.2 implies that the operators A_i are Picard with $F_{A_i} = \{x_i^\star\}, i = \overline{1,3}$.

(b) From the condition (ii) we have that $A_1 \leq A_2 \leq A_3$ and from (iii) we obtain that A_2 is an increasing operator. Then, from Lemma 2.1 we have the conclusion.

4. **Data dependence: Continuity**

Now we consider the equations

$$x(t) = g_1(t) + \int_{-\infty}^{t} K_1(t,s,x(s))ds,\ t \in \mathbb{R} \tag{4.1}$$

$$x(t) = g_2(t) + \int_{-\infty}^{t} K_2(t,s,x(s))ds,\ t \in \mathbb{R}. \tag{4.2}$$

We have

Theorem 4.1. *We suppose that*

(1) g_1, g_2, K_1, K_2 *satisfy the conditions* (i) *in Theorem 2.2;*
(2) *there exists* $\eta_1 >$ *such that*

$$|g_1(t) - g_2(t)| \leq \eta_1,$$

for all $t \in \mathbb{R}$*;*

(ii) *there exists a function* $\eta_2(t,s)$ *and* $\eta_3 > 0$ *such that*

$$\int_{-\infty}^{t} \eta_2(t,s)ds \leq \eta_3,$$

$$|K_1(t,s,u) - K_2(t,s,u)| \leq \eta_2(t,s),$$
for all $-\infty < s \leq t < \infty$, $u \in \mathbb{R}$.

Then

(a) *the equations (4.1), (4.2) have a unique solution* $x_1^\star$ *respectively* $x_2^\star$*;*

(b) $\|x_1^\star - x_2^\star\| \leq \frac{\eta_1+\eta_3}{1-\alpha}$.

Proof. (a) We define the operators

$$A_i : P_T \to P_T,$$

$$A_i(x)(t) = g_i(t) + \int_{-\infty}^{t} K_i(t,s,x(s))ds, i = \overline{1,2}.$$

The condition (i) from Theorem 2.2 implies that the operators A_i are Picard with $F_{A_i} = \{x_i^\star\}, i = \overline{1,2}$.

(b) Because

$$|A_1(x)(t) - A_2(x)(t)| \leq |g_1(t) - g_2(t)| + \int_{-\infty}^{t} |K_1(t,s,x(s)) - K_2(t,s,x(s))|ds \leq$$

$$\leq \eta_1 + \int_{-\infty}^{t} \eta_2(t,s)ds \leq \eta_1 + \eta_3$$

for all $x \in P_T$ and $t \in \mathbb{R}$, we obtain that

$$\|A_1(x) - A_2(x)\| \leq \eta_1 + \eta_3.$$

Now the proof follows from a well known abstract result([3], [4]).

5. Smooth dependence on parameter

Next we consider the following integral equation

$$x(t) = g(t,\lambda) + \int_{-\infty}^{t} K(t,s,x(s),\lambda)ds, t \in \mathbb{R}, \lambda \in J = [c,d] \subset \mathbb{R}. \tag{5.1}$$

Let $(P_T, \|\cdot\|)$ be the Banach space of continuous scalar $T-$periodic functions, defined on $\mathbb{R} \times J$, with the supremum norm.

We assume that

(H_1) $g, K \in C^1(\mathbb{R} \times J)$ and it verify the conditions (C_1), (C_2);

(H_2) there exists a function $B(t,s)$ such that

$$|\frac{\partial K}{\partial u}(t,s,u,\lambda)| \leq B(t,s),$$

for all $-\infty < s \leq t < \infty$, $u, v \in \mathbb{R}, \lambda \in J$;

(H_3) $\int_{-\infty}^{t} B(t,s)ds$ is defined and $\int_{-\infty}^{t} B(t,s)ds \leq \alpha < 1$.

We define the operator

$$B : P_T \to P_T,$$

$$B(x)(t,\lambda) = g(t,\lambda) + \int_{-\infty}^{t} K(t,s,x(s,\lambda),\lambda)ds.$$

It is clear that, in the conditions $(H_1) - (H_3)$ the operator B is Picard operator. Let $x^\star(\cdot,\lambda)$ be the unique fixed point of the operator B. Then

$$x^\star(t,\lambda) = g(t,\lambda) + \int_{-\infty}^{t} K(t,s,x^\star(s,\lambda),\lambda)ds \tag{5.2}$$

We suppose that there exists $\frac{\partial x^\star}{\partial \lambda}$. Then from (5.2) we have that

$$\frac{\partial x^\star}{\partial \lambda}(t,\lambda) = \frac{\partial g}{\partial \lambda}(t,\lambda) + \int_{-\infty}^{t} [\frac{\partial K}{\partial u}(t,s,x^\star(s,\lambda);\lambda)\frac{\partial x^\star(s,\lambda)}{\partial \lambda} + \frac{\partial K}{\partial \lambda}(t,s,x^\star(s,\lambda);\lambda)]ds$$

This relation suggest us to consider the following operator

$$C : P_T \times P_T \to P_T,$$

$$C(x,y)(t,\lambda) = \frac{\partial g}{\partial \lambda}(t,\lambda) + \int_{-\infty}^{t} [\frac{\partial K}{\partial u}(t,s,x(s,\lambda);\lambda)y(s,\lambda) + \frac{\partial K}{\partial \lambda}(t,s,x(s,\lambda);\lambda)]ds$$

In this way we have the triangular operator

$$A : P_T \times P_T \to P_T \times P_T,$$

$$A(x,y) = (B(x), C(x,y))$$

where B is a Picard operator and $C(x,\cdot) : P_T \to P_T$ is an $\alpha-$contraction.

From the theorem of fiber contraction (see I.A. Rus [5],[6]) we have that the operator A is Picard operator. So, the sequences

$$x_{n+1} = B(x_n), n \in \mathbb{N}$$

$$y_{n+1} = C(x_n, y_n), n \in \mathbb{N}$$

converges uniformly to $(x^\star, y^\star) \in F_A$, for all $x_0, y_0 \in P_T$.

If we take $x_0 = 0$, $y_0 = \frac{\partial x_0}{\partial \lambda} = 0$ then $y_1 = \frac{\partial x_1}{\partial \lambda}$ and by induction we prove that $y_n = \frac{\partial x_n}{\partial \lambda}$, for all $n \in \mathbb{N}^\star$.

Thus

$$x_n \to x^\star, \;\; uniform \;\; as \;\; by \; n \to \infty$$

$$\frac{\partial x_n}{\partial \lambda} \to y^\star, \quad uniform \;\; as \; by \; n \to \infty$$

These imply that there exists $\frac{\partial x^\star}{\partial \lambda}$ and $\frac{\partial x^\star}{\partial \lambda} = y^\star$

From the above considerations, we have the following result

Theorem 5.1. *We consider the integral equation (5.1) in the hypothesis* $(H_1)-(H_3)$. *Then*

(i) *the equation (5.1) has a unique solution* $x^\star(t, \cdot) \in P_T$;

(ii) $x^\star(t, \cdot) \in C^1(J)$, *for all* $t \in \mathbb{R}$.

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