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VARIATIONAL-HEMIVARIATIONAL INEQUALITIES ON UNBOUNDED DOMAINS

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Abstract. This paper is a survey about hemivariational and variational-hemivariational inequalities defined on unbounded domains motivated by certain non-smooth phenomena appearing in Mathematical Physics. The paper contains various results obtained by the authors in the last few years. It is divided into six sections: the first section is a short introduction; in the second section we present some critical points results for locally Lipschitz functions; the third section is dedicated to Motreanu-Panagiotopoulos functionals; in the fourth section we provide some existence results for hemivariational inequalities; in the fifth section we give a multiplicity result for a special class of hemivariational inequalities; and in the last section we give some applications to hemivariational and variational-hemivariational inequalities.

1. Introduction

The study of *variational inequalities* began in the sixties with the pioneering work of Lions and Stampacchia [35]. The connection of this theory with the notion of the subdifferential of a convex function was achieved by Moreau [43], who introduced the notion of convex superpotentials which permitted the formulation and study in the weak form of a wide ranging class of complicated problems in Mechanics and Engineering (see Duvaut and Lions [12]). All the inequality problems studied in that period were related to convex energy functions and therefore were linked with the notion of monotonicity. Motivated by some problems from mechanics, Panagiotopoulos introduced in [50, 51] the notion of nonconvex superpotential by using the generalized gradient of Clarke. Due to the lack of convexity, new types of variational expressions

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were obtained; these are the so-called *Hemivariational Inequalities.* The hemivariational inequalities appears as a generalization of the variational inequalities, but actually they are much more general than these ones, because they are not equivalent to minimum problems. They are no longer connected with monotonicity, but since the main ingredient of their study is based on the notion of Clarke subdifferential of a locally Lipschitz funtion, the theory of hemivariational inequalities appears as a new field of Non-smooth Analysis. For a comprehensive treatment of the hemivariational inequality problems we refer to the monographs Naniewicz and Panagiotopoulos [48] (based on pseudomonotonicity), Motreanu and Panagiotopoulos [46], Motreanu and Rădulescu [47] (based on compactness arguments). In the above works (and in references therein) there are studied elliptic problems on *bounded domains.*

In this paper we treat hemivariational and variational-hemivariational inequalities problems on *unbounded domains* based on the authors' results in the last few years. Note that in the unbounded case the problem is more delicate, due to the lack of compactness in the Sobolev embeddings. First, some old and new results are recalled from critical points theory for locally Lipschitz functions and Motreanu-Panagiotopoulos functionals see [9], [44], [45], [33], [28], [38], [46], [47], [29] with applications to hemivariational and variational-hemivariational inequalities, see [66], [11], [28], [36], [30], [31], [27], [29]. Then, we present for locally Lipschitz functions the Mountain Pass Theorem (MPT) of "zero altitude", the version of MPT which satisfies the Cerami condition, and a version of the three critical points theorem of Ricceri [58]. In the third section we present some critical points results as well as the principle of symmetric criticality for Motreanu-Panagiotopoulos functionals. In the fourth section we give some existence results for a general class of hemivariational inequalities. In section five we prove a multiplicity result for a particular class of hemivariational inequalities while the last section is dedicated to various applications.

2. Critical points results for locally Lipschitz functions

In this section we present some critical points results for locally Lipschitz functions. These results appear in the papers of Motreanu, Varga [44], [45], Kristály, Motreanu and Varga [33] and Kristály, Marzantowicz and Varga [28].

2.1. **Elements of nonsmooth analysis.** Let $(X, \|\cdot\|)$ a real Banach space and $U \subset X$ an open subset. We denote by $\langle\cdot,\cdot\rangle$ the duality mapping between $X^\star$ and X.

Definition 2.1. A function $f : X \to \mathbb{R}$ is **locally Lipschitz** if, for every $x \in X$, there exist a neighborhood U of x and a constant $L > 0$ such that

$$|f(y) - f(z)| \leq L\|y - z\| \quad for\ all \quad y, z \in U.$$

Although it is not necessarily differentiable in the classical sense, a locally Lipschitz function admits a derivative, defined as follows:

Definition 2.2. The **generalized directional derivative** of f at the point $x \in X$ in the direction $y \in X$ is

$$f^\circ(x;y) = \limsup_{z\to x,\ \tau\to 0^+} \frac{f(z+\tau y)-f(z)}{\tau}.$$

The **generalized gradient** of f at $x \in X$ is the set

$$\partial f(x) = \{x^\star \in X^\star :\ \langle x^\star, y\rangle \leq f^\circ(x;y) \text{ for all } y \in X\}.$$

For all $x \in X$, the functional $f^\circ(x,\cdot)$ is subadditive and positively homogeneous: thus, due to the Hahn-Banach theorem, the set $\partial f(x)$ is nonempty. The next Lemma resumes the main properties of the generalized derivatives, which will be useful in the sequel:

Lemma 2.3. *Let $f, g : X \to \mathbb{R}$ be locally Lipschitz functions. Then,*

(f_1) $f^\circ(x;y) = \max\{\langle \xi, y\rangle :\ \xi \in \partial f(x)\}$*;*
(f_2) $(f+g)^\circ(x;y) \leq f^\circ(x;y) + g^\circ(x;y)$*;*
(f_3) $(-f)^\circ(x;y) = f^\circ(x;-y)$.
(f_4) *The function $(x,y) \mapsto \Phi^\circ(x;y)$ is upper semicontinuous.*

This notion extends both that of Gâteux derivative, and that of directional derivative for convex functionals. In particular:

Lemma 2.4. *Let $f : X \to \mathbb{R}$ be a convex, continuous, Gâteaux differentiable functional. Then, f is locally Lipschitz and*

$$\langle f'(x), y\rangle = f^\circ(x;y) \ \ \textit{for all} \ \ x, y \in X.$$

The next definition generalizes the notion of critical point to the non-smooth context:

Proposition 2.5. *The function $\lambda_f(u) = \inf\limits_{w\in\partial f(u)} ||w||_{X^\star}$ is well defined and is lower semicontinuous, i.e. $\liminf\limits_{u\to u_0} \lambda_f(u) \geq \lambda_f(u_0)$.*

Definition 2.6. Let $f : X \to \mathbb{R}$ be a locally Lipschitz function. We say that $u \in X$ is a critical point (in the sense of Chang) of f, if $\lambda_f(u) = 0$, which is equivalent with the fact that $0 \in \partial f(u)$.

Remark 2.7. A point $u \in X$ is critical point of f if $f^\circ(x;y) \geq 0$ for all $y \in X$.

Remark 2.8. Note that every local extremum of f is a critical point of f in the sense above.

Throughout in this paper we use the following notations for the locally Lipschitz function $f : X \to \mathbb{R}$ and a number $c \in \mathbb{R}$:

$$f^c = \{u \in X : f(u) \leq c\};$$

$$f_c = \{u \in X : f(u) \geq c\};$$

$$K_c = \{u \in X : \lambda_f(u) = 0, f(u) = c\};$$

$$(K_c)_\delta = \{u \in X : d(u, K_c) < \delta\};$$

$$(K_c)_\delta^c = X \setminus (K_c)_\delta.$$

In the sequel we introduce the notion of Palais-Smale condition.

Definition 2.9. We say that the locally Lipschitz function $f : X \to \mathbb{R}$ satisfies the Palais-Smale condition at the level c (shortly, $(PS)_c$), if every sequence $\{x_n\} \subset X$ with $f(x_n) \to c$, and $\lambda_f(x_n) \to 0$ when $n \to \infty$, contains a convergent subsequence in X. If we replace the condition $f(x_n) \to c$ with $\{f(x_n)\}$ is bounded we say that the function f satisfies the (PS) condition.

Remark 2.10. The (PS) condition has the following equivalent formulation: The function h satisfies the Palais-Smale condition, if every sequence $\{x_n\}$ in X such that

(PS_1)**:** $\{f(x_n)\}$ bounded;
(PS_2)**:** there exists a sequence $\{\varepsilon_n\}$ in $]0, +\infty[$ with $\varepsilon_n \to 0$ such that $f^\circ(x_n; y - x_n) + \varepsilon_n \|y - x_n\| \geq 0$ for all $y \in X$, $n \in \mathbb{N}$

admits a convergent subsequence.

The following variant of Palais-Smale condition is an extension to the locally Lipschitz case of the one introduced by Ghoussoub and Preiss [20]. We consider a locally Lipschitz function $f : X \to \mathbb{R}$, a real number $c \in \mathbb{R}$ and a subset $B \subset X$.

Definition 2.11. We say that the locally Lipschitz function f satisfies the Palais-Smale condition around B at level c (shortly, $(PS)_{B,c}$), if every sequence $\{x_n\} \subset X$ with $f(x_n) \to c$, $\operatorname{dist}(x_n, B) \to 0$ and $\lambda_f(x_n) \to 0$ when $n \to \infty$, contains a convergent subsequence in X.

In particular, we put $(PS)_c = (PS)_{X,c}$ and simply (PS) if $(PS)_c$ holds for every $c \in \mathbb{R}$.

For a fixed $B \subseteq X$ and a fixed number $\delta > 0$, we denote the closed δ-neighborhood of B by $N_\delta(B)$, that is,

$$N_\delta(B) = \{x \in X : \operatorname{dist}(x, B) \leq \delta\}.$$

Definition 2.12. A generalized normalized pseudo-gradient vector field of the locally Lipschitz $f : X \to \mathbb{R}$ with respect to a subset $B \subset X$ and a number $c \in \mathbb{R}$ is a locally Lipschitz mapping $v : N_\delta(B) \cap f^{-1}[c-\delta, c+\delta] \to X$ with some $\delta > 0$, such that $||v(x)|| \leq 1$ and

$$\langle y^*, v(x)\rangle > \frac{1}{2} \inf_{x \in \mathrm{dom} v} \lambda_f(x) > 0$$

for all $y^* \in \partial f(x)$ and $x \in \mathrm{dom} v := N_\delta(B) \cap f^{-1}[c-\delta, c+\delta]$.

The existence of a generalized normalized pseudo-gradient vector field in the sense of Definition 2.12 is given by the result below. For the proof, see Motreanu-Varga [45].

Lemma 2.13. (Motreanu-Varga [45]) *Let $f : X \to \mathbb{R}$ be a locally Lipschitz function, $c \in \mathbb{R}$ and a closed subset B of X, such that $(PS)_{B,c}$ is satisfied together with $B \cap K_c(f) = \emptyset$ and $B \subset f^c$. Then there exists $\delta > 0$ and a generalized normalized pseudo-gradient vector field $v : N_\delta(B) \cap f^{-1}[c-\delta, c+\delta] \to X$ of f with respect to B and c.*

The following deformation result has been proved by Motreanu and Varga [45].

Theorem 2.14. (Motreanu-Varga [45]) *Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional, $c \in \mathbb{R}$ and a closed subset B of X provided on has $(PS)_{B,c}, B \cap K_c(f) = \emptyset$ and $B \subset f^c$. Let v be a generalized normalized pseudo-gradient vector field of f with respect to B and c. Then for every $\overline{\varepsilon} > 0$ there exist an $\varepsilon \in (0, \overline{\varepsilon})$ and a number $\delta < c$ such that for each closed subset A of X with $A \cap B = \emptyset$ and $A \subset f_{c-\varepsilon_A}$, where*

$$\varepsilon_A := \min(\varepsilon, \varepsilon d(A,B)), \tag{2.1}$$

and $d(A,B) := \inf\{\|x-y\| : \ x \in A, \ y \in B\}$, there is a continuous mapping $\eta_A : \mathbb{R} \times X \to X$ with the properties below

- (i) *$\eta_A(\cdot, x)$ is the solution of the vector field $V_A = -\varphi_A v$ with the initial condition $x \in X$ for some locally Lipschitz function $\varphi_A : X \to [0,1]$ whose support is contained in the set $(X \setminus A)$;*
- (ii) *$\eta_A(t,x) = x$ for all $t \in \mathbb{R}$ and $x \in A \cup f^{c-\overline{\varepsilon}} \cup f_{c+\overline{\varepsilon}}$;*
- (iii) *for every $\delta \leq d \leq c$ one has $\eta_A(1, B \cap f^d) \subset f^{d-\varepsilon}$.*

Proof. Let us note that the existence of a normalized generalized pseudo-gradient vector field $v : N_{3\delta_1}(B) \cap f^{-1}[c - 3\varepsilon_1, c + 3\varepsilon_1] \to X$ of f with respect to B and c is assured by Lemma 2.13, for some constants $\delta_1 > 0$ and $\varepsilon_1 > 0$. Consequently, a constant, $\sigma_1 > 0$ can be found such that

$$\langle y^*, v(x)\rangle > \frac{1}{2}\sigma_1, \ \forall y^* \in \partial f(x), \ x \in N_{3\delta_1}(B) \cap f_{c-3\varepsilon_1} \cap f^{c+3\varepsilon_1}. \tag{2.2}$$

We claim that the result of Theorem 2.14 holds for every $\varepsilon > 0$ with

$$\varepsilon < \min\{\overline{\varepsilon}, \varepsilon_1, \frac{1}{2}\sigma_1, \frac{1}{2}\sigma_1\delta_1\}. \tag{2.3}$$

In order to check the claim in (2.3) let us fix two locally Lipschitz functions $\varphi,\ \psi :X \to [0,1]$ satisfying

$$\varphi = 1 \text{ on } N_{\delta_1}(B) \cap f^{c+\varepsilon_1} \cap f_{c-\varepsilon_1};$$
$$\varphi = 0 \quad \text{on} \quad X \setminus (N_{2\delta_1}(B) \cap f^{c+2\varepsilon_1} \cap f_{c-2\varepsilon_1});$$
$$\psi = 0 \quad \text{on} \quad f^{c-\overline{\varepsilon}} \cup f_{c+\overline{\varepsilon}};$$
$$\psi = 1 \quad \text{on} \quad f^{c+\varepsilon_0} \cap f_{c-\varepsilon_0},$$

for some ε_0 with

$$\varepsilon < \varepsilon_0 < \min(\overline{\varepsilon}, \varepsilon_1). \tag{2.4}$$

Then we are able to construct the locally Lipschitz vector field $V : X \to X$ by setting

$$V(x) = \begin{cases} -\delta_1\varphi(x)\psi(x)v(x), & \forall x \in N_{3\delta_1}(B) \cap f_{c-3\varepsilon_1} \cap f^{c+3\varepsilon_1}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

Using (2.5) we see that the vector field V is locally Lipschitz and bounded, namely

$$\|V(x)\| \leq \delta_1, \quad x \in X. \tag{2.6}$$

From (2.2), (2.5) and (2.6) we derive

$$-\langle y^*, V(x)\rangle = \delta_1\langle y^*, v(x)\rangle \geq \frac{1}{2}\delta_1\sigma_1,\ \ \forall x \in N_{\delta_1}(B)\cap f_{c-\varepsilon_0}\cap f^{c+\varepsilon_0},\ y^* \in \partial f(x). \tag{2.7}$$

In view of (2.6) we may consider the global flow $\gamma : \mathbb{R} \times X \to X$ of V defined by (2.5), i.e.

$$\frac{d\gamma}{dt}(t, x) = V(\gamma(t, x)),\ \forall (t, x) \in \mathbb{R} \times X,$$
$$\gamma(0, x) = x,\ \forall x \in X.$$

In the next we set

$$B_1 := \gamma([0, 1]) \times B). \tag{2.8}$$

We notice that B_1 in (2.8) is a closed subset of X. To see this let $y_n = \gamma(t_n, x_n) \in B_1$ be a sequence with $t_n \in [0, 1]$, $x_n \in B$ and $y_n \to y$ in X. Passing to a subsequence we can suppose that $t_n \to t \in [0, 1]$ in $\mathbb{R}$. Putting $u_n = \gamma(t, x_n)$ we get

$$\|u_n - y_n\| = \|\gamma(t, x_n) - \gamma(t_n, x_n)\| = \|\int_{t_n}^{t} \frac{d}{dt}\gamma(\tau, x_n)d\tau\| \leq \delta_1|t_n - t|,$$

where (2.6) has been used. Since $u_n \to y$ in X, it turns out that $x_n \to \gamma(-t, y) \in B$. Finally, we obtain $y = \gamma(t, \gamma(-t, y)) \in B_1$ which establishes the closedness of B_1.

The next step is to justify that $f(\gamma(t,x))$ is a decreasing function of $t \in \mathbb{R}$, for each $x \in X$. Toward this, by applying Lebourg's mean value theorem and the chain rule for generalized gradients we infer for arbitrary real numbers $t > t_0$ the following inclusions

$$f(t,x) - f(t_0,x) \in \partial_t(f(\gamma(t,x)))\Big|_{t=\tau}$$
$$\subset \partial f(\gamma(\tau,x))\frac{d\gamma}{dt}(\tau,x)(t-t_0) = \partial f(\gamma(\tau,x))V(\gamma(\tau,x))(t-t_0)$$

with some $\tau \in (t_0,t)$, where the notation ∂_t stands for the generalized gradient with respect to t. By (2.2) and (2.5) we derive that $f(t,x) \leq f(t_0,x)$. Now we prove the relation

$$A \cap B_1 = \emptyset. \tag{2.9}$$

To check (2.9), we admit by contradiction that there exist $x_0 \in B$ and $t_0 \in [0,1]$ provided $\gamma(t_0,x_0) \in A$. Since A and B are disjoint we have necesarilly that $t_0 > 0$.

From the relations $A \subset f_{c-\varepsilon_A}$ and $B \subset f^c$ we deduce

$$c - \varepsilon_A \leq f(\gamma(t_0,x_0)) \leq f(\gamma(t,x_0)) \leq f(x_0) \leq c,\ \forall t \in [0,t_0]. \tag{2.10}$$

It turns out that

$$\gamma(t,x_0) \in N_{\delta_1}(B) \cap f^c \cap f_{c-\varepsilon_A},\quad \forall t \in [0,t_0].$$

On the other hand from (2.6) we infer the estimate

$$d(A,B) \leq \|\gamma(t_0,x_0) - x_0\| = \|\int_0^{t_0} V(\gamma(s,x_0))ds\| \leq \delta_1 t_0.$$

If we denote $h(t) = f(\gamma(t,x_0))$, then h is a locally Lipschitz function, and (2.5), (2.7) allow to write

$$h'(s) \leq \max\{\langle y^*, \frac{d\gamma}{ds}(s,x)\rangle :\ y^* \in \partial f(\gamma(s,x))\}$$
$$= \max\{\langle y^*, V(\gamma(s,x))\rangle :\ y^* \in \partial f(\gamma(s,x))\} \leq -\frac{1}{2}\delta_1\sigma_1$$

for a.e. $s \in [0,t_0]$. Therefore, by virtue of (2.3), we have the following estimate

$$f(\gamma(t_0,x_0)) - f(x_0) = h(t_0) - h(0) = \int_0^{t_0} h'(s)ds \leq$$
$$-\frac{1}{2}\delta_1\sigma_1 t_0 < -\delta_1\varepsilon t_0 \leq -\varepsilon d(A,B) \leq -\varepsilon_A. \tag{2.11}$$

The contradiction between (2.10) and (2.11) shows that the property (2.9) is actually true. Taking into account (2.9) there is a locally Lipschitz function $\psi_A : X \to \mathbb{R}$ veryfying $\psi_A = 0$ on a neighborhood of A and $\psi_A = 1$ on B_1. Then we define the homotopy $\eta_A : \mathbb{R}\times X \to X$ as being the global flow of the vector field $V_A = \psi_A V$. The assertion (i) is clear from the construction of η_A because one can take $\varphi_A = -\delta_1\psi_A\varphi\psi$. Assertion (ii) follows easily because $V_A = 0$ on $A \cup f^{c-\overline{\varepsilon}} \cup f_{c+\overline{\varepsilon}}$. We show that (iii) is valid for $\delta = c+\varepsilon-\varepsilon_0$ with ε described in (2.3) and ε_0 in (2.4). To this end we argue by contradiction. Suppose that for some $d \in [\delta, c]$ there exists $x \in B \cap f^d$ such that

$$f(\eta_A(1,x)) > d - \varepsilon. \tag{2.12}$$

Using the fact that $\psi_A = 1$ on B_1 we deduce

$$\eta_A(t,x) = \gamma(t,x) \in N_{\delta_1}(B) \cap f^d \cap f_{d-\varepsilon}, \quad \forall t \in [0,1].$$

Then a reasoning similar to the one in (2.11) can be carried out to write

$$f(\eta_A(1,x)) - f(x) \leq -\frac{1}{2}\delta_1\sigma_1 < -\varepsilon.$$

This contradicts the relation (2.12) because $f(x) \leq d$. The proof of the assertion (iii) is complete. □

In this section we present a general minimax principle for locally Lipschitz functions. This result appears in the paper of Motreanu and Varga [45].

Theorem 2.15. (Motreanu-Varga [45]) *Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional and $B \subseteq X$ a closed set such that $c := \inf_B f > -\infty$ and f satisfies $(PS)_{B,c}$. Let $\mathcal{M}$ be a nonempty family of subsets M of X such that*

$$c = \inf_{M\in\mathcal{M}} \sup_{x\in M} f(x). \tag{2.13}$$

Assume that for a generalized normalized pseudo-gradient vector field $\widehat{v}$ of f with respect to B and c the following hypothesis holds

(H) *for each set $M \in \mathcal{M}$ and each number $\varepsilon > 0$ with $f|_M < c+\varepsilon$ there exists a closed subset A of X with $f|_A \leq c+\varepsilon_A$ (see (2.1)), and $A \cap B = \emptyset$ such that for each locally Lipschitz function $\varphi_A : X \to [0,1]$ with supp $\varphi_A \subset (X \setminus A) \cap$ supp $\widehat{v}$ the global flow ξ_A of $\varphi_A\widehat{v}$ satisfies $\xi_A(1,M) \cap B \neq \emptyset$.*

Then the assertions below are true

(i) *$c = \inf_B f$ is attained;*
(ii) *$K_c(f) \setminus A \neq \emptyset$ for each set A entering* (H)*;*
(iii) *$K_c(f) \cap B \neq \emptyset$.*

Proof. The assertions (i) and (ii) are direct consequences of the property (iii). The proof of (iii) is achieved arguing by contradiction. Accordingly, we suppose $K_{-c}(-f) \cap B = \emptyset$. By hypothesis we know that $B \subset (-f)_{-c}$, so Theorem 2.14 can be applied for $-f$ and $-c$ (in place of f and c, respectively). Thus Theorem 2.14 yields an $\varepsilon > 0$ with the properties there stated. Then from the minimax description of c, by means of $\mathcal{M}$, we obtain the existence of a set $M \in \mathcal{M}$ satisfying $f|_M < c+\varepsilon$. Corresponding to M, assumption (H) allows to find a closed set $A \subset X \setminus B$ which satisfies $A \subset (-f)^{-c-\varepsilon_A}$ and the linking property formulated in (H). Theorem 2.14 gives rise to the deformation $\eta_A \in \mathcal{C}(\mathbb{R} \times X, X)$ which verifies $\eta_A(1, B \cap (-f)_{-c}) \subset (-f)_{-c-\varepsilon}$. This reads as

$$\eta_A(1,B) \subset f^{c+\varepsilon}. \tag{2.14}$$

By Theorem 2.14 and assumption (H) it is seen that

$$\xi_A(t,x) = \eta_A(-t,x), \tag{2.15}$$

for all $(t,x) \in \mathbb{R} \times X$. As shown in (H) one has the intersection property

$$\xi(1,M) \cap B \neq \emptyset.$$

Combining with (2.15) it turns out

$$\eta_A(1,B) \cap M \neq \emptyset.$$

Taking into account (2.13) we obtain the existence of some point $x_0 \in M$ with $f(x_0) \geq c+\varepsilon$. This contradicts the choice of the set M. □

Corollary 2.16. (Motreanu-Varga [45]) *Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional satisfying (PS) and let a family $\mathcal{M}$ of subsets M of X be such that c defined by (2.13) is a real number. Assume that the hypothesis below holds*

(H') *for each $M \in \mathcal{M}$ there exists a closed set A in X with $f|_A < c$ such that for every homeomorphism h of X with $h|_A = id_A$ one has $h(M) \cap f^c \neq \emptyset$.*

Then c in (2.13) is a critical value of f and $K_c(f) \cap A = \emptyset$ for every A in (H')*.*

Proof. We consider the global flow ξ_A (see (2.14)) and we apply Theorem 2.15 with $B = f^c$. It is clear that (H') implies (H) because $A \subset M \setminus B$ and $\xi_A(1,\cdot)$ is a homeomorphism of X with $\xi_A(1,\cdot) = id$ on A. Then Theorem 2.15 concludes the proof. □

Theorem 2.15 is suitable for applications to multiple linking problems.

Definition 2.17. Let Q, Q_0 be closed subsets of X, with $Q_0 \neq \emptyset$, $Q_0 \subset Q$, and let S be a subset of X such that $Q_0 \cap S = \emptyset$. We say that the pair (Q, Q_0) links with S if for each mapping $g \in \mathcal{C}(Q,X)$ with $g|_{Q_0} = id|_{Q_0}$ one has $g(Q) \cap S \neq \emptyset$.

Corollary 2.18. (Motreanu-Varga [45]) *Given the subsets Q, Q_0, S of the real Banach space X we assume that (Q,Q_0) links with S in X in the sense above. Let $f : X \to \mathbb{R}$ be a locally Lipschitz functional such that $\sup_Q f < \infty$ and, for some number $\alpha \in \mathbb{R}_+$,*

$$Q_0 \subset f_\alpha, \quad S \subset f^\alpha.$$

Then assuming that for the minimax value

$$c = \inf_{g\in\Gamma} \sup_{x\in Q} f(g(x)),$$

where

$$\Gamma = \{g \in \mathcal{C}(Q, X) : g|_{Q_0} = id|_{Q_0}\},$$

$(PS)_{S,c}$ *is satisfied, the following properties hold*

(i) $c \geq \alpha$;
(ii) $K_c(f) \setminus Q_0 \neq \emptyset$;
(iii) $K_c(f) \cap S \neq \emptyset$ *if* $c = \alpha$.

Proof. Since the case $\alpha < c$ follows immediately we discuss only the situation where $\alpha = c$. The conclusion is readily obtained from Theorem 2.15 by choosing $\mathcal{M} = \{g(Q) : g \in \Gamma\}$ and $B = S$. □

A direct consequence of this corollary is the following.

Corollary 2.19. (Mountain pass theorem; zero altitude) *Let* $f : X \to \mathbb{R}$ *be a locally Lipschitz function on a Banach space satisfying* $(PS)_c$ *for every* $c \in \mathbb{R}$ *and the conditions:*

(i) $f(x) \geq \alpha \geq f(0)$ *for all* $||x|| = \rho$ *where* α *and* $\rho > 0$ *are constants;*
(ii) *there is* $e \in X$ *with* $||e|| > \rho$ *and* $f(e) \leq \alpha$.

Then the number

$$c = \inf_{g\in\Gamma} \max_{u\in[0,e]} f(g(u)),$$

where $[0, e]$ *is the closed line segment in* X *joining* 0 *and* e *and*

$$\Gamma = \{g \in \mathcal{C}([0, e], X) \ : \ g(0) = 0, \quad g(e) = e\},$$

is a critical value of f *with* $c \geq \alpha$.

Proof. It is sufficient to take in Corollary 2.18 the following choices $Q = [0, e], Q_0 = \{0, e\}$ and $S = \{x \in X \ : \ ||x|| = \rho \ \}$. □

A direct consequence of the above corollary is locally Lipschitz version of Pucci-Serrin Mountain Pass theorem, see [52].

Theorem 2.20. *Let* X *be a Banach space,* $h : X \to \mathbb{R}$ *a locally Lipschitz functional, satisfying the Palais-Smale condition,* x *and* y *two local minima of* h*. Then,* h *has a critical point in* X *different from* x *and* y*.*

In the next we prove a common generalization of some results of Chang [9] and Kourogenis-Papageorgiou [23]. For this see the paper of Kristály-Motreanu-Varga [33]. Let us consider $f : X \to \mathbb{R}$ to be a locally Lipschitz function.

Definition 2.21. We say that f satisfies the (C)-condition at level c (in short $(C)_c$) if every sequence $\{x_n\} \subset X$ such that $f(x_n) \to c$ and $(1 + \|x_n\|)\lambda_f(x_n) \to 0$ has a convergent subsequence.

It is clear that $(PS)_c$ implies $(C)_c$. Our approach is based on the following idea. We consider a globally Lipschitz functional $\varphi : X \to \mathbb{R}$ such that $\varphi(x) \geq 1, \forall x \in X$ (or, $\varphi(x) \geq \alpha$, for some $\alpha > 0$).

Definition 2.22. We say that the function f satisfies the $(\varphi - C)$-condition at level c (in short, $(\varphi - C)_c$) if every sequence $\{x_n\} \subset X$ such that $f(x_n) \to c$ and $\varphi(x_n)\lambda_f(x_n) \to 0$ has a convergent subsequence.

The $(\varphi - C)_c$-condition contains the $(PS)_c$ and $(C)_c$ compactness conditions, respectively. Indeed if $\varphi \equiv 1$ we get the $(PS)_c$-condition and if $\varphi(x) = 1 + \|x\|$ we have the $(C)_c$-condition.

We need the following result in order to obtain the existence of a suitable locally Lipschitz vector field.

Lemma 2.23. (Kristály-Motreanu-Varga [33]) *Let X be a Banach space and let $f : X \to \mathbb{R}$ be a locally Lipschitz function satisfying the $(\varphi - C)_c$-condition, where $\varphi : X \to \mathbb{R}$ is a globally Lipschitz function such that $\varphi(x) \geq 1, \quad \forall x \in X$. Then for each $\delta > 0$ there exist constants $\gamma, \varepsilon > 0$ and a locally Lipschitz vector field*

$$v : f^{-1}([c - \varepsilon, c + \varepsilon]) \cap (K_c)^c_\delta \to X$$

such that for each $x \in f^{-1}([c - \varepsilon, c + \varepsilon]) \cap (K_c)^c_\delta$ one has

$$||v(x)|| \leq \varphi(x) \tag{2.16}$$

$$\langle y^*, v(x)\rangle \geq \frac{\gamma}{2} \text{ for all } y^* \in \partial f(x). \tag{2.17}$$

In the sequel we shall prove a very general deformation result which unifies several results of this kind it appears in the paper of Kristály, Motreanu and Varga [33].

Theorem 2.24. (Kristály-Motreanu-Varga [33]) *Let $f : X \to \mathbb{R}$ be a locally Lipschitz function on the Banach space X satisfying the $(\varphi - C)_c$-condition, with $c \in \mathbb{R}$ and a globally Lipschitz function $\varphi : X \to \mathbb{R}$ with Lipschitz constant $L > 0$ and $\varphi(x) \geq 1$, $\forall\ x \in X$. Then for every $\varepsilon_0 > 0$ and every neighborhood U of K_c (if $K_c = \emptyset$, then we choose $U = \emptyset$) there exist a number $0 < \varepsilon < \varepsilon_0$ and a continuous function $\eta : X \times [0, 1] \to X$, such that for every $(x, t) \in X \times [0, 1]$ we have:*

(a) $\|\eta(x, t) - x\| \leq \varphi(x)te^{Lt}$*;*

(b) $\eta(x, t) = x$ *for every* $x \notin f^{-1}([c - \varepsilon_0, c + \varepsilon_0])$ *and* $t \in [0, 1]$*;*

(c) $f(\eta(x, t)) \leq f(x)$*;*

(d) $\eta(x,t) \neq x \Rightarrow f(\eta(x,t)) < f(x)$.
(e) $\eta(f^{c+\varepsilon},1) \subset f^{c-\varepsilon} \cup U$;
(f) $\eta(f^{c+\varepsilon} \setminus U, 1) \subset f^{c-\varepsilon}$.

Proof. Fix $\varepsilon_0 > 0$ and a neighborhood U of K_c. From the compactness of K_c we can find $\delta > 0$ such that $(K_c)_{3\delta} \subseteq U$. Moreover, the proof of Lemma 2.23 guarantees the existence of $\gamma > 0$ and $0 < \overline{\varepsilon} < \varepsilon_0$ such that $\varphi(x)\lambda_f(x) \geq \gamma$ for all $x \in f^{-1}([c-\overline{\varepsilon}, c+\overline{\varepsilon}]) \cap (K_c)^c_\delta$. We consider the following two closed sets:

$$A = \{x \in X : |f(x) - c| \geq \overline{\varepsilon}\} \cup \overline{(K_c)_\delta} \tag{2.18}$$

$$B = \{x \in X : |f(x) - c| \leq \frac{\overline{\varepsilon}}{2}\} \cap (K_c)^c_{2\delta}. \tag{2.19}$$

Because $A \cap B = \emptyset$ there exists a locally Lipschitz function $\psi : X \to [0,1]$ such that $\psi = 0$ on a closed neighborhood of A, say $\tilde{A}$, disjoint of B, $\psi|_B = 1$ and $0 \leq \psi \leq 1$. For instance, we can take $\psi(x) = \dfrac{d(x,\tilde{A})}{d(x,\tilde{A}) + d(x,B)}$, $\forall\, x \in X$.

Let $V : X \to X$ be defined by

$$V(x) = \begin{cases} -\psi(x)\cdot v(x), & x \in f^{-1}([c-\overline{\varepsilon}, c+\overline{\varepsilon}]) \cap (K_c)^c_\delta; \\ 0, & \text{otherwise}, \end{cases} \tag{2.20}$$

where $v(x)$ is constructed in Lemma 2.23. The vector field V is locally Lipschitz and by the same lemma, for $x \in f^{-1}([c-\overline{\varepsilon}, c+\overline{\varepsilon}]) \cap (K_c)^c_\delta$ we have

$$\|V(x)\| = \psi(x)\cdot\|v(x)\| \leq \varphi(x) \tag{2.21}$$

$$\langle y^*, V(x)\rangle = -\psi(x)\cdot\langle y^*, v(x)\rangle \leq -\psi(x)\frac{\gamma}{2},\ \forall y^* \in \partial f(x). \tag{2.22}$$

Since V is locally Lipschitz and $\|V(x)\| \leq \varphi(0) + L\|x\|$, the following Cauchy problem:

$$\begin{cases} \dot{\eta}(x,t) = V(\eta(x,t)) & \text{a.e. on } [0,1] \\ \eta(x,0) = x \end{cases} \tag{2.23}$$

has a unique solution $\eta(x,\cdot)$ on $\mathbb{R}$, for each $x \in X$. By (2.21) we have that:

$$\|\eta(x,t) - x\| \leq \int_0^t \|V(\eta(x,s))\|ds \leq \int_0^t \varphi(\eta(x,s))ds =$$

$$= \int_0^t [\varphi(\eta(x,s)) - \varphi(x)]ds + \int_0^t \varphi(x)ds \leq$$

$$\leq L\cdot\int_0^t \|\eta(x,s) - x\|ds + \varphi(x)t.$$

Using Gronwall's inequality we get $\|\eta(x,t) - x\| \leq \varphi(x)t\cdot e^{Lt}$, therefore the assertion (a) is proved. If $x \notin f^{-1}([c-\overline{\varepsilon}, c+\overline{\varepsilon}])$, then $x \in A$, so $\psi(x) = 0$. By (2.20) it follows that $V(x) = 0$ and from (2.23) we obtain that $\eta(x,t) = x$, for each $t \in [0,1]$. This yields (b).

Next, for a fixed $x \in X$, let us consider the function $h_x : [0,1] \to \mathbb{R}$ given by $h_x(t) = f(\eta(x,t))$. Using the chain rule we have

$$\frac{d}{dt}h_x(t) \leq \max\Big\{\langle y^*, \frac{d}{dt}\eta(x,t)\rangle : y^* \in \partial f(\eta(x,t))\Big\} =$$

$$= \max\Big\{\langle y^*, V(\eta(x,t))\rangle : y^* \in \partial f(\eta(x,t))\Big\} \text{ a.e. on } [0,1].$$

Therefore, taking into account (2.22), we infer

$$\frac{d}{dt}h_x(t) \leq -\psi(\eta(x,t))\frac{\gamma}{2} \leq 0 \text{ if } \eta(x,t) \in f^{-1}([c-\bar{\varepsilon}, c+\bar{\varepsilon}]) \cap (K_c)^c_\delta, \tag{2.24}$$

and clearly, by (2.20)

$$\frac{d}{dt}h_x(t) \leq 0, \quad \text{if} \quad \eta(x,t) \notin f^{-1}([c-\bar{\varepsilon}, c+\bar{\varepsilon}]) \cap (K_c)^c_\delta.$$

Hence property (c) holds true.

In order to prove property (d), suppose that $\eta(x,t) \neq x$. First, we show that

$$\eta(x,s) \in f^{-1}([c-\bar{\varepsilon}, c+\bar{\varepsilon}]) \cap (K_c)^c_\delta\,, \quad \forall s \in [0,t]. \tag{2.25}$$

On the contrary, there would exist $s_0 \in [0,t]$ such that $\eta(x,s_0) \in A$. This implies that $V(\eta(x,s_0)) = 0$. Using the uniqueness of solution to the Cauchy problem formed by the equation in (2.23) and the initial condition with the initial value $\eta(x,s_0)$, we see that

$$\eta(x,\tau+s_0) = \eta(x,s_0), \quad \forall \tau \in \mathbb{R}.$$

Letting $\tau = t - s_0$ and $\tau = -s_0$ one obtains $\eta(x,t) = x$, which contradicts our assumption. Thus the claim in (2.25) is true.

Using (2.24) and (2.25) it follows that

$$f(x) - f(\eta(x,t)) = -\int_0^t \frac{d}{ds}h_x(s)\,ds \geq \frac{\gamma}{2}\int_0^t \psi(\eta(x,s))ds. \tag{2.26}$$

We show that there is $s \in [0,t]$ such that

$$\psi(\eta(x,s)) \neq 0. \tag{2.27}$$

For, otherwise, if $\psi(\eta(x,s)) = 0$, $\forall s \in [0,t]$, then $V(\eta(x,s)) = 0$, $\forall s \in [0,t]$. By (2.23), we get that $\eta(x,\cdot)$ is constant on $[0,t]$, which contradicts $\eta(x,t) \neq x$. It results that (2.27) is valid. Since $\psi \geq 0$, from (2.26) and (2.27) we infer that $f(\eta(x,t)) < f(x)$, which proves assertion (d).

We show now assertion (e). Let $\rho > 0$ such that $\overline{(K_c)_{3\delta}} \subset B(0,\rho)$. We choose

$$0 < \varepsilon \leq \min\left\{\frac{\bar{\varepsilon}}{2}, \frac{\gamma}{4}, \frac{\delta\gamma}{8}\mathrm{e}^{-\mathrm{L}}(\varphi(0) + \mathrm{L}\rho)^{-1}\right\}. \tag{2.28}$$

We argue by contradiction. Let $x \in f^{c+\varepsilon}$ such that $f(\eta(x,1)) > c-\varepsilon$ and $\eta(x,1) \notin U$. Since, by (c), $f(\eta(x,t)) \leq f(x) \leq c+\varepsilon$ and $f(\eta(x,t)) \geq f(\eta(x,1))$ for each $t \in [0,1]$, we get

$$c-\varepsilon < f(\eta(x,t)) \leq c+\varepsilon, \quad \forall t \in [0,1]. \tag{2.29}$$

We claim that

$$\eta(\{x\} \times [0,1]) \cap (K_c)_{2\delta} \neq \emptyset. \tag{2.30}$$

Suppose that (2.30) does not hold. This means that

$$\eta(\{x\} \times [0,1]) \cap (K_c)_{2\delta} = \emptyset. \tag{2.31}$$

First, we show that

$$\eta(x,t) \in B, \quad \forall t \in [0,1]. \tag{2.32}$$

The fact that $\eta(x,t) \in f^{-1}([c-\frac{\bar{\varepsilon}}{2}, c+\frac{\bar{\varepsilon}}{2}])$ follows from (2.28) and (2.29). By (2.31) one has that $\eta(x,t) \in (K_c)^c_{2\delta}$. Consequently, from (2.19) we conclude that (2.32) is established. On the basis of (2.32) and (2.24) we may write

$$f(x) - f(\eta(x,1)) = h_x(0) - h_x(1) = -\int_0^1 \frac{d}{dt} h_x(t) dt \geq \int_0^1 \frac{\gamma}{2} \psi(\eta(x,t)) dt.$$

Then, combining (2.32) and the definition of ψ it is clear that

$$f(x) - f(\eta(x,1)) \geq \frac{\gamma}{2}. \tag{2.33}$$

On the other hand, from (2.29) we obtain that

$$f(x) - f(\eta(x,1)) < 2\varepsilon. \tag{2.34}$$

From (2.33) and (2.34) we get $\frac{\gamma}{2} < 2\varepsilon$, which contradicts (2.28). This justifies (2.30).

The next step in the proof is to show that there exist $0 \leq t_1 < t_2 \leq 1$ such that

$$\mathrm{dist}(\eta(x,t_1), K_c) = 2\delta, \quad \mathrm{dist}(\eta(x,t_2), K_c) = 3\delta \tag{2.35}$$

and

$$2\delta < \mathrm{dist}(\eta(x,t), K_c) < 3\delta, \quad \forall t_1 < t < t_2. \tag{2.36}$$

Denote $g(t) = \mathrm{dist}(\eta(x,t), K_c)$, $\forall t \in [0,1]$. In view of (2.30) we have that $\{t \in [0,1] : g(t) \leq 2\delta\} \neq \emptyset$. Thus it is permitted to consider

$$t_1 = \sup\{t \in [0,1] \ : \ g(t) \leq 2\delta\}.$$

Since it is known that $(K_c)_{3\delta} \subset U$ and $\eta(x,1) \notin U$, we derive that $\eta(x,1) \notin (K_c)_{3\delta}$. This means that $g(1) \geq 3\delta$. Since $g(t_1) \leq 2\delta$ it is necessary to have $t_1 < 1$. The definition of t_1 implies $g(t) > 2\delta$ for all $t \in (t_1, 1]$ (which is the first inequality in (2.36)). Letting $t \downarrow t_1$ we deduce that $g(t_1) \geq 2\delta$. We obtain that $g(t_1) = 2\delta$, so the first part in (2.35) is proved. Taking into account that $g(1) \geq 3\delta$, we see that $\{t \in [t_1, 1] \,:\, g(t) \geq 3\delta\}$ is nonempty. Then we can define

$$t_2 = \inf\{t \in [t_1, 1] \,:\, g(t) \geq 3\delta\}.$$

Since $g(t_2) \geq 3\delta$ and $g(t_1) = 2\delta$ it is clear that $t_1 < t_2$. By the definition of t_2 we have that $g(t) < 3\delta$ for all $t_1 \leq t < t_2$, so (2.36) holds. In addition, letting $t \uparrow t_2$, we get $g(t_2) = 3\delta$, so (2.35) holds, too.

Let us show that

$$t_2 - t_1 < \frac{4\varepsilon}{\gamma}\,. \tag{2.37}$$

From (2.36) it follows that $\eta(x,t) \notin (K_c)_{2\delta}$, $\forall t \in [t_1, t_2]$, while (2.29) and (2.28) imply $\eta(x,t) \in f^{-1}([c - \frac{\bar{\varepsilon}}{2}, c + \frac{\bar{\varepsilon}}{2}])$, $\forall t \in [t_1, t_2]$. The definition of the set B in (2.19) yields

$$\eta(x,t) \in B, \quad \forall t \in [t_1, t_2].$$

Using the definition of ψ, (2.24) and (2.29) we see that

$$\begin{aligned}\frac{\gamma}{2}(t_2 - t_1) &= \frac{\gamma}{2}\int_{t_1}^{t_2} \psi(\eta(x,t))dt \leq -\int_{t_1}^{t_2} \frac{d}{dt}h_x(t)dt \\ &= h_x(t_1) - h_x(t_2) = f(\eta(x,t_1)) - f(\eta(x,t_2)) < 2\varepsilon.\end{aligned}$$

Therefore (2.37) is proved.

We need the following inequality

$$\|\eta(x,t_2) - \eta(x,t_1)\| \geq \delta. \tag{2.38}$$

To check (2.38) consider a point $v \in K_c$ so that

$$\mathrm{dist}(\eta(x,t_1), K_c) = \|\eta(x,t_1) - v\| = 2\delta.$$

Here the compactness of K_c and the first part in (2.35) have been used. Then, on the basis of the second part in (2.35) we can write

$$\|\eta(x,t_2) - \eta(x,t_1)\| \geq \|\eta(x,t_2) - v\| - \|\eta(x,t_1) - v\| \geq 3\delta - 2\delta = \delta.$$

Therefore (2.38) holds.

Using (2.23), (2.21) and the Lipschtzianess of φ we can write

$$\begin{aligned}\|\eta(x,t_2) - \eta(x,t_1)\| &\leq \int_{t_1}^{t_2} \|V(\eta(x,s))\|ds \leq \int_{t_1}^{t_2} \varphi(\eta(x,s))ds \\ &= \int_{t_1}^{t_2} [\varphi(\eta(x,s)) - \varphi(\eta(x,t_1))]ds + \varphi(\eta(x,t_1))(t_2 - t_1) \\ &\leq \int_{t_1}^{t_2} L\|\eta(x,s) - \eta(x,t_1)\|ds + \varphi(\eta(x,t_1))(t_2 - t_1).\end{aligned} \tag{2.39}$$

By (2.39) and Gronwall's inequality we get

$$\|\eta(x,t_2)-\eta(x,t_1)\| \le \varphi(\eta(x,t_1))(t_2-t_1)e^{L(t_2-t_1)}. \tag{2.40}$$

From (2.38), (2.40), (2.37) and the Lipschtzianess of φ we deduce that

$$\begin{aligned}\delta \le \|\eta(x,t_2)-\eta(x,t_1)\| &< \frac{4\varepsilon}{\gamma}e^L\varphi(\eta(x,t_1))\\ &\le \frac{4\varepsilon}{\gamma}e^L(\varphi(0)+L\|\eta(x,t_1)\|).\end{aligned} \tag{2.41}$$

In view of (2.35) and the choice of ρ to satisfy $\overline{(K_c)_{3\delta}} \subset B(0,\rho)$ we have $\eta(x,t_1) \in (K_c)_{3\delta} \subset B(0,\rho)$. This property and (2.28) yield from (2.41) that

$$\delta \le \frac{4\varepsilon}{\gamma}e^L(\varphi(0)+L\rho) \le \frac{\delta}{2}\,,$$

which is a contradiction. This proves (e).

In order to show (f), since $(K_c)_{3\delta} \subset U$ it is enough to prove that

$$\eta(f^{c+\varepsilon}\setminus (K_c)_{3\delta},1) \subset f^{c-\varepsilon}. \tag{2.42}$$

Let us denote

$$C=(f^{c+\varepsilon}\setminus f^{c-\varepsilon})\cap (K_c)^c_{3\delta}\,.$$

To check (2.42), we note that it is sufficient to verify that

$$\eta(x,1)\in f^{c-\varepsilon}, \quad \forall x\in C, \tag{2.43}$$

because for $x \in f^{c-\varepsilon}$ we have $f(\eta(x,1)) \le f(x) \le c-\varepsilon$, due to the nondecreasing monotonicity of $f(\eta(x,\cdot))$.

To show (2.43), denote

$$D=(f^{c+\varepsilon}\setminus f^{c-\varepsilon})\cap (K_c)^c_{\frac{5}{2}\delta}\,.$$

First, we verify that

$$\forall x\in C,\ \exists\, t_x\in (0,\frac{4\varepsilon}{\gamma}] \ \text{ such that } \ \eta(x,t_x)\notin D. \tag{2.44}$$

To this end, we prove the inclusion below

$$\{t>0\ :\ \eta(x,\tau)\in D,\ \forall \tau\in[0,t]\}\subset (0,\frac{4\varepsilon}{\gamma}),\ \forall x\in C. \tag{2.45}$$

Indeed, if $\eta(x,\tau)$ is in $D \subset B$, $\forall \tau \in [0,t]$, we have $\psi(\eta(x,\tau)) = 1$, $\forall \tau \in [0,t]$. Therefore, by (2.24), we have $\frac{d}{d\tau}h_x(\tau) \le -\frac{\gamma}{2}$, $\forall \tau \in [0,t]$. From this and (2.29) we obtain

$$2\varepsilon > h_x(0)-h_x(t) = -\int_0^t \frac{d}{d\tau}h_x(\tau)d\tau \ge \frac{\gamma}{2}t,$$

so $t < \frac{4\varepsilon}{\gamma}$. Thus (2.45) is satisfied.

We are now in the position to prove (2.44). We proceed arguing by contradiction. Assuming that there exist $x \in C$ such that $\eta(x,t) \in D$, $\forall t \in (0, \frac{4\varepsilon}{\gamma}]$, by (2.45), we arrive at the contradiction

$$\frac{4\varepsilon}{\gamma} \in \{t > 0 \, : \, \eta(x,\tau) \in D, \; \forall \tau \in [0,t]\} \subset (0, \frac{4\varepsilon}{\gamma}),$$

which proves (2.44).

Let us show that for every $x \in C$, it is true that

$$\eta(\{x\} \times [0,1]) \cap (K_c)_{\frac{5}{2}\delta} \neq \emptyset \Rightarrow \exists\, t_0 \in (0, t_3] \; \text{ such that } \; \eta(x,t_0) \in f^{c-\varepsilon}, \tag{2.46}$$

with

$$t_3 = \inf\{t \in [0,1] : \operatorname{dist}(\eta(x,t), K_c) \leq \frac{5}{2}\delta\},$$

where the set $\{t \in [0,1] : \; \operatorname{dist}(\eta(x,t), K_c) \leq \frac{5}{2}\delta\}$ is nonempty in view of (2.36). If (2.46) were not true it would exist $x \in C$ with $\eta(\{x\} \times [0,1]) \cap (K_c)_{\frac{5}{2}\delta} \neq \emptyset$ and $f(\eta(x,t)) > c - \varepsilon$, $\forall t \in [0, t_3]$. Hence $\eta(x,t) \in D$, $\forall t \in [0,t_3]$. This follows from the definition of t_3 and since $x \in C$. The inclusion in (2.45) implies that

$$t_3 < \frac{4\varepsilon}{\gamma}. \tag{2.47}$$

Introduce

$$t_4 = \sup\{t \in [0, t_3] : \; \operatorname{dist}(\eta(x,t), K_c) \geq 3\delta\}.$$

Since $x \in C$, then $x \in (K_c)^c_{3\delta}$, thus the set $\{t \in [0,t_3] : \; \operatorname{dist}(\eta(x,t), K_c) \geq 3\delta\}$ is nonempty. By the definitions of t_3 and t_4 it follows that

$$\eta(x,t) \in (f^{c+\varepsilon} \setminus f^{c-\varepsilon}) \cap ((K_c)_{3\delta} \setminus (K_c)_{\frac{5}{2}\delta}), \; \forall t \in [t_4, t_3].$$

We remark that

$$\|\eta(x,t_3) - \eta(x,t_4)\| \geq \frac{\delta}{2}\,. \tag{2.48}$$

Indeed, by the definition of t_4 we have

$$\begin{aligned} \|\eta(x,t_3) - \eta(x,t_4)\| &\geq \|\eta(x,t_4) - v\| - \|\eta(x,t_3) - v\| \\ &\geq 3\delta - \|\eta(x,t_3) - v\|, \quad \forall v \in K_c. \end{aligned}$$

This leads to

$$\|\eta(x,t_3) - \eta(x,t_4)\| \geq 3\delta - \operatorname{dist}(\eta(x,t_3), K_c) = 3\delta - \frac{5}{2}\delta = \frac{\delta}{2}\,,$$

so (2.48) is verified.

Using (2.23), (2.21) and the Lipschtzianess of φ we can write

$$\|\eta(x,t_3) - \eta(x,t_4)\| \leq \int_{t_4}^{t_3} \|V(\eta(x,s))\| ds \leq \int_{t_4}^{t_3} \varphi(\eta(x,s)) ds$$

$$= \int_{t_4}^{t_3} [\varphi(\eta(x,s)) - \varphi(\eta(x,t_4))]ds + \varphi(\eta(x,t_4))(t_3 - t_4)$$

$$\leq \int_{t_4}^{t_3} L\|\eta(x,s) - \eta(x,t_4)\|ds + \varphi(\eta(x,t_4))(t_3 - t_4).$$

By Gronwall's inequality we get

$$\|\eta(x,t_3) - \eta(x,t_4)\| \leq \varphi(\eta(x,t_4))(t_3 - t_4)e^{L(t_3-t_4)}. \tag{2.49}$$

Using (2.48), (2.49), the Lipschitzianess of φ, the inclusion $\overline{(K_c)_{3\delta}} \subset B(0,\rho)$ and (2.47), we have that

$$\frac{\delta}{2} \leq \|\eta(x,t_3) - \eta(x,t_4)\| \leq e^{L(t_3-t_4)}\varphi(\eta(x,t_4))(t_3 - t_4)$$

$$\leq e^L(\varphi(0) + L\|\eta(x,t_4)\|)t_3 < e^L(\varphi(0) + L\rho)\frac{4\varepsilon}{\gamma}\,.$$

This contradicts the choice of ε in (2.28), therefore (2.46) is true.

In order to complete the proof of (f), let $x \in C$. From (2.44), there exists $t_x \in (0, \frac{4\varepsilon}{\gamma}]$ such that $\eta(x,t_x) \notin D$. This means that

$$\eta(x,t_x) \in (X \setminus f^{c+\varepsilon}) \cup f^{c-\varepsilon} \cup (K_c)_{\frac{5}{2}\delta}\,.$$

On the other hand, $\eta(x,t_x) \in f^{c+\varepsilon}$ since, as $x \in C$, $f(\eta(x,t_x)) \leq f(x) \leq c + \varepsilon$. Consequently, we deduce that $\eta(x,t_x) \in f^{c-\varepsilon} \cup (K_c)_{\frac{5}{2}\delta}$. Two cases arise:

1) $\eta(x,t_x) \in f^{c-\varepsilon}$;

2) $\eta(x,t_x) \in (K_c)_{\frac{5}{2}\delta}$.

In case 1) we have directly that

$$f(\eta(x,1)) \leq f(\eta(x,t_x)) \leq c - \varepsilon,$$

which ensures the desired conclusion.

It remains to treat case 2). In this situation, we make use of property (2.46). Therefore, we find $t_0 \in (0,t_3]$ such that $\eta(x,t_0) \in f^{c-\varepsilon}$. Thus we may write $f(\eta(x,1)) \leq f(\eta(x,t_0)) \leq c - \varepsilon$. The proof is complete. □

Remark 2.25. If we choose $\varphi(x) = 1$ or $\varphi(x) = 1 + \|x\|$ then we obtain the deformation lemmas of Chang [9] and Kourogenis-Papageorgiou [24], respectively.

In the next we present a a general linking type result for locally Lipschitz functions which satisfy the generalized $(\varphi - C)_c$ condition. Let X be a Banach space and $A, C \subseteq X$ two sets.

Definition 2.26. We say that C links A, if $A \cap C = \emptyset$, and C is not contractible in $X \setminus A$.

Theorem 2.27. (Kristály-Motreanu-Varga [33]) *If $A, C \subseteq X$ are nonempty, A is closed, C links A, Γ_C is the set of all contractions of C, and $f : X \to \mathbb{R}$ is a locally Lipschitz which satisfies the $(\varphi - C)_c$-condition with*

$$c = \inf_{h \in \Gamma_C} \sup_{[0,1]\times C} f \circ h < \infty \quad \text{and} \quad \sup_{x \in C} f(x) \leq \inf_{x \in A} f(x),$$

then $c \geq \inf_{x \in A} f(x)$ and c is a critical value of f. Moreover, if $c = \inf_{x \in A} f(x)$, then there exists $x \in A$ such that $x \in K_c$.

Proof. Since by hypothesis C links A, for every $h \in \Gamma_C$ we have $h([0,1] \times C) \neq \emptyset$. So we infer that $c \geq \inf_{x \in A} f(x)$.

First we assume that $\inf_{x \in A} f(x) < c$. Suppose that $K_c = \emptyset$. Let $U = \emptyset$ and let $\varepsilon > 0$ and $\eta : [0,1] \times X \to X$ be as in Theorem 2.24. Also from the definition of c, we can find $h \in \Gamma_C$ such that

$$f(h(t,x)) \leq c + \varepsilon \text{ for all } t \in [0,1] \text{ and } x \in C. \tag{2.50}$$

Let $H : [0,1] \times C \to X$ defined by

$$H(t,x) = \begin{cases} \eta(2t, x), & \text{if} \quad 0 \leq t \leq \dfrac{1}{2} \\ \\ \eta(1, h(2t-1, x)), & \text{if} \quad \dfrac{1}{2} \leq t \leq 1. \end{cases}$$

It is easy to check that $H \in \Gamma_C$ and from d) and c) of Theorem 2.24 we obtain that for every $x \in C$ we have

$$f(H(t,x)) = f(\eta(2t,x)) \leq f(x) \leq \sup_{x \in C} f(x) < c, \text{ if } t \in \left[0, \frac{1}{2}\right]$$

$$f(H(t,x)) = f(\eta(1, h(2t-1,x))) \leq c - \varepsilon < c, \text{ if } t \in \left[\frac{1}{2}, 1\right]$$

and from (2.50) we get

$$h(t,x) \in f^{c+\varepsilon} \text{ for every } t \in [0,1].$$

So we have contradicted the definition of c. This proves that $K_c \neq \emptyset$, when $c > \inf_{x \in A} f(x)$.

Next assume that $c = \inf_{x \in A} f(x)$. We need to show that $K_c \cap A \neq \emptyset$. Suppose the contrary and let U be a neighborhood of K_c with $U \cap A = \emptyset$. Let $\varepsilon > 0$ and $\eta : [0,1] \times X \to X$ be as in Theorem 2.24. As before let $h \in \Gamma_C$ such that $f(h(t,x)) \leq$

$c+\varepsilon$ for all $(t,x)\in[0,1]\times C$. Then we define $H:[0,1]\times C\to X$ by

$$H(t,x)=\begin{cases}\eta(2t,x), & \text{if} \quad 0\le t\le \dfrac{1}{2}\\[2ex] \eta(1,h(2t-1,x)), & \text{if} \quad \dfrac{1}{2}\le t\le 1.\end{cases}$$

Again, we have $H\in\Gamma_C$. From Theorem 2.24 follows that for all $0\le t\le \dfrac{1}{2}$ and all $x\in C$, we have

$$\eta(2t,x)=x \text{ or } f(\eta(2t,x))<f(x)\le \inf_{x\in A} f(x)=c$$

which implies

$$\eta(2t,x)\in C_X A \text{ for all } t\in\left[0,\frac{1}{2}\right] \text{ and all } x\in C.$$

For all $t\in\left[\dfrac{1}{2},1\right]$ and all $x\in C$, we have from d) Theorem 2.24

$$\eta(1,h(2t-1,x))\subseteq f^{c-\varepsilon}\cup U$$

while $(f^{c-\varepsilon}\cup U)\cap A=\emptyset$.

So H is a contraction of C in $X\setminus A$, which is a contradiction. This completely proves the theorem. □

In the next we prove a variant of Mountain Pass Theorem.

Theorem 2.28. (Kristály-Motreanu-Varga [33]) *Let X be a Banach space, $f:X\to\mathbb{R}$ be a locally Lipschitz function and $\varphi:X\to\mathbb{R}$ a globally Lipschitz function such that $\varphi(x)\ge 1$, $\forall\ x\in X$. Suppose that there exist $x_1\in X$ and $r>0$ such that $\|x_1\|>r$ and*

(i) $\max\{f(0),f(x_1)\}\le\inf\{f(x):\ \|x\|=r\}$

(ii) the function f satisfies the $(\varphi-C)_c$-condition ($c\in\mathbb{R}$), where

$$c=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]} f(\gamma(t)),$$

with

$$\Gamma=\{\gamma\in C([0,1],X):\ \gamma(0)=0,\ \gamma(1)=x_1\}.$$

Then the minimax value c in (ii) is a critical value of f. Moreover, if $c=\inf\{f(x):\ \|x\|=r\}$, there exist a critical point x of f with $f(x)=c$ and $\|x\|=r$.

Proof. We will apply Theorem 2.27 with $A=\{x\in X:\|x\|=r\}$ and $C=\{0,x_1\}$. Clearly C links A and $c<\infty$. Let $\gamma\in\Gamma$ and define

$$h(t,x)=\begin{cases}\gamma(t), & \text{if} \quad x=0\\ x_1, & \text{if} \quad x=x_1\end{cases}$$

Then $h \in \Gamma_C$. Therefore

$$\inf_{\overline{h}\in\Gamma_C} \sup_{[0,1]\times C} f(\overline{h}(t,x)) \leq f(h(t,x)) \leq c. \tag{2.51}$$

On the other hand, if $h \in \Gamma_C$, then

$$\gamma(t) = \begin{cases} h(2t,0), & \text{if} \quad t \in \left[0, \frac{1}{2}\right] \\ \\ h(2-2t,x_1), & \text{if} \quad t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

belongs to Γ and so

$$\inf_{h\in\Gamma_C} \sup_{[0,1]\times C} f(h(t,x)) \geq c. \tag{2.52}$$

By (2.51) and (2.52) we have

$$c = \inf_{h\in\Gamma_C} \sup_{[0,1]\times C} f(h(t,x))$$

and so we can apply Theorem 2.27 and finish the proof. $\square$

2.2. **Multiple critical points results.** In this subsection we present a generalization of the three critical points theorem of Ricceri [58] to locally Lipschitz functions which appears in the paper of Kristály-Marzantowicz-Varga [28]. To do this, we first recall a topological result of Ricceri [59].

Theorem 2.29. (Ricceri [59, Theorem 4]) *Let X be a real, reflexive Banach space, let $\Lambda \subseteq \mathbb{R}$ be an interval, and let $\varphi : X \times \Lambda \to \mathbb{R}$ be a function satisfying the following conditions:*

1. *$\varphi(x, \cdot)$ is concave in Λ for all $x \in X$;*
2. *$\varphi(\cdot, \lambda)$ is continuous, coercive and sequentially weakly lower semicontinuous in X for all $\lambda \in \Lambda$;*
3. $\beta_1 := \sup\limits_{\lambda\in\Lambda} \inf\limits_{x\in X} \varphi(x,\lambda) < \inf\limits_{x\in X} \sup\limits_{\lambda\in\Lambda} \varphi(x,\lambda) =: \beta_2$.

Then, for each $\sigma > \beta_1$ there exists a non-empty open set $\Lambda_0 \subset \Lambda$ with the following property: for every $\lambda \in \Lambda_0$ and every sequentially weakly lower semicontinuous function $\Phi : X \to \mathbb{R}$, there exists $\mu_0 > 0$ such that, for each $\mu \in]0, \mu_0[$, the function $\varphi(\cdot, \lambda) + \mu\Phi(\cdot)$ has at least two local minima lying in the set $\{x \in X : \varphi(x, \lambda) < \sigma\}$.

The main result of this subsection is the following.

Theorem 2.30. (Kristály-Marzantowicz-Varga [28]) *Let $(X, \|\cdot\|)$ be a real reflexive Banach space and $\tilde{X}_i$ $(i = 1, 2)$ be two Banach spaces such that the embeddings $X \hookrightarrow \tilde{X}_i$ are compact. Let Λ be a real interval, $h : [0, \infty) \to [0, \infty)$ be a non-decreasing convex function, and let $\Phi_i : \tilde{X}_i \to \mathbb{R}$ $(i = 1, 2)$ be two locally Lipschitz functions such* *that* $E_{\lambda,\mu} = h(\|\cdot\|) + \lambda\Phi_1 + \mu g \circ \Phi_2$ *restricted to* X *satisfies the* $(PS)_c$*-condition for every* $c \in \mathbb{R}$, $\lambda \in \Lambda$, $\mu \in [0, |\lambda| + 1]$ *and* $g \in \mathcal{G}_\tau$, $\tau \geq 0$. *Assume that* $h(\|\cdot\|) + \lambda\Phi_1$ *is coercive on* X *for all* $\lambda \in \Lambda$ *and that there exists* $\rho \in \mathbb{R}$ *such that*

$$\sup_{\lambda\in\Lambda} \inf_{x\in X} [h(\|x\|) + \lambda(\Phi_1(x) + \rho)] < \inf_{x\in X} \sup_{\lambda\in\Lambda} [h(\|x\|) + \lambda(\Phi_1(x) + \rho)]. \tag{2.53}$$

Then, there exist a non-empty open set $A \subset \Lambda$ *and* $r > 0$ *with the property that for every* $\lambda \in A$ *there exists* $\mu_0 \in]0, |\lambda| + 1]$ *such that, for each* $\mu \in [0, \mu_0]$ *the functional* $\mathcal{E}_{\lambda,\mu} = h(\|\cdot\|) + \lambda\Phi_1 + \mu\Phi_2$ *has at least three critical points in* X *whose norms are less than* r.

Proof. Since h is a non-decreasing convex function, $X \ni x \mapsto h(\|x\|)$ is also convex; thus, $h(\|\cdot\|)$ is sequentially weakly lower semicontinuous on X, see H. Brézis [7, Corollaire III.8]. From the fact that the embeddings $X \hookrightarrow \tilde{X}_i$ $(i = 1, 2)$ are compact and $\Phi_i : \tilde{X}_i \to \mathbb{R}$ $(i = 1, 2)$ are locally Lipschitz functions, it follows that the function $E_{\lambda,\mu}$ as well as $\varphi : X \times \Lambda \to \mathbb{R}$ (in the first variable) given by

$$\varphi(x, \lambda) = h(\|x\|) + \lambda(\Phi_1(x) + \rho)$$

are sequentially weakly lower semicontinuous on X.

The function φ satisfies the hypotheses of Theorem 2.29. Fix $\sigma > \sup_\Lambda \inf_X \varphi$ and consider a nonempty open set Λ_0 with the property expressed in Theorem 2.29. Let $A = [a, b] \subset \Lambda_0$.

Fix $\lambda \in [a, b]$; then, for every $\tau \geq 0$ and $g_\tau \in \mathcal{G}_\tau$, there exists $\mu_\tau > 0$ such that, for any $\mu \in]0, \mu_\tau[$, the functional $E^\tau_{\lambda,\mu} = h(\|\cdot\|) + \lambda\Phi_1 + \mu g_\tau \circ \Phi_2$ restricted to X has two local minima, say x_1^τ, x_2^τ, lying in the set $\{x \in X : \varphi(x, \lambda) < \sigma\}$.

Note that

$$\begin{aligned} \bigcup_{\lambda\in[a,b]} \{x \in X : \varphi(x, \lambda) < \sigma\} \quad \subset \quad & \{x \in X : h(\|x\|) + a\Phi_1(x) < \sigma - a\rho\} \\ & \cup \{x \in X : h(\|x\|) + b\Phi_1(x) < \sigma - b\rho\}. \end{aligned}$$

Because the function $h(\|\cdot\|) + \lambda\Phi_1$ is coercive on X, the set on the right-side is bounded. Consequently, there is some $\eta > 0$, such that

$$\bigcup_{\lambda\in[a,b]} \{x \in X : \varphi(x, \lambda) < \sigma\} \subset B_\eta, \tag{2.54}$$

where $B_\eta = \{x \in X : \|x\| < \eta\}$. Therefore,

$$x_1^\tau, x_2^\tau \in B_\eta.$$

Now, set $c^\star = \sup_{t\in[0,\eta]} h(t) + \max\{|a|,|b|\} \sup_{B_\eta} |\Phi_1|$ and fix $r > \eta$ large enough such that for any $\lambda \in [a,b]$ to have

$$\{x \in X : h(\|x\|) + \lambda\Phi_1(x) \leq c^\star + 2\} \subset B_r. \tag{2.55}$$

Let $r^\star = \sup_{B_r} |\Phi_2|$ and correspondingly, fix a function $g = g_{r^*} \in \mathcal{G}_{r^*}$. Let us define $\mu_0 = \min\{|\lambda|+1, \frac{1}{1+\sup|g|}\}$. Since the functional $E_{\lambda,\mu} = E^{r^*}_{\lambda,\mu} = h(\|\cdot\|)+\lambda\Phi_1+\mu g_{r^*}\circ\Phi_2$ restricted to X satisfies the $(PS)_c$ condition for every $c \in \mathbb{R}$, $\mu \in [0,\mu_0]$, and $x_1 = x_1^{r^*}, x_2 = x_2^{r^*}$ are local minima of $E_{\lambda,\mu}$, we may apply Corollary 2.19, obtaining that

$$c_{\lambda,\mu} = \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} E_{\lambda,\mu}(\gamma(s)) \geq \max\{E_{\lambda,\mu}(x_1), E_{\lambda,\mu}(x_2)\} \tag{2.56}$$

is a critical value for $E_{\lambda,\mu}$, where Γ is the family of continuous paths $\gamma : [0,1] \to X$ joining x_1 and x_2. Therefore, there exists $x_3 \in X$ such that

$$c_{\lambda,\mu} = E_{\lambda,\mu}(x_3) \;\text{ and }\; 0 \in \partial E_{\lambda,\mu}(x_3).$$

If we consider the path $\gamma \in \Gamma$ given by $\gamma(s) = x_1 + s(x_2 - x_1) \subset B_\eta$we have

$$\begin{aligned}
h(\|x_3\|) + \lambda\Phi_1(x_3) &= E_{\lambda,\mu}(x_3) - \mu g(\Phi_2(x_3)) \\
&= c_{\lambda,\mu} - \mu g(\Phi_2(x_3)) \\
&\leq \sup_{s\in[0,1]} (h(\|\gamma(s)\|) + \lambda\Phi_1(\gamma(s)) + \mu g(\Phi_2(\gamma(s)))) - \mu g(\Phi_2(x_3)) \\
&\leq \sup_{t\in[0,\eta]} h(t) + \max\{|a|,|b|\} \sup_{B_\eta} |\Phi_1| + 2\mu_0 \sup|g| \\
&\leq c^\star + 2.
\end{aligned}$$

From (2.55) it follows that $x_3 \in B_r$. Therefore, x_i, $i = 1,2,3$ are critical points for $E_{\lambda,\mu}$, all belonging to the ball B_r. It remains to prove that these elements are critical points not only for $E_{\lambda,\mu}$ but also for $\mathcal{E}_{\lambda,\mu} = h(\|\cdot\|) + \lambda\Phi_1 + \mu\Phi_2$. Let $x = x_i$, $i \in \{1,2,3\}$. Since $x \in B_r$, we have that $|\Phi_2(x)| \leq r^*$. Note that $g(t) = t$ on $[-r^*, r^*]$; thus, $g(\Phi_2(x)) = \Phi_2(x)$. Consequently, on the open set B_r the functionals $E_{\lambda,\mu}$ and $\mathcal{E}_{\lambda,\mu}$ coincide, which completes the proof.

At the end of this section we recall the following non-smooth version of Ricceri [62, Theorem 2.5] which is proved by Marano and Motreanu [37].

Theorem 2.31. (Marano-Motreanu, [37, Theorem 1.1]) *Let $(X,\|\cdot\|)$ be a reflexive real Banach space, and $\tilde{X}$ another real Banach spaces such that X is compactly embedded into $\tilde{X}$. Let $\Phi : \tilde{X} \to \mathbb{R}$ and $\Psi : X \to \mathbb{R}$ be two locally Lipschitz functions, such that Ψ is weakly sequentially lower semicontinuous and coercive. For every $\rho > \inf_X \Psi$,* *put*

$$\varphi(\rho) = \inf_{u \in \Psi^{-1}(]-\infty,\rho[)} \frac{\Phi(u) - \inf_{v \in \overline{(\Psi^{-1}(]-\infty,\rho[))}_w} \Phi(v)}{\rho - \Psi(u)}, \tag{2.57}$$

where $\overline{(\Psi^{-1}(]-\infty,\rho[))}_w$ *is the closure of* $\Psi^{-1}(]-\infty,\rho[)$ *in the weak topology. Furthermore, set*

$$\gamma := \liminf_{\rho \to +\infty} \varphi(\rho), \qquad \delta := \liminf_{\rho \to (\inf_X \Psi)^+} \varphi(\rho). \tag{2.58}$$

Then, the following conclusions hold.

(A) *If* $\gamma < +\infty$ *then, for every* $\lambda > \gamma$, *either*
- (A1) $\Phi + \lambda\Psi$ *possesses a global minimum, or*
- (A2) *there is a sequence* $\{u_n\}$ *of critical points of* $\Phi + \lambda\Psi$ *such that* $\lim_{n\to+\infty} \Psi(u_n) = +\infty$.

(B) *If* $\delta < +\infty$ *then, for every* $\lambda > \delta$, *either*
- (B1) $\Phi + \lambda\Psi$ *possesses a local minimum, which is also a global minimum of* Ψ, *or*
- (B2) *there is a sequence* $\{u_n\}$ *of pairwise distinct critical points of* $\Phi + \lambda\Psi$, *with* $\lim_{n\to+\infty} \Psi(u_n) = \inf_X \Psi$, *weakly converging to a global minimum of* Ψ.

3. **Motreanu-Panagiotopoulos functionals**

In this section we present some results from the critical point theory for Motreanu-Panagiotopoulos type functionals. For details we refer the reader to the monographs of Motreanu-Panagiotopoulos [46], Motreanu-Rădulescu [47], Gasinski-Papageorgiou [18] and the papers of Marano and Motreamu [38], [37]. At the end of this section we present the Principle of Symmetric Criticality for this class of functionals following the paper of Kristály-Varga-Varga [29].

3.1. **Critical point results.** Let $\mathcal{I} = h + \psi$, with $h : X \to \mathbb{R}$ locally Lipschitz and $\psi : X \to (-\infty, +\infty]$ convex, proper (i.e., $\psi \not\equiv +\infty$), and lower semicontinuous. $\mathcal{I}$ is a *Motreanu-Panagiotopoulos type* functional, see [46, Chapter 3].

Definition 3.1. ([46, Definition 3.1]) An element $u \in X$ is said to be a critical point of $\mathcal{I} = h + \psi$, if

$$h^0(u; v - u) + \psi(v) - \psi(u) \geq 0, \forall v \in X.$$

In this case, $\mathcal{I}(u)$ is a critical value of $\mathcal{I}$.

We have the following result, see Gasinski-Papgeourgiu [18], Remark 2.3.1.

Proposition 3.2. *An element $u \in X$ is a critical point of $\mathcal{I} = h + \psi$, if and only if $0 \in \partial h(u) + \partial\psi(u)$, where $\partial\psi(u)$ denotes the subdifferential of the convex function ψ at u, i.e.*

$$\partial\psi(u) = \{x^* \in X^* : \psi(v) - \psi(u) \geq \langle x^*, v - u\rangle_X \text{ for every } v \in X\}.$$

Definition 3.3. ([46, Definition 3.2]) The functional $\mathcal{I} = h + \psi$ is said to satisfy the Palais-Smale condition at level $c \in \mathbb{R}$ (*shortly*, $(PS)_c$), if every sequence (u_n) from X satisfying $\mathcal{I}(u_n) \to c$ and

$$h^0(u_n; v - u_n) + \psi(v) - \psi(u_n) \geq -\varepsilon_n\|v - u_n\|, \forall v \in X,$$

for a sequence (ε_n) in $[0, \infty)$ with $\varepsilon_n \to 0$, contains a convergent subsequence. If $(PS)_c$ is verified for all $c \in \mathbb{R}$, $\mathcal{I}$ is said to satisfy the Palais-Smale condition (*shortly*,(PS)).

The next result is a non-smooth version of the Mountain Pass Theorem, see Corollary 3.2 from [46].

Theorem 3.4. (Motreanu-Panagiotopoulos [46]) *Assume that the functional $I : X \to (-\infty, +\infty]$ defined by $I = h + \psi$, satisfies (PS), $I(0) = 0$, and*

- (i) *there exist constants $\alpha > 0$ and $\rho > 0$, such that $I(u) \geq \alpha$ for all $||u|| = \rho$;*
- (ii) *there exists $e \in X$, with $||e|| > \rho$ and $I(e) \leq 0$.*

Then, the number

$$c = \inf_{f\in\Gamma} \sup_{t\in[0,1]} I(f(t)),$$

where

$$\Gamma = \{f \in C([0, 1], X) \ : \ f(0) = 0, \ f(1) = e \ \},$$

is a critical value of I with $c \geq \alpha$.

In the next we present the three critical points theorem of Ricceri [55] for Motreanu-Panagiotopoulos functionals. This result was proved by Marano and Motreanu [38, Theorem B].

Let $h_1, h_2 : X \to \mathbb{R}$ be locally Lipschitz functions, and let $\psi_1 : X \to]-\infty, +\infty]$ be a convex, proper, lower semicontinuous function. Then the function $h_1 + \psi_1 + \lambda h_2$ is a Motreanu-Panagiotopoulos type functional for every $\lambda \in \mathbb{R}$.

Theorem 3.5. (Marano-Motreanu [38]) *Suppose that $(X, ||\cdot||)$ is a separable and reflexive Banach space. Let $I_1 = h_1 + \psi_1$, $I_2 = h_2$, and let $\Lambda \subseteq \mathbb{R}$ be an interval. We assume that:*

- (a_1) *h_1 is weakly sequentially lower semicontinuous and h_2 is weakly sequentially continuous;*

(a_2) *for every* $\lambda \in \Lambda$ *the function* $I_1 + \lambda I_2$ *fulfils* $(PS)_c, c \in \mathbb{R}$*, and*

$$\lim_{||u||\to+\infty} (I_1(u) + \lambda I_2(u)) = +\infty;$$

(a_3) *there exists a continuous concave function* $h : \Lambda \to \mathbb{R}$ *satisfying*

$$\sup_{\lambda\in\Lambda} \inf_{u\in X} (I_1(u) + \lambda I_2(u) + h(\lambda)) < \inf_{u\in X} \sup_{\lambda\in\Lambda} (I_1(u) + \lambda I_2(u) + h(\lambda)).$$

Then, there exists an open interval $\Lambda_0 \subset \Lambda$*, such that for each* $\lambda \in \Lambda_0$ *the function* $I_1 + \lambda I_2$ *has at least three critical points in* X*.*

3.2. **Principle of Symmetric Criticality.** We now prove the Principle of Symmetric Criticality for Motreanu-Panagiotopoulos functionals. This result simultaneously generalizes the Principle of Symmetric Criticality in its standard form, see Palais [49] for smooth functionals; the result of Krawcewicz and Marzantowicz [25] for locally Lipschitz functions; and the result of Kobayashi and Ôtani [22] for Szulkin-type functionals. The results of this subsection is contained in the paper of Kristály, Varga and Varga [29].

Let G be a topological group which acts *linearly* on X, i.e., the action $G \times X \to X : [g, u] \mapsto gu$ is continuous and for every $g \in G$, the map $u \mapsto gu$ is linear. The group G induces an action of the same type on the dual space X^* defined by $\langle gx^*, u\rangle_X = \langle x^*, g^{-1}u\rangle_X$ for every $g \in G$, $u \in X$ and $x^* \in X^*$. A function $h : X \to \mathbb{R} \cup \{+\infty\}$ is $G-$*invariant* if $h(gu) = h(u)$ for every $g \in G$ and $u \in X$. A set $K \subseteq X$ (or $K \subseteq X^*$) is $G-$*invariant* if $gK = \{gu : u \in K\} \subseteq K$ for every $g \in G$. Let

$$\Sigma = \{u \in X : gu = u \text{ for every } g \in G\}$$

the *fixed point set of* X *under* G.

Now we recall some facts from [22]. Let

$$\Phi(X) = \{\psi : X \to \mathbb{R} \cup \{\infty\} : \ \psi \text{ is convex, proper, lower semicontinuous}\};$$

$$\Phi_G(X) = \{\psi \in \Phi(X) : \ \psi \text{ is } G-\text{invariant}\};$$

$$\Gamma_G(X^*) = \{K \subseteq X^* : \ K \text{ is } G-\text{invariant, weak}^*\text{−closed, convex}\}.$$

Proposition 3.6. ([22, Theorem 3.16]) *Assume that a compact group* G *acts linearly on a reflexiv Banach space* X*. Then for every* $K \in \Gamma_G(X^*)$ *and* $\psi \in \Phi_G(X)$ *one has*

$$K|_\Sigma \cap \partial(\psi|_\Sigma)(u) \neq \emptyset \Rightarrow K \cap \partial\psi(u) \neq \emptyset, \ u \in \Sigma, \tag{3.1}$$

where $K|_\Sigma = \{x^*|_\Sigma : x^* \in K\}$ *with* $\langle x^*|_\Sigma, u\rangle_\Sigma = \langle x^*, u\rangle_X$*,* $u \in \Sigma$*.*

Let $A : X \to X$ be the *averaging operator* over G, defined by

$$Au = \int_G gud\mu(g),\ u \in X, \tag{3.2}$$

where μ is the normalized Haar measure on G. The relation (3.2) can reads as follows

$$\langle x^*, Au\rangle_X = \int_G \langle x^*, gu\rangle_X d\mu(g),\ u \in X,\ x^* \in X^*. \tag{3.3}$$

It is easy to verify that A is a continuous linear projection from X to Σ and for every G-invariant closed convex set $K \subseteq X$ we have $A(K) \subseteq K$. The adjoint operator $A^* : \Sigma^* \to X^*$ of $A : X \to \Sigma$ is defined by

$$\langle A^*w^*, z\rangle_X = \langle w^*, Az\rangle_\Sigma,\ z \in X,\ w^* \in \Sigma^*. \tag{3.4}$$

Lemma 3.7. *Let $h : X \to \mathbb{R}$ be a G-invariant locally Lipschitz function and $u \in \Sigma$. Then*

(a) $\partial(h|_\Sigma)(u) \subseteq \partial h(u)|_\Sigma$.
(b) $\partial h(u) \in \Gamma_G(X^*)$.

Proof. (a) Let us fix $w^* \in \partial(h|_\Sigma)(u)$. Then by definition, one has

$$\langle w^*, v\rangle_\Sigma \le (h|_\Sigma)^0(u; v) \text{ for every } v \in \Sigma.$$

First, a simple estimation shows that $(h|_\Sigma)^0(u; v) \le h^0(u; v)$ for every $v \in \Sigma$. Thus, applying the above inequality for $v = Az \in \Sigma$ with $z \in X$ arbitrarily fixed, by (3.4) one has

$$\langle A^*w^*, z\rangle_X = \langle w^*, Az\rangle_\Sigma \le h^0(u; Az). \tag{3.5}$$

Using [10, Proposition 2.1.2 (b)] and (3.3), we get

$$\begin{aligned} h^0(u; Az) &= \max\{\langle x^*, Az\rangle_X :\ x^* \in \partial h(u)\} \\ &= \max\{\int_G \langle x^*, gz\rangle_X d\mu(g) :\ x^* \in \partial h(u)\} \\ &\le \int_G h^0(u; gz)d\mu(g) = \int_G h^0(g^{-1}u; z)d\mu(g) = \int_G h^0(u; z)d\mu(g) \\ &= h^0(u; z). \end{aligned}$$

Combining this relation with (3.5), we conclude that $A^*w^* \in \partial h(u)$. Since $w^* = A^*w^*|_\Sigma$, we obtain that $w^* \in \partial h(u)|_\Sigma$, completing the proof of (a).

(b) Since $\partial h(u)$ is a nonempty, convex and weak*-compact subset of X^* (see [10, Proposition 2.1.2 (a)]), it is enough to prove that $\partial h(u)$ is G-invariant, i.e., $g\partial h(u) \subseteq \partial h(u)$ for every $g \in G$. To this end, let us fix $g \in G$ and $x^* \in \partial h(u)$. Then, for every $z \in X$ we have

$$\langle gx^*, z\rangle_X = \langle x^*, g^{-1}z\rangle_X \le h^0(u; g^{-1}z) = h^0(gu; z) = h^0(u; z),$$

i.e., $gx^* \in \partial h(u)$. □

Theorem 3.8. (Kristály-Varga-Varga [29]) *Let X be a reflexiv Banach space and $\mathcal{I} = h + \psi : X \to \mathbb{R} \cup \{+\infty\}$ be a Motreanu-Panagiotopoulos type functional. If a compact group G acts linearly on X, and the functionals h and ψ are $G-$invariant, then every critical point of $\mathcal{I}|_\Sigma$ is also a critical point of $\mathcal{I}$.*

Proof. Let $u \in \Sigma$ be a critical point of $\mathcal{I}|_\Sigma$. Thanks to Proposition 3.2 one has $0 \in \partial(h|_\Sigma)(u) + \partial(\psi|_\Sigma)(u)$. Moreover, due to Lemma 3.7(a) we have

$$\emptyset \neq -\partial(h|_\Sigma)(u) \cap \partial(\psi|_\Sigma)(u) \subseteq -\partial h(u)|_\Sigma \cap \partial(\psi|_\Sigma)(u).$$

By choosing $K = \partial h(u)$ in Proposition 3.6 and taking into account Lemma 3.7(b), relation (3.1) implies that $\emptyset \neq -\partial h(u) \cap \partial\psi(u)$. Thus, in particular $0 \in \partial h(u) + \partial\psi(u)$, i.e., u is indeed a critical point of $\mathcal{I}$. □

A direct consequence of this theorem is the following proved by Krawcewicz and Marzantowicz [25].

Remark 3.9. (Krawcewicz-Marzantowicz [25]) Let $f : X \to \mathbb{R}$ be a G-invariant locally Lipschitz function and $u \in X^G$ a fixed point. Then $u \in X^G$ is a critical point of f if and only if u is a critical point of $f^G = f|_{X^G} : X^G \to \mathbb{R}$.

4. **Application to hemivaritional inequalities**

4.1. **Formulation of the problem.**
In this section we prove some existence results for a general class of hemivariational inequalities. These results appear in the paper of Kristály [27] and Dályai-Varga [11].

Let $(X, \|\cdot\|)$ be a real, separable, reflexive Banach space, and let $(X^\star, \|\cdot\|_\star)$ be its dual. We consider $\Omega \subset \mathbb{R}^N$ an unbounded domain. Also assume that the inclusion $X \hookrightarrow L^l(\Omega)$ is continuous with the embedding constants $C(l)$, where $l \in [p, p^\star]$ $(p \geq 2, p^\star = \frac{Np}{N-p})$.

Let us denote by $\|\cdot\|_l$ the norm of $L^l(\Omega)$. In this section we suppose that the following condition holds:

(CE)**:** X is compactly embedded in $L^r(\Omega)$ for some $r \in [p, p^\star[$

Let $A : X \to X^\star$ be a potential operator with the potential $a : X \to \mathbb{R}$, i.e. a is Gâteaux differentiable and

$$\lim_{t \to 0} \frac{a(u + tv) - a(u)}{t} = \langle A(u), v \rangle,$$

for every $u, v \in X$. Here $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $X^\star$ and X. For a potential we always assume that $a(0) = 0$. We suppose that $A : X \to X^\star$ satisfies the following properties:

- A is hemicontinuous, i.e. A is continuous on line segments in X and $X^\star$ equipped with the weak topology.
- A is homogeneous of degree $p-1$, i.e. for every $u \in X$ and $t > 0$ we have $A(tu) = t^{p-1}A(u)$. Consequently, for a homogeneous hemicontinuous operator of degree $p-1$, we have $a(u) = \frac{1}{p}\langle A(u), u\rangle$.
- $A : X \to X^\star$ is a strongly monotone operator, i.e. there exists a function $\kappa : [0,\infty) \to [0,\infty)$ which is positive on $(0,\infty)$ and $\lim_{t\to\infty} \kappa(t) = \infty$ and such that for all $u, v \in X$,

$$\langle A(u) - A(v), u - v\rangle \geq \kappa(\|u - v\|)\|u - v\| \, .$$

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a measurable function which satisfies the following growth condition:

(F1) $|f(x,s)| \leq c(|s|^{p-1} + |s|^{r-1})$, for a.e. $x \in \Omega$, for all $s \in \mathbb{R}$

Let $F : \Omega \times \mathbb{R} \to \mathbb{R}$ be the function defined by

$$F(x,u) = \int_0^u f(x,s)ds, \quad \text{for a.e. } x \in \Omega, \ \forall s \in \mathbb{R}. \tag{4.1}$$

For a.e. $x \in \Omega$ and for every $u, v \in \mathbb{R}$, we have:

$$|F(x,u) - F(x,v)| \leq c_1|u - v| \left(|u|^{p-1} + |v|^{p-1} + |u|^{r-1} + |v|^{r-1}\right), \tag{4.2}$$

where c_1 is a constant which depends only of u and v. Therefore, the function $F(x,\cdot)$ is locally Lipschitz and we can define the partial Clarke derivative, i.e.

$$F_2^0(x,u;w) = \limsup_{y\to u,\ t\to 0^+} \frac{F(x,y+tw) - F(x,y)}{t}, \tag{4.3}$$

for every $u, w \in \mathbb{R}$ and for a.e. $x \in \mathbb{R}$.

Now, we formulate the hemivariational inequality problem that will be studied in the next:

Find $u \in X$ such that

$$\langle Au, v\rangle + \int_\Omega F_2^0(x, u(x); -v(x))dx \geq 0, \quad \forall\, v \in X. \tag{4.4}$$

To study the existence of solutions of the problem (4.4) we introduce the energy functional $\Psi : X \to \mathbb{R}$ defined by

$$\Psi(u) = a(u) - \Phi(u),$$

where $a(u) = \frac{1}{p}\langle A(u), u\rangle$ and $\Phi(u) = \int_\Omega F(x, u(x))dx$.

Remark 4.1. In Proposition 4.6 we will prove that the critical points of the functional Ψ are solution of the problem (4.4).

To study the existence of the critical point of the function Ψ is necessary to impose some conditions on the function f:

(F2) There exists $\alpha > p$, $\lambda \in [0, \frac{\kappa(1)(\alpha-p)}{C^p(p)}[$ and a continuous function $g : \mathbb{R} \to \mathbb{R}_+$, such that for a.e. $x \in \mathbb{R}^N$ and for all $u \in \mathbb{R}$ we have

$$\alpha F(x,u) + F_2^0(x,u;-u) \leq g(u), \tag{4.5}$$

where $\lim_{|u|\to\infty} g(u)/|u|^p = \lambda$.

(F2') There exists $\alpha \in (\max\{p, p^\star \frac{r-p}{p^\star - p}\}, p^\star)$ and a constant $C > 0$ such that for a.e. $x \in \Omega$ and for all $u \in \mathbb{R}$ we have

$$-C|u|^\alpha \geq F(x,u) + \frac{1}{p}F_2^0(x,u;-u). \tag{4.6}$$

Next, we impose further assumptions on f. First we define two functions by

$$\underline{f}(x,s) = \lim_{\delta\to 0^+} \text{essinf}\{f(x,t) : |t-s| < \delta\},$$
$$\overline{f}(x,s) = \lim_{\delta\to 0^+} \text{esssup}\{f(x,t) : |t-s| < \delta\},$$

for every $s \in \mathbb{R}$ and for a.e. $x \in \Omega$. It is clear that the function $\underline{f}(x,\cdot)$ is lower semicontinuous and $\overline{f}(x,\cdot)$ is upper semicontinuous. The following hypothesis on f was introduced by Chang [9].

(F3) The functions $\underline{f}, \overline{f}$ are N-measurable, i.e. for every measurable function $u : \Omega \to \mathbb{R}$ the functions $x \mapsto \underline{f}(x,u(x)), x \mapsto \overline{f}(x,u(x))$ are measurable.

(F4) For every $\varepsilon > 0$, there exists $c(\varepsilon) > 0$ such that for a.e. $x \in \Omega$ and for every $s \in \mathbb{R}$ we have

$$|f(x,s)| \leq \varepsilon|s|^{p-1} + c(\varepsilon)|s|^{r-1}.$$

(F5) For the $\alpha \in (p, p^\star)$ from condition (F2), there exists a $c^\star > 0$ such that for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$ we have

$$F(x,u) \geq c^\star(|u|^\alpha - |u|^p).$$

Remark 4.2. We observe that if we impose the following condition on f,

(F4') $\lim_{\varepsilon\to 0^+} \text{esssup}\{\frac{|f(x,s)|}{|s|^p} : (x,s) \in \Omega \times (-\varepsilon, \varepsilon)\} = 0$,

then this condition with (F1) imply (F4).

4.2. **Some basic lemmas.** Before to study the hemivariational inequality (4.4) we prove some auxiliary lemmas. The results of this subsection appear in the paper of Dályai-Varga [11]. So, we consider the function $\Phi : X \to \mathbb{R}$ by

$$\Phi(u) = \int_\Omega F(x,u(x))dx, \quad \forall\, u \in X, \tag{4.7}$$

where $F(x,u) = \int_0^u f(x,s)ds, \quad$ for a.e. $x \in \Omega$, $\forall s \in \mathbb{R}$..

Remark 4.3. For simplicity we denote $h(u) = c|u|^{p-1}$ and in the next two results we use only that the function h is monotone increasing, convex and $h(0) = 0$.

The following results appears in the paper of Kristály [27] and Dályai-Varga [11].

Proposition 4.4. *The function* $\Phi : X \to \mathbb{R}$*, defined by* $\Phi(u) = \int_\Omega F(x,u(x))dx$ *is locally Lipschitz on bounded sets of* X*.*

Proof. For every $u, v \in X$, with $\|u\|, \|v\| < r$, we have

$$\begin{aligned}
&\|\Phi(u) - \Phi(v)\| \\
&\le \int_\Omega |F(x,u(x)) - F(x,v(x))|dx \\
&\le c_1 \int_\Omega |u(x) - v(x)|[h(|u(x)|) + h(|v(x)|)] \\
&\le c_2\Big(\int_\Omega |u(x) - v(x)|^p\Big)^{1/p}\Big[\Big(\int_\Omega (h(|u(x)|)^{p'}dx\Big)^{1/p'} + \Big(\int_\Omega (h(|v(x)|)^{p'}dx\Big)^{1/p'}\Big] \\
&\le c_2\|u - v\|_p[\|h(|u|)\|_{p'} + \|h(|v|)\|_{p'}) \\
&\le C(u,v)\|u - v\|,
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and we used the Hölder inequality, the subadditivity of the norm $\|\cdot\|_{p'}$ and the fact that the inclusion $X \hookrightarrow L^p(\Omega)$ is continuous. We observe that $C(u,v)$ is a constant which depends only of u and v. $\square$

Proposition 4.5. (Kristály [27] and Dályai-Varga [11]) *If condition* (F1) *holds, then for every* $u, v \in X$*, we have*

$$\Phi^0(u;v) \le \int_\Omega F_2^0(x,u(x);v(x))dx. \tag{4.8}$$

Proof. It is sufficient to prove the proposition for the function f, which satisfies only the growth condition $|f(x,s)| \le c|u|^{p-1}$ from Remark 4.3. Let us fix the elements $u, v \in X$. The function $F(x,\cdot)$ is locally Lipschitz and therefore continuous. Thus $F_2^0(x,u(x);v(x))$ can be expressed as the upper limit of $\big(F(x,y+tv(x)) - F(x,y)\big)/t$, where $t \to 0^+$ takes rational values and $y \to u(x)$ takes values in a countable subset of $\mathbb{R}$. Therefore, the map $x \to F_2^0(x,u(x);v(x))$ is measurable as the "countable limsup" of measurable functions in x. From condition (F1) we get that the function $x \to F_2^0(x,u(x);v(x))$ is from $L^1(\mathbb{R}^N)$.

Using the fact that the Banach space X is separable, there exists a sequence $w_n \in X$ with $\|w_n - u\| \to 0$ and a real number sequence $t_n \to 0^+$, such that

$$\Phi^0(u,v) = \lim_{n\to\infty} \frac{\Phi(w_n + t_n v) - \Phi(w_n)}{t_n}. \tag{4.9}$$

Since the inclusion $X \hookrightarrow L^p(\mathbb{R}^N)$ is continuous, we get $\|w_n - u\|_p \to 0$. Using [7, Theorem IV.9], there exists a subsequence of (w_n) denoted in the same way, such that $w_n(x) \to u(x)$ a.e. $x \in \mathbb{R}^N$. Now, let $\varphi_n : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ be the function defined by

$$\begin{aligned}\varphi_n(x) = &-\frac{F(x, w_n(x) + t_n v(x)) - F(x, w_n(x))}{t_n}\\ &+ c_1|v(x)|[h(|w_n(x) + t_n v(x)|) + h(|w_n(x)|)].\end{aligned}$$

We see that the the functions φ_n are measurable and non-negative. If we apply Fatou's lemma, we get

$$\int_\Omega \liminf_{n\to\infty} \varphi_n(x)dx \le \liminf_{n\to\infty} \int_\Omega \varphi_n(x)dx.$$

This inequality is equivalent to

$$\int_\Omega \limsup_{n\to\infty} [-\varphi_n(x)]dx \ge \limsup_{n\to\infty} \int_\Omega [-\varphi_n(x)]dx. \tag{4.10}$$

For simplicity in the calculus we introduce the following notation:

(i) $\varphi_n^1(x) = \frac{F(x,w_n(x)+t_nv(x)) - F(x,w_n(x))}{t_n}$;
(ii) $\varphi_n^2(x) = c_1|v(x)|[h(|w_n(x) + t_n v(x)|) + h(|w_n(x)|)]$.

With these notation, we have $\varphi_n(x) = -\varphi_n^1(x) + \varphi_n^2(x)$.

Now we prove the existence of limit $b = \lim_{n\to\infty} \int_\Omega \varphi_n^2(x)dx$. Using the facts that the inclusion $X \hookrightarrow L^p(\Omega)$ is continuous and $\|w_n - u\| \to 0$, we get $\|w_n - u\|_p \to 0$. Using [7, Theorem IV.9], there exist a positive function $g \in L^p(\Omega)$, such that $|w_n(x)| \le g(x)$ a.e. $x \in \Omega$. Considering that the function h is monotone increasing, we get

$$|\varphi_n^2(x)| \le c_1|v(x)|[h(g(x) + |v(x)|) + h(g(x))], \quad \text{a.e. } x \in \Omega.$$

Moreover, $\varphi_n^2(x) \to 2c_1|v(x)|h(|u(x)|)$ for a.e. $x \in \Omega$. Thus, using the Lebesque dominated convergence theorem, we have

$$b = \lim_{n\to\infty} \int_\Omega \varphi_n^2(x)dx = \int_\Omega 2c_1|v(x)|h(|u(x)|)dx. \tag{4.11}$$

If we denote by $I_1 = \limsup_{n\to\infty} \int_\Omega [-\varphi_n(x)]dx$, then using (4.9) and (4.11), we have

$$I_1 = \limsup_{n\to\infty} \int_\Omega [-\varphi_n(x)]dx = \Phi^0(u;v) - b. \tag{4.12}$$

Next we estimate the expression $I_2 = \int_\Omega \limsup_{n\to\infty}[-\varphi_n(x)]dx$. We have the inequality

$$\int_\Omega \limsup_{n\to\infty}[\varphi_n^1(x)]dx - \int_\Omega \lim_{n\to\infty} \varphi_n^2(x)dx \geq I_2. \tag{4.13}$$

Using the fact that $w_n(x) \to u(x)$ a.e. $x \in \Omega$ and $t_n \to 0^+$, we get

$$\int_\Omega \lim_{n\to\infty} \varphi_n^2(x)dx = 2c_1 \int_\Omega |v(x)|h(|u(x)|)dx.$$

On the other hand,

$$\begin{aligned}\int_\Omega \limsup_{n\to\infty} \varphi_n^1(x)dx &\leq \int_\Omega \limsup_{y\to u(x),\, t\to 0^+} \frac{F(x, y + tv(x)) - F(x, y)}{t}dx \\ &= \int_\Omega F_2^0(x, u(x); v(x))dx.\end{aligned}$$

Using relations (4.10), (4.12), (4.13) and the above estimates, we obtain the desired result. □

Now we prove that the critical points of the function $\Psi : X \to \mathbb{R}$ defined by $\Psi(u) = a(u) - \Phi(u)$ are solutions of problem (4.4).

Proposition 4.6. *If* $0 \in \partial\Psi(u)$*, then* u *solves the problem* (4.4).

Proof. Because $0 \in \partial\Psi(u)$, we have $\Psi^0(u; v) \geq 0$ for every $v \in X$. Using the Proposition 4.5 and a property of Clarke derivative we obtain

$$\begin{aligned}0 \leq \Psi^0(u; v) &\leq \langle u, v\rangle + (-\Phi)^0(u; v) \\ &= \langle A(u), v\rangle + \Phi^0(u; -v) \\ &\leq \langle A(u), v\rangle + \int_{\mathbb{R}^N} F_2^0(x, u(x), -v(x))dx,\end{aligned}$$

for every $v \in X$. □

4.3. **The Palais-Smale and Cerami compactness conditions.** In this subsection we study the situation when the function Ψ satisfies the $(PS)_c$ and $(CPS)_c$ conditions. We have the following result.

Proposition 4.7. *Let* $(u_n) \subset X$ *be a* $(PS)_c$ *sequence for the function* $\Psi : X \to \mathbb{R}$*. If the conditions (F1) and (F2) are fulfilled, then the sequence* (u_n) *is bounded in* X*.*

Proof. Because $(u_n) \subset X$ is a $(PS)_c$ sequence for the function Ψ, we have $\Psi(u_n) \to c$ and $\lambda_\Psi(u_n) \to 0$. From the condition $\Psi(u_n) \to c$ we get $c + 1 \geq \Psi(u_n)$ for sufficiently large $n \in \mathbb{N}$.

Because $\lambda_\Psi(u_n) \to 0$, $\|u_n\| \geq \|u_n\|\lambda_\Psi(u_n)$ for every sufficiently large $n \in \mathbb{N}$. From the definition of $\lambda_\Psi(u_n)$ results the existence of an element $z^\star_{u_n} \in \partial\Psi(u_n)$, such that $\lambda_\Psi(u_n) = \|z^\star_{u_n}\|_\star$. For every $v \in X$, we have $|z^\star_{u_n}(v)| \leq \|z^\star_{u_n}\|_\star \|v\|$, therefore $\|z^\star_{u_n}\|_\star \|v\| \geq -z^\star_{u_n}(v)$. If we take $v = u_n$, then $\|z^\star_{u_n}\|_\star \|u_n\| \geq -z^\star_{u_n}(u_n)$.

Using the properties $\Psi^0(u,v) = \max\{z^\star(v) : z^\star \in \partial\Psi(u)\ \}$ for every $v \in X$, we have $-z^\star(v) \geq -\Psi^0(u,v)$ for all $z^\star \in \partial\Psi(u)$ and $v \in X$. If we take $u = v = u_n$ and $z^\star = z^\star_{u_n}$, we get $-z^\star_{u_n}(u_n) \geq -\Psi^0(u_n,u_n)$. Therefore, for every $\alpha > 0$, we have

$$\frac{1}{\alpha}\|u_n\| \geq \frac{1}{\alpha}\|z^\star_{u_n}\|_\star\|u_n\| \geq -\frac{1}{\alpha}\Psi^0(u_n,u_n).$$

When we add the above inequality with $c+1 \geq \Psi(u_n)$, we obtain

$$c+1+\frac{1}{\alpha}\|u_n\| \geq \Psi(u_n) - \frac{1}{\alpha}\Psi^0(u_n;u_n).$$

Using the above inequality, $\Psi^0(u,v) \leq \langle A(u),v\rangle + \Phi^0(u,-v)$, and Proposition 4.5 we get

$$\begin{aligned}
&c+1+\frac{1}{\alpha}\|u_n\| \\
&\geq \Psi(u_n) - \frac{1}{\alpha}\Psi^0(u_n;u_n) \\
&= \frac{1}{p}\langle A(u_n),u_n\rangle - \Phi(u_n) - \frac{1}{\alpha}\left(\langle A(u_n),u_n\rangle + \Phi^0(u_n;-u_n)\right) \\
&\geq (\frac{1}{p}-\frac{1}{\alpha})\langle A(u_n),u_n\rangle - \int_\Omega \left[F(x,u_n(x)) + \frac{1}{\alpha}F_2^0(x,u_n(x);-u_n(x))\right]dx \\
&\geq (\frac{1}{p}-\frac{1}{\alpha})\langle A(u_n),u_n\rangle - \frac{1}{\alpha}\int_\Omega g(u_n(x))dx.
\end{aligned}$$

The relation $\lim_{|u|\to\infty}\frac{g(u)}{|u|^p} = \lambda$ assures the existence of a constant M, such that $\int_\Omega g(u_n(x))dx \leq M + \lambda\int_\Omega |u_n(x)|^p dx$. We use again that the inclusion $X \hookrightarrow L^p(\Omega)$ is continuous, that $a(u) = \frac{1}{p}\langle A(u),u\rangle$ and that

$$a(u) = \|u\|^p\langle A(\frac{u}{\|u\|}), \frac{u}{\|u\|}\rangle \geq \kappa(1)\|u\|^p,$$

to obtain

$$\begin{aligned}
c+1+\|u_n\| &\geq (\frac{1}{p}-\frac{1}{\alpha})\langle A(u_n),u_n\rangle - \frac{\lambda C^p(p)}{\alpha}\|u_n\|^p - \frac{M}{\alpha} \\
&\geq \frac{\kappa(1)(\alpha-p)-\lambda C^p(p)}{\alpha}\|u_n\|^p - \frac{M}{\alpha}.
\end{aligned}$$

From the above inequality, it results that the sequence (u_n) is bounded. $\square$

Proposition 4.8. *If conditions (F1), (F2') and (F4) hold, then every $(CPS)_c(c>0)$ sequence $(u_n) \subset X$ for the function $\Psi : X \to \mathbb{R}$ is bounded in X.*

Proof. Let $(u_n) \subset X$ be a $(CPS)_c$ $(c > 0)$ sequence for the function Ψ, i.e. $\Psi(u_n) \to c$ and $(1 + \|u_n\|)\lambda_\Psi(u_n) \to 0$. From $(1 + \|u_n\|)\lambda_\Psi(u_n) \to 0$, we get $\|u_n\|\lambda_\Psi(u_n) \to 0$ and $\lambda_\Psi(u_n) \to 0$. As in Proposition 4.7, there exists $z^\star_{u_n} \in \partial\Psi(u_n)$ such that

$$\frac{1}{p}\|z^\star_{u_n}\|_\star \|u_n\| \geq -\Psi^0(u_n; \frac{1}{p}u_n).$$

From this inequality, Proposition 4.5, condition (F2') and the property $\Psi^0(u; v) \leq \langle Au, v\rangle + \Phi^0(u; -v)$ we get

$$\begin{aligned} c + 1 &\geq \Psi(u_n) - \frac{1}{p}\Psi^0(u_n; u_n) \\ &\geq a(u_n) - \Phi(u_n) - \frac{1}{p}\left[\langle Au_n, u_n\rangle + \Phi^0(u_n; -u_n)\right] \\ &\geq -\int_\Omega \left[F(x, u_n(x)) + \frac{1}{p}F_2^0(x, u_n(x); -u_n(x))\right] dx \\ &\geq C\|u_n\|_\alpha^\alpha. \end{aligned}$$

Therefore, the sequence (u_n) is bounded in $L^\alpha(\Omega)$. From the condition (F4) follows that, for every $\varepsilon > 0$, there exists $c(\varepsilon) > 0$, such that for a.e. $x \in \mathbb{R}^N$,

$$F(x, u(x)) \leq \frac{\varepsilon}{p}|u(x)|^p + \frac{c(\varepsilon)}{r}|u(x)|^r.$$

After integration, we obtain

$$\Phi(u) \leq \frac{\varepsilon}{p}\|u\|_p^p + \frac{c(\varepsilon)}{r}\|u\|_r^r.$$

Using the above inequality, the expression of Ψ, and $\|u\|_p \leq C(p)\|u\|$, we obtain

$$\frac{\kappa(1) - \varepsilon C^p(p)}{p}\|u\|^p \leq \Psi(u) + \frac{c(\varepsilon)}{r}\|u\|_r^r \leq c + 1 + \|u\|_r^r.$$

Now, we study the behaviour of the sequence $(\|u_n\|_r)$. We have the following two cases:

(i) If $r = \alpha$, then it is easy to see that the sequence $(\|u_n\|_r)$ is bounded in $\mathbb{R}$.

(ii) If $r \in (\alpha, p^\star)$ and $\alpha > p^\star \frac{r-p}{p^\star - p}$, then we have

$$\|u\|_r^r \leq \|u\|_\alpha^{(1-s)\alpha} \cdot \|u\|_{p^\star}^{sp^\star},$$

where $r = (1-s)\alpha + sp^\star, s \in (0, 1)$.

Using the inequality $\|u\|_{p^\star}^{sp^\star} \leq C^{sp^\star}(p)\|u\|^{sp^\star}$, we obtain

$$\frac{\kappa(1) - \varepsilon C^p(p)}{p}\|u\|^p \leq c + 1 + \frac{c(\varepsilon)}{r}\|u\|_\alpha^{(1-s)\alpha}\|u\|^{sp^\star}. \tag{4.14}$$

When in the inequality (4.14) we take $\varepsilon \in \left(0, \frac{\kappa(1)}{C^p(p)}\right)$ and use b), we obtain that the sequence (u_n) is bounded in X. $\square$

The main result of this section is as follows.

Theorem 4.9. (Dályai-Varga [11])

1. *If the conditions (CE),(F1)-(F4) hold, then Ψ satisfies the $(PS)_c$ condition for every $c \in \mathbb{R}$.*
2. *If the conditions (CE),(F1), (F2'), (F3), and (F4) hold, then Ψ satisfies the $(CPS)_c$ condition for every $c > 0$.*

Proof. Let $(u_n) \subset X$ be a $(PS)_c(c \in \mathbb{R})$ or a $(CPS)_c(c > 0)$ sequence for the function $\Psi(u_n)$. Using Propositions 4.7, 4.8 it follows that the sequence (u_n) is a bounded in X. Because X is reflexive Banach space follows the existence of an element $u \in X$, such that $u_n \rightharpoonup u$ weakly in X. Because the inclusions $X \hookrightarrow L^r(\mathbb{R}^N)$ is compact, we have that $u_n \to u$ strongly in $L^r(\mathbb{R}^N)$.

Next we estimate the expressions $I_n^1 = \Psi^0(u_n; u_n - u)$ and $I_n^2 = \Psi^0(u; u - u_n)$. First we estimate the expression $I_n^2 = \Psi^0(u; u - u_n)$. We know that $\Psi^0(u; v) = \max\{z^\star(v) : z^\star \in \partial\Psi(u)\}$, $\forall\, v \in X$. Therefore, there exists $z_u^\star \in \partial\Psi(u)$, such that $\Psi^0(u; v) = z_u^\star(v)$ for all $v \in X$. From the above relation and from the fact that $u_n \rightharpoonup u$ weakly in X, we get $\Psi^0(u; u - u_n) = z_u^\star(u - u_n) \to 0$.

Now, we estimate the expression $I_n^1 = \Psi^0(u_n; u_n - u)$. From $\lambda_\Psi(u_n) \to 0$ follows the existence of a positive real numbers sequence $\mu_n \to 0$, such that $\Psi^0(u_n, u_n - u) + \mu_n\|u_n - u\| \geq 0$.

Now, we estimate the expression $I_n = \Phi^0(u_n; u - u_n) + \Phi^\circ(u; u - u_n)$. For the simplicity in calculus we introduce the notations $h_1(s) = |s|^{p-1}$ and $h_2(s) = |s|^r$. For this we observe that if we use the continuity of the functions h_1 and h_2, the condition (F4) implies that for every $\varepsilon > 0$, there exists a $c(\varepsilon) > 0$ such that

$$\max\left\{|\underline{f}(x,s)|, |\overline{f}(x,s)|\right\} \leq \varepsilon h_1(s) + c(\varepsilon)h_2(s), \tag{4.15}$$

for a.e. $x \in \mathbb{R}^N$ and for all $s \in \mathbb{R}$. Using this relation and Proposition 4.5, we have

$$\begin{aligned}
I_n &= \Phi^0(u_n; u - u_n) + \Phi(u; u - u_n) \\
&\leq \int_\Omega \left[F_2^0(x, u_n(x); u_n(x) - u(x)) + F_2^0(x, u(x); u(x) - u_n(x))\right] dx \\
&\leq \int_\Omega \left[\underline{f}(x, u_n(x))(u_n(x) - u(x)) + \overline{f}(x, u(x))(u(x) - u_n(x))\right] dx \\
&\leq 2\varepsilon \int_\Omega \left[h_1(u(x)) + h_1(u_n(x))\right] |u_n(x) - u(x)| dx \\
&\quad + 2c_\varepsilon \int_\Omega \left[(h_2(u(x)) + h_2(u_n(x))\right] |u_n(x) - u(x)| dx.
\end{aligned}$$

Using Hölder inequality and that the inclusion $X \hookrightarrow L^p(\Omega)$ is continuous, we get

$$\begin{aligned} I_n &\le 2\varepsilon C(p)\|u_n - u\|(\|h_1(u)\|_{p'} + \|h_1(u_n)\|_{p'}) \\ &\quad + 2c(\varepsilon)\|u_n - u\|_r(\|h_2(u)\|_{r'} + \|h_2(u_n)\|_{r'}), \end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{r} + \frac{1}{r'} = 1$. Using the fact that the inclusion $X \hookrightarrow L^r(\Omega)$ is compact, we get that $\|u_n - u\|_r \to 0$ as $n \to \infty$. For $\varepsilon \to 0^+$ and $n \to \infty$ we obtain that $I_n \to 0$.

Finally, we use the inequality $\Psi^0(u; v) \le \langle A(u), v\rangle + \Phi^0(u; -v)$. If we replace v with $-v$, we get $\Psi^0(u, -v) \le -\langle A(u), v\rangle + \Phi^0(u; v)$, therefore $\langle A(u), v\rangle \le \Phi^0(u; v) - \Psi^0(u, -v)$.

In the above inequality we replace u and v by $u = u_n, v = u - u_n$ then $u = u, v = u_n - u$ and we get

$$\begin{aligned} \langle A(u_n), u - u_n\rangle &\le \Phi^0(u_n, u - u_n) - \Psi^0(u_n; u_n - u), \\ \langle A(u), u_n - u\rangle &\le \Phi^0(u, u_n - u) - \Psi^0(u, u - u_n). \end{aligned}$$

Adding these relations, we have the following key inequality:

$$\|u_n - u\|\kappa(u_n - u) \le \langle A(u_n - u), u_n - u\rangle$$

$$\le \left[\Phi^0(u_n; u - u_n) + \Phi(u; u - u_n)\right] - \Psi^0(u_n; u_n - u) - \Psi^0(u; u - u_n) = I_n - I_n^1 - I_n^2.$$

Using the above relation and the estimations of I_n, I_n^1 and I_n^2, we obtain

$$\|u_n - u\|\kappa(u_n - u) \le I_n + \mu_n\|u_n - u\| - z_u^\star(u_n - u).$$

If $n \to \infty$, from the above inequality we obtain the assertion of the theorem. □

4.4. **Existence result.** The main result of this subsection is the following.

Theorem 4.10. (Dályai-Varga [11])

1. *If conditions (CE),(F1)-(F5) hold, then problem* (4.4) *has a nontrivial solution.*
2. *If conditions (CE), (F1),(F2'), (F3), and (F4) hold, then problem* (4.4) *has a nontrivial solution.*

Proof. Using (1) in Theorem 4.9, and conditions (F1)-(F4), it follows that the functional $\Psi(u) = \frac{1}{p}\langle A(u), u\rangle - \Phi(u)$ satisfies the $(PS)_c$ condition for every $c \in \mathbb{R}$. From Corollary 2.19 we verify the following geometric hypotheses:

$$\exists\, \alpha, \rho > 0, \quad \text{such that } \Psi(u) \ge \beta \text{ on } B_\rho(0) = \{u \in X : \|u\| = \rho\}, \tag{4.16}$$

$$\Psi(0) = 0 \quad \text{and there exists } v \in H \setminus B_\rho(0) \text{ such that } \Psi(v) \le 0. \tag{4.17}$$

For the proof of relation (4.16), we use the relation (F4), i.e. $|f(x,s)| \leq \varepsilon|s|^{p-1} + c(\varepsilon)|s|^{r-1}$. Integrating this inequality and using that the inclusions $X \hookrightarrow L^p(\mathbb{R}^N)$, $X \hookrightarrow L^r(\mathbb{R}^N)$ are continuous, we get that

$$\begin{aligned}\Psi(u) &\geq \frac{\kappa(1) - \varepsilon C(p)}{p}\langle A(u), u\rangle - \frac{1}{r}c(\varepsilon)C(r)\|u\|_r^r \\ &\geq \frac{\kappa(1) - \varepsilon C(p)}{p}\|u\|^p - \frac{1}{r}c(\varepsilon)C(r)\|u\|^r.\end{aligned}$$

The right member of the inequality is a function $\chi : \mathbb{R}_+ \to \mathbb{R}$ of the form $\chi(t) = At^p - Bt^r$, where $A = \frac{\kappa(1)-\varepsilon C(p)}{p}$, $B = \frac{1}{r}c(\varepsilon)C(r)$. The function χ attains its global maximum in the point $t_M = (\frac{pA}{rB})^{\frac{1}{r-p}}$. When we take $\rho = t_M$ and $\beta \in]0, \chi(t_M)]$, it is easy to see that the condition (4.16) is fulfilled.

From (F5) we have $\Psi(u) \leq \frac{1}{p}\langle A(u), u\rangle + c^\star\|u\|_p^p - c^\star\|u\|_\alpha^\alpha$. If we fix an element $v \in H \setminus \{0\}$ and in place of u we put tv, then we have

$$\Psi(tv) \leq (\frac{1}{p}\langle A(v), v\rangle + c^\star\|v\|_p^p)t^p - c^\star t^\alpha\|v\|_\alpha^\alpha.$$

From this we see that if t is large enough, $tv \notin B_\rho(0)$ and $\Psi(tv) < 0$. So, the condition (4.17) is satisfied and Corollary 2.19 assures the existence of a nontrivial critical point of Ψ.

Now when we use (2) in Theorem 4.9, from conditions (F1), (F2'), (F3), and (F4), we get that the function Ψ satisfies the condition $(CPS)_c$ for every $c > 0$. Now, we use Theorem 2.28, which assures the existence of a nontrivial critical point for the function Ψ. It is sufficient to prove only the relation (4.17), because (4.16) is proved in the same way.

To prove the relation (4.17) we fix an element $u \in X$ and we define the function $h : (0, +\infty) \to \mathbb{R}$ by $h(t) = \frac{1}{t}F(x, t^{1/p}u) - C\frac{p}{\alpha-p}t^{\frac{\alpha}{p}-1}|u|^\alpha$. The function h is locally Lipschitz. We fix a number $t > 1$, and from the Lebourg's main value theorem follows the existence of an element $\tau \in (1, t)$ such that

$$h(t) - h(1) \in \partial_t h(\tau)(t-1),$$

where ∂_t denotes the generalized gradient of Clarke with respect to $t \in \mathbb{R}$. From the Chain Rules we have

$$\partial_t F(x, t^{1/p}u) \subset \frac{1}{p}\partial F(x, t^{1/p}u)t^{\frac{1}{p}-1}u.$$

Also we have

$$\partial_t h(t) \subset -\frac{1}{t^2}F(x, t^{1/p}u) + \frac{1}{t}\partial F(x, t^{1/p}u)t^{\frac{1}{p}-1}u - Ct^{\frac{\alpha}{p}-2}|u|^\alpha.$$

Therefore,

$$\begin{aligned} h(t) - h(1) &\subset \partial_t h(\tau)(t-1) \\ &\subset -\frac{1}{t^2}\left[F(x, t^{1/p}u) - t^{1/p}u\partial F(x, t^{1/p}u) + C|t^{1/p}u|^\alpha\right](t-1). \end{aligned}$$

Using the relation (F2'), we obtain that $h(t) \geq h(1)$; therefore,

$$\frac{1}{t}F(x, t^{1/p}u) - C\frac{p}{\alpha - p}t^{\frac{\alpha}{p}-1}|u|^\alpha \geq F(x,u) - C\frac{p}{\alpha - p}|u|^\alpha.$$

From this inequality, we get

$$F(x, t^{1/p}) \geq tF(x,u) + C\frac{p}{\alpha - p}[t^{\alpha/p} - t]|u|^\alpha, \tag{4.18}$$

for every $t > 1$ and $u \in \mathbb{R}$. Let us fix an element $u_0 \in X \setminus \{0\}$; then for every $t > 1$, we have

$$\begin{aligned} \Psi(t^{1/p}u_0) &= \frac{1}{p}\langle A(t^{1/p}u_0), t^{1/p}u_0\rangle - \int_{\mathbb{R}^N} F(x, t^{1/p}u_0(x))dx \\ &\leq \frac{t}{p}\langle Au_0, u_0\rangle - t\int_{\mathbb{R}^N} F(x, u_0(x))dx - C\frac{p}{\alpha - p}[t^{\alpha/p} - t]\|u_0\|_\alpha^\alpha. \end{aligned}$$

If t is sufficiently large, then for $v_0 = t^{1/p}u_0$ we have $\Psi(v_0) \leq 0$. This ends the proof. □

In general the inclusion $X \hookrightarrow L^r(\Omega)$ is not compact and we impose some invariant properties. So, let G be the compact topological group $O(N)$ or a subgroup of $O(N)$. We suppose that G acts continuously and linear isometrically on the Banach space X. We denote by

$$X^G = \{u \in H : gx = x \text{ for all } g \in G\}$$

the fixed point set of the action G on X. It is well known that X^G is a closed subspace of X. In several applications the condition (CE) is replaced by the condition

(CEG) The embeddings $X^G \hookrightarrow L^r(\mathbb{R}^N)$ are compact $(p < r < p^\star)$.

We suppose that the potential $a : X \to \mathbb{R}$ of the operator $A : X \to X^\star$ is G-invariant and the next condition for the function $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ holds:

(F6) For a.e. $x \in \mathbb{R}^N$ and for every $g \in G, s \in \mathbb{R}$ we have $f(gx, s) = f(x, s)$.

If we use the Principle of Symmetric Criticality for locally Lipschitz functions, see Remark 3.9, from the above theorem we obtain the following corollary, which is useful in the applications.

Corollary 4.11. *We suppose that the potential $a : X \to \mathbb{R}$ is G-invariant and (F6) is satisfied. Then the following assertions hold.*

(a) *If the conditions (CEG),(F1)-(F5) are fulfilled, then problem* (4.4) *has a nontrivial solution.*
(b) *If the conditions (CEG), (F1), (F2'), F3), and (F4) are fulfilled, then problem* (4.4) *has a nontrivial solution.*

5. **A multiplicity result for hemivariational inequalities**

In this section we state a multiplicity result for a particular hemivariational inequality. These results appear in the paper of Faraci, Iannizzotto, Lisei and Varga [15]. Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be an unbounded domain with smooth boundary $\partial\Omega$, $p \in]1, N[$ be a real number. Throughout in this section X denotes a separable, uniformly convex Banach space with strictly convex topological dual; moreover, we assume that the condition (CE) holds. In the sequel, X will denote a (real) Banach space (with norm $\|\cdot\|$) and $X^\star$ its topological dual (with norm $\|\cdot\|_\star$); by $\langle\cdot,\cdot\rangle$ we will denote the duality pairing between $X^\star$ and X.

The next Lemma introduces the *duality mapping* on the space X, related to the weight function $t \to t^{p-1}$:

Lemma 5.1. ([8], Propositions 2.2.2, 2.2.4) *Let X be a Banach space with strictly convex dual, $p > 1$ a real number. Then, there exists a mapping $A : X \to X^\star$ such that for all $x \in X$*

(DM_1)**:** $\|A(x)\|_\star = \|x\|^{p-1}$;
(DM_2)**:** $\langle A(x), x\rangle = \|A(x)\|_\star \|x\|$.

Moreover, for all $x, y \in X$

$$\langle A(x) - A(y), x - y\rangle \geq \left(\|x\|^{p-1} - \|y\|^{p-1}\right)(\|x\| - \|y\|).$$

The functional $x \to \dfrac{\|x\|^p}{p}$ is Gâteaux differentiable with derivative A.

Let $F : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz, non-zero function such that $F(0) = 0$ and

(F)**:** there exist $k > 0$, $q \in]0, p-1[$ such that $|\xi| \leq k|s|^q$ for all $s \in \mathbb{R}$, $\xi \in \partial F(s)$.

Let $b : \Omega \to \mathbb{R}$ be a non-negative, not zero function such that

(b)**:** $b \in L^1(\Omega) \cap L^\infty(\Omega) \cap L^\nu(\Omega)$, where $\nu = \dfrac{r}{r-(q+1)}$.

The problem studied in this section is the following.

Find $u_0 \in X$, $\lambda > 0$ such that

$$(P_\lambda) \qquad \langle A(u - u_0), v\rangle + \lambda \int_\Omega b(x) F^\circ(u(x); -v(x))dx \geq 0 \ \text{ for all } v \in X$$

Our approach to problem (P_λ) is variational. Given $u_0 \in X$ and $\lambda > 0$, the energy functional $I : X \to \mathbb{R}$ associated to the problem (P_λ) is defined by

$$I(u) = \frac{\|u - u_0\|^p}{p} - \lambda J(u).$$

As in Proposition 4.6 follows that the critical points of I are solutions of the problem (P_λ).

Let us define the functional $J : X \to \mathbb{R}$ by

$$J(u) = \int_\Omega b(x)F(u(x))dx$$

for all $u \in X$.

Lemma 5.2. *The functional J is well-defined, locally Lipschitz, sequentially weakly continuous and satisfies*

$$J^\circ(u; v) \le \int_\Omega b(x)F^\circ(u(x); v(x))dx \quad \textit{for all } u, v \in X.$$

Proof. In the same way as in Proposition 4.4 follows that J is locally Lipschitz and from Proposition 4.5 follows the inequality. We prove now that J is sequentially weakly continuous: let $\{u_n\}$ be a sequence in X, weakly convergent to some $\bar{u} \in X$. Due to condition (CE), there is a subsequence, still denoted by $\{u_n\}$, such that $\|u_n - \bar{u}\|_r \to 0$; then, by well-known results, we may assume that $u_n \to \bar{u}$ a.e. in Ω and there exists a positive function $g \in L^r(\Omega)$ such that $|u_n(x)| \le g(x)$ for all $n \in \mathbb{N}$ and almost all $x \in \Omega$. By the Lebesgue Theorem, $\{J(u_n)\}$ tends to $J(\bar{u})$. ☐

Before to prove the main result of this section we recall two results.

Theorem 5.3. ([60, Theorem 1 and Remark 1]) *Let X be a topological space, Λ a real interval, and $f : X \times \Lambda \to \mathbb{R}$ a function satisfying the following conditions:*

- (A_1) *for every $x \in X$, the function $f(x, \cdot)$ is quasi-concave and continuous;*
- (A_2) *for every $\lambda \in \Lambda$, the function $f(\cdot, \lambda)$ is lower semicontinuous and each of its local minima is a global minimum;*
- (A_3) *there exist $\rho_0 > \sup_\Lambda \inf_X f$ and $\lambda_0 \in \Lambda$ such that $\{x \in X \ : \ f(x, \lambda_0) \le \rho_0\}$ is compact.*

Then,

$$\sup_\Lambda \inf_X f = \inf_X \sup_\Lambda f.$$

Theorem 5.4. ([65, Theorem 2], [13, Lemma 1]) *Let X be a uniformly convex Banach space, with strictly convex topological dual, M a sequentially weakly closed, non-convex subset of X.*

Then, for any convex, dense subset S of X, there exists $x_0 \in S$ such that the set

$$\{y \in M \ : \ \|y - x_0\| = d(x_0, M)\}$$

has at least two points.

The main result of this section is the following and appear in the paper of Faraci, Iannizzotto, Lisei, and Varga [15].

Theorem 5.5. (Faraci-Iannizzotto-Lisei-Varga [15]) *Let $\Omega \subset \mathbb{R}^N$ be an unbounded domain with smooth boundary $\partial\Omega$ ($N \geq 2$), $p \in]1, N[$ be a real number, X be a separable, uniformly convex Banach space with strictly convex topological dual, satisfying (E). Let $F : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz, non-zero function satisfying $F(0) = 0$ and (F), $b : \Omega \to \mathbb{R}$ be a non-negative, not zero function satisfying (b).*

Then, for every $\sigma \in]\inf_X J, \sup_X J[$ and every $u_0 \in J^{-1}(]-\infty, \sigma[)$ one of the following conditions is true:

(B_1) *there exists $\lambda > 0$ such that the problem (P_λ) has at least three solutions in X;*

(B_2) *there exists $v \in J^{-1}(\sigma)$ such that, for all $u \in J^{-1}([\sigma, +\infty[)$, $u \neq v$,*

$$\|u - u_0\| > \|v - u_0\|.$$

Proof. Fix σ and u_0 as in the thesis, and assume that (B_1) does not hold: we shall prove that (B_2) is true.

Putting $\Lambda = [0, +\infty[$ and endowing X with the weak topology, we define the function $f : X \times \Lambda \to \mathbb{R}$ by

$$f(u, \lambda) = \frac{\|u - u_0\|^p}{p} + \lambda(\sigma - J(u)),$$

which satisfies all the hypotheses of Theorem 5.3. Indeed, conditions (A_1), (A_3) are trivial.

In examining condition (A_2), let $\lambda \geq 0$ be fixed: we first observe that, by Lemma 5.2, the functional $f(\cdot, \lambda)$ is sequentially weakly lower semicontinuous (l.s.c.).

Moreover, $f(\cdot, \lambda)$ is coercive: indeed, for all $u \in X$ we have

$$f(u, \lambda) \geq \|u\|^p \left(\frac{\|u - u_0\|^p}{p\ \|u\|^p} - \lambda\ k\ c_r^{q+1} \|b\|_\nu \|u\|^{(q+1)-p} \right) + \lambda\sigma,$$

and the latter goes to $+\infty$ as $\|u\| \to +\infty$. As a consequence of the Eberlein-Smulyan theorem, the outcome is that $f(\cdot, \lambda)$ is weakly l.s.c..

We need to check that every local minimum of $f(\cdot, \lambda)$ is a global minimum. Arguing by contradiction, suppose that $f(\cdot, \lambda)$ admits a local, non global minimum; besides, being coercive, it has a global minimum too, that is, it has two strong local minima.

We now prove that $f(\cdot,\lambda)$ fulfills the Palais-Smale condition: let $\{u_n\}$ be a sequence satisfying (PS_1), (PS_2). From (PS_1), together with the coercivity of $f(\cdot,\lambda)$, it follows that $\{u_n\}$ is bounded, hence we can find a subsequence, which we still denote $\{u_n\}$, weakly convergent to a point $\bar{u} \in X$. By condition (CE) we can choose $\{u_n\}$ to be convergent to $\bar{u}$ with respect to the norm of $L^r(\Omega)$.

Fix $\varepsilon > 0$. As the sequence $\{\varepsilon_n\}$ from (PS_2) tends to 0, for $n \in \mathbb{N}$ big enough we have

$$\varepsilon_n \|u_n - \bar{u}\| < \frac{\varepsilon}{2},$$

so, from (PS_2) and Lemma 5.2 it follows

$$\begin{aligned} 0 &\leq f^\circ(u_n,\lambda;\bar{u}-u_n) + \frac{\varepsilon}{2} \\ &\leq \langle A(u_n-u_0), \bar{u}-u_n\rangle + \lambda \int_\Omega b(x) F^\circ(u_n(x); u_n(x)-\bar{u}(x))dx + \frac{\varepsilon}{2} \end{aligned}$$

($f^\circ(\cdot,\lambda;\cdot)$ denotes the generalized directional derivative of the locally Lipschitz functional $f(\cdot,\lambda)$). Moreover, for n big enough

$$\begin{aligned} \left| \int_\Omega b(x) F^\circ(u_n(x); u_n(x)-\bar{u}(x))dx \right| &\leq k \int_\Omega b(x)|u_n(x)|^q |u_n(x)-\bar{u}(x)|dx \\ &\leq k\, c_r^q \|b\|_\nu \|u_n\|^q \|u_n - \bar{u}\|_r < \frac{\varepsilon}{2\lambda}. \end{aligned}$$

Hence

$$\langle A(u_n-u_0), u_n-\bar{u}\rangle < \varepsilon$$

for $n \in \mathbb{N}$ big enough. On the other hand, $\langle A(\bar{u}-u_0), u_n-\bar{u}\rangle$ tends to zero as n goes to infinity. From the previous computations, it follows that

$$\limsup_n \langle A(u_n-u_0) - A(\bar{u}-u_0), u_n-\bar{u}\rangle \leq 0. \tag{5.1}$$

Applying Lemma 5.1, we obtain that

$$\begin{aligned} &\langle A(u_n-u_0) - A(\bar{u}-u_0), u_n-\bar{u}\rangle \\ &\geq \left(\|u_n-u_0\|^{p-1} - \|\bar{u}-u_0\|^{p-1}\right)\left(\|u_n-u_0\| - \|\bar{u}-u_0\|\right) \geq 0. \end{aligned}$$

From the previous inequality and (5.1), we deduce that $\|u_n - u_0\| \to \|\bar{u}-u_0\|$ and this, together with the weak convergence, implies that $\{u_n\}$ tends to $\bar{u}$ in X: that is, the Palais-Smale condition is fulfilled.

Then, we can apply Theorem 2.20, deducing that $f(\cdot,\lambda)$ (or equivalently the energy functional I) admits a third critical point: by Proposition 4.6, the inequality (P_λ) should have at least three solutions in X, against our assumption. Thus, condition (A_2) is fulfilled.

Now Theorem 5.3 assures that

$$\sup_{\lambda\in\Lambda}\inf_{u\in X} f(u,\lambda) = \inf_{u\in X}\sup_{\lambda\in\Lambda} f(u,\lambda) =: \alpha. \tag{5.2}$$

Notice that the function $\lambda \to \inf_{u\in X} f(u,\lambda)$ is upper semicontinuous in Λ, and tends to $-\infty$ as $\lambda \to +\infty$ (since $\sigma < \sup_X J$): hence, it attains its supremum in $\lambda^\star \in \Lambda$, that is

$$\alpha = \inf_{u\in X}\left(\frac{\|u-u_0\|^p}{p} + \lambda^\star(\sigma - J(u))\right). \tag{5.3}$$

The infimum in the right hand side of (5.2) is easily determined as

$$\alpha = \inf_{u\in J^{-1}([\sigma,+\infty[)} \frac{\|u-u_0\|^p}{p} = \frac{\|v-u_0\|^p}{p}$$

for some $v \in J^{-1}([\sigma,+\infty[)$.

It is easily seen that $v \in J^{-1}(\sigma)$. Hence

$$\alpha = \inf_{u\in J^{-1}(\sigma)} \frac{\|u-u_0\|^p}{p} \qquad \text{(in particular } \alpha > 0). \tag{5.4}$$

By (5.3) and (5.4) it follows that

$$\inf_{u\in X}\left(\frac{\|u-u_0\|^p}{p} - \lambda^\star J(u)\right) = \inf_{u\in J^{-1}(\sigma)}\left(\frac{\|u-u_0\|^p}{p} - \lambda^\star J(u)\right). \tag{5.5}$$

We deduce that $\lambda^\star > 0$: if $\lambda^\star = 0$, indeed, (5.5) would become $\alpha = 0$, against (5.4).

Now we can prove (B_2). Arguing by contradiction, let $w \in J^{-1}([\sigma,+\infty[)\setminus\{v\}$ be such that $\|w-u_0\| = \|v-u_0\|$. As above, we have that $w \in J^{-1}(\sigma)$, and so both w and v are global minima of the functional I (for $\lambda = \lambda^\star$) over $J^{-1}(\sigma)$, hence, by (5.5), over X. Thus, applying Theorem 2.20, we obtain that I has at least three critical points, against the assumption that (B_1) does not hold (recall that $\lambda^\star$ is positive). This concludes the proof. $\square$

In the next Corollary, the alternative of Theorem 5.5 is resolved, under a very general assumption on the functional J, and so we are led to a multiplicity result for the hemivariational inequality (P_λ) (for suitable data u_0, λ).

Corollary 5.6. (Faraci-Iannizzotto-Lisei-Varga [15]) *Let Ω, p, X, F, b be as in Theorem 5.5 and let S be a convex, dense subset of X. Moreover, let $J^{-1}([\sigma,+\infty[)$ be not convex for some $\sigma \in]\inf_X J, \sup_X J[$.*
Then, there exist $u_0 \in J^{-1}(]-\infty,\sigma[) \cap S$ and $\lambda > 0$ such that problem (P_λ) admits at least three solutions in X.

Proof. Since J is sequentially weakly continuous (Lemma 5.2), the set $M = J^{-1}([\sigma,+\infty[)$ is sequentially weakly closed.

By Theorem 5.4 we get that, for some $u_0 \in S$, there exist two distinct points $v_1, v_2 \in M$ satisfying

$$\|v_1 - u_0\| = \|v_2 - u_0\| = \mathrm{dist}(u_0, M).$$

Clearly $u_0 \notin M$, that is, $J(u_0) < \sigma$. In the framework of Theorem 5.5, condition (B_2) is false, so (B_1) must be true: there exists $\lambda > 0$ such that (P_λ) has at least three solutions in X. $\square$

6. **Applications**

6.1. **Existence results for a particular hemivariational inequality.** In this subsection we give some concrete applications of Theorem 4.10. In the first two examples we suppose that X is a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$.

Let $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be a measurable function as in the section 4, i.e. satisfies the conditions (F1), (F2), (F'2) and (F3)-(F5).

Application 1. We consider the function $V \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$ which satisfies the following conditions:

(a) $V(x) > 0$ for all $x \in \mathbb{R}^N$
(b) $V(x) \to +\infty$ as $|x| \to +\infty$.

Let X be the Hilbert space defined by

$$X = \{u \in H^1(\mathbb{R}^N) : \int (|\nabla u(x)|^2 + V(x)|u(x)|^2) dx < \infty\},$$

with the inner product

$$\langle u, v \rangle = \int (\nabla u \nabla v + V(x) uv) dx.$$

It is well known that if the conditions (a) and (b) are fulfilled then the inclusion $X \hookrightarrow L^2(\mathbb{R}^N)$ is compact, see [17], therefore the condition (CE) is satisfied.

Now we formulate the problem.

Find a positive $u \in X$ such that for every $v \in X$ we have

$$\int_{\mathbb{R}^N} (\nabla u \nabla v + V(x) uv) dx + \int_{\mathbb{R}^N} F_2^0(x, u(x); -v(x)) dx \geq 0. \tag{6.1}$$

We have the following result.

Corollary 6.1.
1. *If conditions (F1)-(F5) and (a)-(b) hold, then problem* (6.1) *has a nontrivial positive solution.*
2. *If conditions (F1),(F2'), (F3), (F4) and (a)-(b) hold, then problem* (6.1) *has a nontrivial positive solution.*

Proof. We replace the function f by $f_+ : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ defined by

$$f_+(x,u) = \begin{cases} f(x,u) & \text{if } u \geq 0; \\ 0, & \text{if } u < 0 \end{cases} \tag{6.2}$$

and use (2) in Theorem 4.10. ☐

Application 2. Now, we consider $Au := -\triangle u + |x|^2 u$ for $u \in D(A)$, where

$$D(A) := \{u \in L^2(\mathbb{R}^N) : Au \in L^2(\mathbb{R}^N)\}.$$

Here $|\cdot|$ denotes the Euclidian norm of $\mathbb{R}^N$. In this case the Hilbert space X is defined by

$$X = \{\ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\nabla u|^2 + |x|^2 u^2) dx < \infty\},$$

with the inner product

$$\langle u, v\rangle = \int_{\mathbb{R}^N} (\nabla u \nabla v + |x|^2 uv) dx.$$

The inclusion $X \hookrightarrow L^s(\mathbb{R}^N)$ is compact for $s \in [2, \frac{2N}{N-2})$, see Kavian [21, Exercise 20, pp. 278]. Therefore, the condition (CE) is satisfied.

Now, we formulate the next problem.

Find a positive $u \in X$ such that for every $v \in X$ we have

$$\int_{\mathbb{R}^N} (\nabla u \nabla v + |x|^2 uv) dx + \int_{\mathbb{R}^N} F_2^0(x, u(x); -v(x)) dx \geq 0. \tag{6.3}$$

Corollary 6.2.

1. *If conditions (F1)-(F5) hold, then problem* (6.3) *has a positive solution.*
2. *If conditions (F1),(F2'), (F3), and (F4) hold, then problem* (6.3) *has a positive solution.*

Application 3. In this example we suppose that G is a subgroup of the group $O(N)$. Let Ω be an unbounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and the elements of G leave Ω invariant, i.e. $g(\Omega) = \Omega$ for every $g \in G$. We suppose that Ω is compatible with G, see the book of Willem [67, Definition 1.22]. The action of G on $X = W_0^{1,p}(\Omega)$ is defined by

$$gu(x) := u(g^{-1}x).$$

The subspace of invariant function X^G is defined by

$$X^G := \{u \in X : gu = u,\ \forall g \in G\ \}.$$

The norm on X is defined by

$$\|u\| = \Big(\int_\Omega (|\nabla u|^p + |u|^p) dx\Big)^{1/p}.$$

If Ω is compatible with G, then the embeddings $X \hookrightarrow L^s(\Omega)$, with $p < s < p^\star$ are compact, see the paper of Kobayashi and Otani [22]. Therefore the condition (CEG) is satisfied.

We consider the potential $a : X \to \mathbb{R}$ defined by $a(u) = \frac{1}{p}\|u\|^p$. This function is G-invariant because the action of G is isometric on X. The Gateaux differential $A : X \to X^\star$ of the function $a : X \to \mathbb{R}$ is given by

$$\langle Au, v\rangle = \int_\Omega \left(|\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv\right) dx.$$

The operator A is homogeneous of degree $p-1$ and strongly monotone, because $p \geq 2$.

Now, we formulate the following problem.

Find $u \in X \setminus \{0\}$ such that for every $v \in X$ we have

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv\right) dx + \int_\Omega F_2^0(x, u(x); -v(x))dx \geq 0. \tag{6.4}$$

We have the following result.

Corollary 6.3. (a) *If conditions (F1)-(F6) are fulfilled, then problem* (6.4) *has a nontrivial symmetric solution.*

(b) *If conditions (F1), (F2'), (F3), (F4) and (F6) are fulfilled, then problem* (6.4) *has a nontrivial symmetric solution.*

6.2. **Multiplicity results for some hemivariational inequalities.** In this subsection we state a multiplicity result for a particular hemivariational inequality as application of Corollary 5.6. Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be an unbounded domain with smooth boundary $\partial\Omega$, $p \in]1, N[$ be a real number. As in Section 5, let $F : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz, non-zero function such that $F(0) = 0$ and

(F)**:** there exist $k > 0$, $q \in]0, p-1[$ such that $|\xi| \leq k|s|^q$ for all $s \in \mathbb{R}$, $\xi \in \partial F(s)$.

Let $b : \Omega \to \mathbb{R}$ be a non-negative, not zero function such that

(b)**:** $b \in L^1(\Omega) \cap L^\infty(\Omega) \cap L^\nu(\Omega)$, where $\nu = \dfrac{r}{r-(q+1)}$.

We suppose that F is not a quasi-concave function, that is:

(C)**:** there exists $\rho \in]\inf_\mathbb{R} F, \sup_\mathbb{R} F[$ such that $F^{-1}([\rho, +\infty[)$ is not convex.

6.2.1. *First application.* Let $V : \Omega \to \mathbb{R}$ be a continuous potential satisfying the following conditions:

(V_1) $\inf_\Omega V > 0$;

(V_2) for every $M > 0$ the set $\{x \in \Omega \ : \ V(x) \leq M\}$ has finite Lebesgue measure (note that in particular, condition (V_2) is fulfilled whenever V is coercive). We introduce the space

$$X = \left\{ u \in W^{1,p}(\Omega) \ : \ \int_\Omega (|\nabla u(x)|^p + V(x)|u(x)|^p)dx < \infty \right\}$$

endowed with the norm

$$\|u\| = \left(\int_\Omega (|\nabla u(x)|^p + V(x)|u(x)|^p)dx \right)^{\frac{1}{p}}.$$

With the definitions above, for all $u_0 \in X$, $\lambda > 0$, our problem (P_λ) reads as follows:

$$\int_\Omega (|\nabla(u(x) - u_0(x))|^{p-2}\nabla(u(x) - u_0(x)) \cdot \nabla v(x)$$
$$+V(x)|u(x) - u_0(x)|^{p-2}(u(x) - u_0(x))v(x))dx$$
$$+\lambda \int_\Omega b(x)F^\circ(u(x); -v(x))dx \geq 0 \text{ for all } v \in X.$$

We can state the following multiplicity result:

Corollary 6.4. *Let Ω, p, V, X be as above; F, b be as in Theorem 5.5 (with $\nu = p/(p-(q+1))$ in condition (b)); S be a convex, dense subset of X. Moreover, assume that condition (C) is satisfied. Then, there exist $u_0 \in S$ and $\lambda > 0$ such that the problem (P_λ) admits at least three solutions in X.*

Proof. We observe that X is a separable, uniformly convex Banach space with strictly convex topological dual, and that $C_c^\infty(\Omega) \subset X$; moreover, the conditions (V_1), (V_2) guarantee that the space X is compactly embedded in $L^p(\Omega)$ (see [4] for the case $p = 2$), so condition (E) is satisfied with $r = p$. Since b is not zero, there exist a point $x_0 \in \Omega$ and $R > 0$ such that

$$b_1 = \int_B b(x)dx > 0,$$

where B is the open ball centered in x_0 with radius R, contained in Ω.

By condition (C), we can assume, without loss of generality, that there exist real numbers $s_1 < s_2 < s_3$ such that $F(s_1), F(s_3) > \rho$, $F(s_2) < \rho$. Now we prove that the functional J admits a non-convex superlevel set. Choose $\varepsilon > 0$, $R_1 > R$ with

$$\|b\|_\infty M \text{meas}(A) < \varepsilon < b_1|F(s_i) - \rho| \quad (i = 1, 2, 3),$$

where $A = \{x \in \Omega : R < |x - x_0| < R_1\}$ and $M = \max\{|F(t)| : |t| \leq |s_i|,\ i = 1, 2, 3\}$. There exists $u_1 \in C_c^\infty(\Omega)$ such that

$$u_1(x) = \begin{cases} s_1 & \text{if } x \in B \\ 0 & \text{if } x \in \Omega \setminus (A \cup B) \end{cases}$$

and $\|u_1\|_\infty = |s_1|$; define, also, $u_2, u_3 \in C_c^\infty(\Omega)$ by putting $u_2 = (s_2/s_1)u_1$, $u_3 = (s_3/s_1)u_1$ (we assume $s_1 \neq 0$). Thus,

$$\begin{aligned} J(u_1) &= \int_B b(x)F(s_1)dx + \int_A b(x)F(u_1(x))dx \\ &\geq b_1F(s_1) - M\|b\|_\infty \text{meas}(A) \\ &\geq b_1F(s_1) - \varepsilon \\ &> b_1\rho. \end{aligned}$$

Analogously, we get

$$J(u_2) < b_1\rho, \ J(u_3) > b_1\rho.$$

Then, since u_2 lies on the segment joining u_1 and u_3, it is proved that $J^{-1}([b_1\rho, +\infty[)$ is not convex. An application of Corollary 5.6 yields the existence of a function $u_0 \in J^{-1}(]-\infty, b_1\rho[) \cap S$ and $\lambda > 0$ such that (P_λ) has at least three solutions in X. □

Example 6.5. In this example we prove the existence of a continuous function $g : \mathbb{R}^N \to \mathbb{R}$ and a positive λ such that the equation

$$(E_\lambda) \qquad -\Delta u + V(x)u = \lambda b(x)H(u-1)(\ln u - 1) + g(x) \quad \text{ in } \mathbb{R}^N$$

(where V is a positive and coercive potential and H is the Heaviside function) admits at least three solutions in $H^2(\mathbb{R}^N)$ More precisely, let $V : \mathbb{R}^N \to \mathbb{R}$ be a continuous, positive and coercive function, X be as above with $p = 2 < N$, b be as in Theorem 5.5. Recall that the Heaviside function $H : \mathbb{R} \to \mathbb{R}$ is defined by

$$H(s) = \begin{cases} 0 & \text{if } s \leq 0 \\ 1 & \text{if } s > 0 \end{cases},$$

and put

$$f(s) = H(s-1)(\ln s - 1) \quad \text{for all } s \in \mathbb{R}$$

(with obvious meaning for $s \leq 0$). We denote, for all $s \in \mathbb{R}$,

$$f_-(s) = \lim_{\delta \to 0^+} \inf_{|t-s|<\delta} f(t), \quad f_+(s) = \lim_{\delta \to 0^+} \sup_{|t-s|<\delta} f(t).$$

Following Chang [9], for all continuous $g : \mathbb{R}^N \to \mathbb{R}$ and $\lambda > 0$, by a weak solution of (E_λ) we mean a function $u \in H^2(\mathbb{R}^N)$ such that, for almost every $x \in \mathbb{R}^N$,

$$-\Delta u(x) + V(x)u(x) \in g(x) + \lambda b(x)[f_-(u(x)), f_+(u(x))]. \tag{6.5}$$

It is easily seen that the function $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(s) = \int_0^s f(t)dt$$

is locally Lipschitz and satisfies the condition (F) with arbitrary $q \in]0,1[$ for k big enough; moreover, for all $\rho \in]2-e,0]$ the set $F^{-1}([\rho,+\infty[)$ is not convex, so condition (C) is fulfilled. Taking $S = C_c^\infty(\mathbb{R}^N)$, we can apply Corollary 6.4: thus, we find $u_0 \in S$ and $\lambda > 0$ such that the hemivariational inequality

$$\int_{\mathbb{R}^N} (\nabla(u(x) - u_0(x)) \cdot \nabla v(x) + V(x)(u(x) - u_0(x))v(x))\, dx +$$

$$+\lambda \int_{\mathbb{R}^N} b(x)F^\circ(u(x); -v(x))dx \geq 0 \quad \text{for all } v \in X$$

admits at least three solutions in X. Let u be one of these: by standard regularity results, we get $u \in H_0^1(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$; arguing as in [9], we find that u satisfies (6.5) with

$$g(x) = -\Delta u_0(x) + V(x)u_0(x) \quad \text{for all } x \in \mathbb{R}^N.$$

Thus, (E_λ) has at least three weak solutions.

6.2.2. *Second application.* Here we give an application of Corollary 5.6 combined with the Principle of Symmetric Criticality for locally Lipschitz functions. Let Ω be an unbounded domain in $\mathbb{R}^N$ $(N > 2)$ with smooth boundary, such that $0 \in \Omega$, and G be a closed subgroup of $O(N)$ which leaves Ω invariant, i.e. $g(\Omega) = \Omega$ for all $g \in G$. We assume that Ω is *compatible* with G, that is, there exists $r > 0$ such that

$$m(x, r, G) \to \infty \text{ as } \operatorname{dist}(x, \Omega) \leq r, \quad |x| \to \infty,$$

where

$$m(x, r, G) = \sup\{n \in \mathbb{N} : \exists\, g_1, g_2, \cdots g_n \in G \text{ s.t. } B(g_i x, r) \cap B(g_j x, r) = \emptyset \text{ if } i \neq j\}.$$

We consider the space $X = W_0^{1,p}(\Omega)$ endowed with the norm

$$\|u\| = \left(\int_\Omega (|\nabla u(x)|^p + |u(x)|^p)dx\right)^{\frac{1}{p}}.$$

Our problem is the following: *For $u_0 \in X$, $\lambda > 0$, find $u \in X$ such that*

$$\int_\Omega (|\nabla(u(x) - u_0(x))|^{p-2}\nabla(u(x) - u_0(x)) \cdot \nabla v(x)$$

$$+|u(x) - u_0(x)|^{p-2}(u(x) - u_0(x))v(x))dx$$

$$+\lambda \int_\Omega b(x)F^\circ(u(x); -v(x))dx \geq 0 \quad \text{for all } v \in X.$$

We define the action of the group G over the space X as follows:

$$gu(x) = u(g^{-1}x) \text{ for all } g \in G,\ u \in X, x \in \Omega.$$

We observe that G acts *linearly* and *isometrically* on X, i.e., the action $G \times X \to X$ which maps (g, u) into gu is continuous and, for every $g \in G$, the map $u \to gu$ is linear and $||gu|| = ||u||$ for every $u \in X$. The group G induces an action of the same type on the dual space $X^\star$ defined by $\langle gu^\star, u\rangle = \langle u^\star, g^{-1}u\rangle$ for every $g \in G$, $u \in X$ and $u^\star \in X^\star$.

We introduce the set

$$X^G = \{u \in X \ : \ gu = u \text{ for all } g \in G\}$$

of the fixed points of X under the action of G, and observe that X^G is a Banach space (which inherits all the properties of X), whose dual coincides with the fixed point set of $X^\star$ under the action of G, denoted $(X^G)^\star$. From [22, Proposition 4.2], follows that X^G is compactly embedded in $L^r(\Omega)$ for all $r \in]p, p^\star[$.

We have the following result.

Corollary 6.6. *Let Ω, p, X, G be as above, S be a convex, dense subset of X^G. Let F be as in Theorem 5.5 and satisfying condition (C). Also, let $b : \Omega \to \mathbb{R}$ be a non-negative, G-invariant function (that is, $b(gx) = b(x)$ for all $g \in G$, $x \in \Omega$) satisfying condition (b) and such that*

$$\int_B b(x)dx > 0 \qquad (B = B(0,R) \textit{ for some } R > 0 \textit{ small enough}).$$

Then, there exist $u_0 \in S$ and $\lambda > 0$ such that the problem (P_λ) admits at least three solutions lying in X^G.

Proof. We are going to apply Corollary 5.6 to the space X^G and to the functional $J|_{X^G}$: first, we note that X^G is separable and uniformly convex, and that $(X^G)^\star$ is strictly convex (as a subspace of $X^\star$); moreover, the space X^G satisfies condition (CEG) for any $r \in]p, p^\star[$.

In order to see that $J|_{X^G}$ admits a non-convex superlevel set, we argue as in the proof of Corollary 6.4, putting $x_0 = 0$ and choosing the functions $u_1, u_2, u_3 \in C_c^\infty(\Omega)$ radially symmetric (so, in particular, lying in X^G).

Thus, by Corollary 5.6 , there exist $u_0 \in S$ and $\lambda > 0$ such that the energy functional $I|_{X^G}$ has at least three critical points in X^G.

Now we prove that I is G-invariant on X. Let $g \in G$ and $u \in X$; recalling that $u_0 \in X^G$, G acts isometrically over X and b is G-invariant, we obtain the following equalities:

$$\begin{aligned} I(gu) &= \frac{1}{p}\|gu - u_0\|^p - \int_\Omega b(x)F(gu(x))dx \\ &= \frac{1}{p}\|g(u - u_0)\|^p - \int_\Omega b(x)F(u(g^{-1}x))dx \\ &= \frac{1}{p}\|u - u_0\|^p - \int_\Omega b(y)F(u(y))dy = I(u). \end{aligned}$$

Then, applying Theorem 6.13, we deduce that the critical points of $I|_{X^G}$ are actually critical points of I. We can conclude that problem (P_λ) has at least three symmetric solutions. $\square$

Next we give an example, in order to highlight the generality of our hypotheses:

Example 6.7. Put $N = 3$ and define the unbounded domain

$$\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ |x_3| < x_1^2 + x_2^2 + 1\}.$$

Then, consider the closed subgroup of $O(3)$ defined by $G = O(2) \times \{\mathrm{id}\}$, whose action on $X = W_0^{1,p}(\Omega)$ $(1 < p < N)$ is expressed as follows: for all $g = (\tilde{g}, \mathrm{id}) \in G$, and for all $u \in X$, $(x_1, x_2, x_3) \in \Omega$ we set

$$gu(x_1, x_2, x_3) = u(\tilde{g}^{-1}(x_1, x_2), x_3).$$

It is easily seen that Ω is G-invariant and compatible with G, and that the subspace X^G of the fixed points of X under the action of G is the set of all $u \in X$ with a *cylindric symmetry*, that is,

$$u(x_1, x_2, x_3) = u(y_1, y_2, x_3) \ \text{ if } \ x_1^2 + x_2^2 = y_1^2 + y_1^2.$$

Let $q \in]0, p-1[$ be a real number, $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(s) = 1 - \left||s|^{q+1} - 1\right| \qquad \text{for all } s \in \mathbb{R}.$$

It is easily seen that F is a locally Lipschitz function, satisfying $F(0) = 0$ and conditions (F) (with $k = q + 1$) and (C) (for all $\rho \in]0, 1]$).

Moreover, we consider a non-negative function $b : \Omega \to \mathbb{R}$, having a cylindric symmetry and satisfying condition (b) and we assume that b is positive in a neighborhood of 0.

In such a setting, Corollary 6.6 applies: thus, there exist $u_0 \in X^G$, $\lambda > 0$ such that the hemivariational inequality (P_λ) admits at least three solutions, and each of them has a cylindric symmetry.

6.3. **Some differential inclusion problems in $\mathbb{R}^N$.** In this subsection we give two applications for some differential inclusions problems. The first application is a differential inclusion problem with two parameters. This result appears in the paper of Kristály, Marzantowicz and Varga [28].

Let $p > 2$ and $F : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function such that

($\tilde{\mathbf{F}}$1) $\displaystyle\lim_{t \to 0} \frac{\max\{|\xi| : \xi \in \partial F(t)\}}{|t|^{p-1}} = 0;$

($\tilde{\mathbf{F}}$2) $\displaystyle\limsup_{|t| \to +\infty} \frac{F(t)}{|t|^p} \leq 0;$

($\tilde{\mathbf{F}}3$) There exists $\tilde{t} \in \mathbb{R}$ such that $F(\tilde{t}) > 0$, and $F(0) = 0$.

Here we study the differential inclusion problem

$$(\tilde{P}_{\lambda,\mu}) \qquad \begin{cases} -\triangle_p u + |u|^{p-2}u \in \lambda\alpha(x)\partial F(u(x)) + \mu\beta(x)\partial G(u(x)) & \text{on } \mathbb{R}^N, \\ u(x) \to 0 \text{ as } |x| \to \infty, \end{cases}$$

where $p > N \geq 2$, the numbers λ, μ are positive, and $G : \mathbb{R} \to \mathbb{R}$ is any locally Lipschitz function. Furthermore, we assume that $\beta \in L^1(\mathbb{R}^N)$ is any function, and

($\tilde{\alpha}$) $\alpha \in L^1(\mathbb{R}^N) \cap L^\infty_{\text{loc}}(\mathbb{R}^N)$, $\alpha \geq 0$, and $\sup_{R>0} \operatorname{essinf}_{|x|\leq R}\alpha(x) > 0$.

The functional space where our solutions are going to be sought is the usual Sobolev space $W^{1,p}(\mathbb{R}^N)$, endowed with the norm

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u(x)|^p + \int_{\mathbb{R}^N} |u(x)|^p\right)^{1/p}.$$

Definition 6.8. We say that $u \in W^{1,p}(\mathbb{R}^N)$ is a solution of problem $(\tilde{P}_{\lambda,\mu})$, if there exist $\xi_F(x) \in \partial F(u(x))$ and $\xi_G(x) \in \partial G(u(x))$ for almost every $x \in \mathbb{R}^N$ such that for all $v \in W^{1,p}(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv)dx = \lambda \int_{\mathbb{R}^N} \alpha(x)\xi_F v dx + \mu \int_{\mathbb{R}^N} \beta(x)\xi_G v dx. \tag{6.6}$$

Remark 6.9. (a) The terms in the right hand side of (6.6) are well-defined. Indeed, due to Morrey's embedding theorem, i.e., $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ is continuous ($p > N$), we have $u \in L^\infty(\mathbb{R}^N)$. Thus, there exists a compact interval $I \subset \mathbb{R}$ such that $u(x) \in I$ for a.e. $x \in \mathbb{R}^N$. Since the set-valued mapping ∂F is upper-semicontinuous, the set $\partial F(I) \subset \mathbb{R}$ is bounded; let $C_F = \sup |\partial F(I)|$. Therefore,

$$|\int_{\mathbb{R}^N} \alpha(x)\xi_F v dx| \leq C_F\|\alpha\|_{L^1}\|v\|_\infty < \infty.$$

Similar argument holds for the function G.

(b) Since $p > N$, any element $u \in W^{1,p}(\mathbb{R}^N)$ is homoclinic, i.e., $u(x) \to 0$ as $|x| \to \infty$, see Brézis [7, Théorème IX.12].

Remark 6.10. An upper bound for the embedding constant c_∞ of $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$, is $2p(p-N)^{-1}$ (see [7]), i.e. $c_\infty \leq 2p(p-N)^{-1}$.

Remark 6.11. Every function $u \in W^{1,p}(\mathbb{R}^N)$ ($p > N$) admits a continuous representation, see [7, p. 166]; in the sequel, we will replace u by this element.

Note that no hypothesis on the growth of G is assumed; therefore, the last term in $(\tilde{P}_{\lambda,\mu})$ may have an arbitrary growth. However, assumption ($\tilde{\alpha}$) together with ($\tilde{\mathbf{F}}3$) guarantee the existence of non-trivial solutions for $(\tilde{P}_{\lambda,\mu})$. The embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ is continuous (due to Morrey's theorem ($p > N$)), bit it is not compact. We overcome this gap by introducing the subspace of radially symmetric functions of $W^{1,p}(\mathbb{R}^N)$. The action of the orthogonal group $O(N)$ on $W^{1,p}(\mathbb{R}^N)$ can be defined by $(gu)(x) = u(g^{-1}x)$, for every $g \in O(N)$, $u \in W^{1,p}(\mathbb{R}^N)$, $x \in \mathbb{R}^N$. It is clear that this group acts linearly and isometrically; in particular $\|gu\| = \|u\|$ for every $g \in O(N)$ and $u \in W^{1,p}(\mathbb{R}^N)$.

We denote by

$$W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N) = \{u \in W^{1,p}(\mathbb{R}^N) : gu = u \text{ for all } g \in O(N)\},$$

the subspace of radially symmetric functions of $W^{1,p}(\mathbb{R}^N)$.

We have the following result, which is contained in the paper of Kristály [30].

Proposition 6.12. (Kristály [30]) *The embedding* $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ *is compact whenever* $2 \leq N < p < \infty$.

Proof. Let u_n be a bounded sequence in $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$. Up to a subsequence, $u_n \rightharpoonup u$ in $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ for some $u \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$. Let $\rho > 0$ be an arbitrarily fixed number. Due to the radially symmetric properties of u and u_n, we have

$$\|u_n - u\|_{W^{1,p}(B_N(g_1 y,\rho))} = \|u_n - u\|_{W^{1,p}(B_N(g_2 y,\rho))} \tag{6.7}$$

for every $g_1, g_2 \in O(N)$ and $y \in \mathbb{R}^N$. For a fixed $y \in \mathbb{R}^N$, we can define

$$\begin{aligned} m(y,\rho) = \sup\{n \in \mathbb{N} \quad : \quad & \exists g_i \in O(N),\ i \in \{1,...,n\} \text{ such that} \\ & B_N(g_i y,\rho) \cap B_N(g_j y,\rho) = \emptyset,\ \forall\ i \neq j\}. \end{aligned}$$

By virtue of (6.7), for every $y \in \mathbb{R}^N$ and $n \in \mathbb{N}$, we have

$$\|u_n - u\|_{W^{1,p}(B_N(y,\rho))} \leq \frac{\|u_n - u\|_{W^{1,p}}}{m(y,\rho)} \leq \frac{\sup_{n\in\mathbb{N}} \|u_n\|_{W_{1,p}} + \|u\|_{W^{1,p}}}{m(y,\rho)}.$$

The right hand side does not depend on n, and $m(y,\rho) \to +\infty$ whenever $|y| \to +\infty$ (ρ is kept fixed, and $N \geq 2$). Thus, for every $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that for every $y \in \mathbb{R}^N$ with $|y| \geq R_\varepsilon$ one has

$$\|u_n - u\|_{W^{1,p}(B_N(y,\rho))} < \varepsilon(2S_\rho)^{-1} \quad \text{for every } n \in \mathbb{N}, \tag{6.8}$$

where $S_\rho > 0$ is the embedding constant of $W^{1,p}(B_N(0,\rho)) \hookrightarrow C^0(B_N[0,\rho])$. Moreover, we observe that the embedding constant for $W^{1,p}(B_N(y,\rho)) \hookrightarrow C^0(B_N[y,\rho])$ can be chosen S_ρ as well, *independent* of the position of the point $y \in \mathbb{R}^N$. This fact can be concluded either by a simple translation of the functions $u \in W^{1,p}(B_N(y,\rho))$ into $B_N(0,\rho)$, i.e. $\tilde{u}(\cdot) = u(\cdot - y) \in W^{1,p}(B_N(0,\rho))$ (thus $\|u\|_{W^{1,p}(B_N(y,\rho))} = \|\tilde{u}\|_{W^{1,p}(B_N(0,\rho))}$ and $\|u\|_{C^0(B_N[y,\rho])} = \|\tilde{u}\|_{C^0(B_N[0,\rho])}$); or, by the invariance with respect to rigid motions of the cone property of the balls $B_N(y,\rho)$ when ρ is kept fixed.

Thus, in view of (6.8), one has that

$$\sup_{|y|\geq R_\varepsilon} \|u_n - u\|_{C^0(B_N[y,\rho])} \leq \varepsilon/2 \quad \text{for every } n \in \mathbb{N}. \tag{6.9}$$

On the other hand, since $u_n \rightharpoonup u$ in $W^{1,p}_{\rm rad}(\mathbb{R}^N)$, then in particular, by Rellich theorem it follows that $u_n \to u$ in $C^0(B_N[0,R_\varepsilon])$, i.e., there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\|u_n - u\|_{C^0(B_N[0,R_\varepsilon])} < \varepsilon \quad \text{for every } n \geq n_\varepsilon. \tag{6.10}$$

Combining (6.9) with (6.10), one concludes that $\|u_n - u\|_{L^\infty} < \varepsilon$ for every $n \geq n_\varepsilon$, i.e., $u_n \to u$ in $L^\infty(\mathbb{R}^N)$. This ends the proof. □

An alternate proof of Proposition 6.12. Lions [34, Lemme II.1] provided us with a Strauss-type estimation (see [63]) for radially symmetric functions of $W^{1,p}(\mathbb{R}^N)$; namely, for every $u \in W^{1,p}_{\rm rad}(\mathbb{R}^N)$ we have

$$|u(x)| \leq p^{1/p}(\text{Area}S^{N-1})^{-1/p}\|u\|_{W^{1,p}}|x|^{(1-N)/p}, \quad x \neq 0, \tag{6.11}$$

where S^{N-1} is the N-dimensional unit sphere.

Now, let $\{u_n\}$ be a sequence in $W^{1,p}_{\rm rad}(\mathbb{R}^N)$ which converges weakly to some $u \in W^{1,p}_{\rm rad}(\mathbb{R}^N)$. By applying inequality (6.11) for $u_n - u$, and taking into account that $\|u_n - u\|_{W^{1,p}}$ is bounded, and $N \geq 2$, then for every $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that

$$\|u_n - u\|_{L^\infty(|x|\geq R_\varepsilon)} \leq C|R_\varepsilon|^{(1-N)/p} < \varepsilon, \quad \forall n \in \mathbb{N},$$

where $C > 0$ does not depend on n. The rest is similar as above. □

Let $\Phi_1, \Phi_2 : L^\infty(\mathbb{R}^N) \to \mathbb{R}$ be defined by

$$\Phi_1(u) = -\int_{\mathbb{R}^N} \alpha(x)F(u(x))dx \quad \text{and} \quad \Phi_2(u) = -\int_{\mathbb{R}^N} \beta(x)G(u(x))dx.$$

Since $\alpha, \beta \in L^1(\mathbb{R}^N)$, the functionals Φ_1, Φ_2 are well-defined and locally Lipschitz, see Clarke [10, p. 79-81]. Moreover, we have

$$\partial\Phi_1(u) \subseteq -\int_{\mathbb{R}^N} \alpha(x)\partial F(u(x))dx, \quad \partial\Phi_2(u) \subseteq -\int_{\mathbb{R}^N} \beta(x)\partial G(u(x))dx.$$

The energy functional $\mathcal{E}_{\lambda,\mu} : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ associated to problem $(\tilde{P}_{\lambda,\mu})$, is given by

$$\mathcal{E}_{\lambda,\mu}(u) = \frac{1}{p}\|u\|^p + \lambda\Phi_1(u) + \mu\Phi_2(u), \quad u \in W^{1,p}(\mathbb{R}^N).$$

It is clear that the critical points of the functional $\mathcal{E}_{\lambda,\mu}$ are solutions of the problem $(\tilde{P}_{\lambda,\mu})$ in the sense of Definition 6.8.

Since α, β are radially symmetric, then $\mathcal{E}_{\lambda,\mu}$ is $O(N)$-invariant, i.e. $\mathcal{E}_{\lambda,\mu}(gu) = \mathcal{E}_{\lambda,\mu}(u)$ for every $g \in O(N)$ and $u \in W^{1,p}(\mathbb{R}^N)$. Therefore, we may apply a non-smooth version of the *principle of symmetric criticality,* proved by Krawcewicz-Marzantowicz [25], for locally Lipschitz functions, see Remark 3.9.

Proposition 6.13. *Any critical point of* $\mathcal{E}^{\rm rad}_{\lambda,\mu} = \mathcal{E}_{\lambda,\mu}|_{W^{1,p}_{\rm rad}(\mathbb{R}^N)}$ *will be also a critical point of* $\mathcal{E}_{\lambda,\mu}$.

In the proof of the main result we use, the following result.

Proposition 6.14. $\lim_{t\to 0^+} \frac{\inf\{\Phi_1(u):\ u\in W^{1,p}_{\rm rad}(\mathbb{R}^N),\ \|u\|^p<pt\}}{t} = 0.$

Proof. Due to ($\tilde{\mathbf{F}}1$), for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that

$$|\xi| \leq \varepsilon|t|^{p-1}, \quad \forall t \in [-\delta(\varepsilon), \delta(\varepsilon)],\ \forall \xi \in \partial F(t). \tag{6.12}$$

For any $0 < t \leq \frac{1}{p}\left(\frac{\delta(\varepsilon)}{c_\infty}\right)^p$ define the set

$$S_t = \{\ u \in W^{1,p}_{\rm rad}(\mathbb{R}^N) : \|u\|^p < pt\},$$

where $c_\infty > 0$ denotes the best constant in the embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$.

Note that $u \in S_t$ implies that $\|u\|_\infty \leq \delta(\varepsilon)$; indeed, we have $\|u\|_\infty \leq c_\infty\|u\| < c_\infty(pt)^{1/p} \leq \delta(\varepsilon)$. Fix $u \in S_t$; for a.e. $x \in \mathbb{R}^N$, Lebourg's mean value theorem and (6.12) imply the existence of $\xi_x \in \partial F(\theta_x u(x))$ for some $0 < \theta_x < 1$ such that

$$F(u(x)) = F(u(x)) - F(0) = \xi_x u(x) \leq |\xi_x| \cdot |u(x)| \leq \varepsilon|u(x)|^p.$$

Consequently, for every $u \in S_t$ we have

$$\begin{aligned}\Phi_1(u) &= -\int_{\mathbb{R}^N} \alpha(x)F(u(x))dx \geq -\varepsilon\int_{\mathbb{R}^N} \alpha(x)|u(x)|^p dx \\ &\geq -\varepsilon\|\alpha\|_{L^1}\|u\|^p_\infty \geq -\varepsilon\|\alpha\|_{L^1}c^p_\infty\|u\|^p \\ &\geq -\varepsilon\|\alpha\|_{L^1}c^p_\infty pt.\end{aligned}$$

Therefore, for every $0 < t \leq \frac{1}{p}\left(\frac{\delta(\varepsilon)}{c_\infty}\right)^p$ we have

$$0 \geq \frac{\inf_{u\in S_t}\Phi_1(u)}{t} \geq -\varepsilon\|\alpha\|_{L^1}c^p_\infty p.$$

Since $\varepsilon > 0$ is arbitrary, we obtain the required limit. ☐

The main result of this subsection appear in the paper Kristály, Marzantowicz and Varga [28].

Theorem 6.15. (Kristály-Marzantowicz-Varga [28]) *Assume that* $p > N \geq 2$. *Let* $\alpha, \beta \in L^1(\mathbb{R}^N)$ *be two radial functions,* α *fulfilling* $(\tilde{\alpha})$, *and let* $F, G : \mathbb{R} \to \mathbb{R}$ *be two locally Lipschitz functions,* F *satisfying the conditions* ($\tilde{\mathbf{F}}1$)-($\tilde{\mathbf{F}}3$)*. Then there exists a non-degenerate compact interval* $[a, b] \subset]0, +\infty[$ *and a number* $\tilde{r} > 0$, *such that for every* $\lambda \in [a, b]$ *there exists* $\mu_0 \in]0, \lambda + 1]$ *such that for each* $\mu \in [0, \mu_0]$, *the problem* $(\tilde{P}_{\lambda,\mu})$ *has at least three distinct, radially symmetric solutions with* L^∞*-norms less than* $\tilde{r}$.

Proof. We are going to apply Theorem 2.30 by choosing $X = W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$, $\tilde{X}_1 = \tilde{X}_2 = L^\infty(\mathbb{R}^N)$, $\Lambda = [0,+\infty)$, $h(t) = t^p/p$, $t \geq 0$.

Fix $g \in \mathcal{G}_\tau$ ($\tau \geq 0$), $\lambda \in \Lambda$, $\mu \in [0, \lambda+1]$, and $c \in \mathbb{R}$. We prove that the functional $E_{\lambda,\mu} : W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$E_{\lambda,\mu}(u) = \frac{1}{p}\|u\|^p + \lambda\Phi_1(u) + \mu(g \circ \Phi_2)(u), \quad u \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N),$$

satisfies the $(PS)_c$ condition.

Note first that the function $\frac{1}{p}\|\cdot\|^p + \lambda\Phi_1$ is coercive on $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$. To prove this, let $0 < \varepsilon < (p\|\alpha\|_1 c_\infty^p \lambda)^{-1}$. Then, on account of ($\tilde{\mathbf{F}}2$), there exists $\delta(\varepsilon) > 0$ such that

$$F(t) \leq \varepsilon|t|^p, \quad \forall |t| > \delta(\varepsilon).$$

Consequently, for every $u \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ we have

$$\Phi_1(u) = -\int_{\mathbb{R}^N} \alpha(x)F(u(x))dx$$

$$= -\int_{\{x\in\mathbb{R}^N : |u(x)|>\delta(\varepsilon)\}} \alpha(x)F(u(x))dx - \int_{\{x\in\mathbb{R}^N : |u(x)|\leq\delta(\varepsilon)\}} \alpha(x)F(u(x))dx$$

$$\geq -\varepsilon\int_{\{x\in\mathbb{R}^N : |u(x)|>\delta(\varepsilon)\}} \alpha(x)|u(x)|^p dx - \max_{|t|\leq\delta(\varepsilon)} |F(t)| \int_{\{x\in\mathbb{R}^N : |u(x)|\leq\delta(\varepsilon)\}} \alpha(x)dx$$

$$\geq -\varepsilon\|\alpha\|_{L^1} c_\infty^p \|u\|^p - \|\alpha\|_{L^1} \max_{|t|\leq\delta(\varepsilon)} |F(t)|.$$

Now, we have

$$\frac{1}{p}\|u\|^p + \lambda\Phi_1(u) \geq \left(\frac{1}{p} - \varepsilon\lambda\|\alpha\|_{L^1} c_\infty^p\right)\|u\|^p - \lambda\|\alpha\|_{L^1} \max_{|t|\leq\delta(\varepsilon)} |F(t)|,$$

which clearly implies the coercivity of $\frac{1}{p}\|\cdot\|^p + \lambda\Phi_1$.

As an immediate consequence, the functional $E_{\lambda,\mu}$ is also coercive on $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$. Therefore, it is enough to consider a bounded sequence $\{u_n\} \subset W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ such that

$$E^\circ_{\lambda,\mu}(u_n; v - u_n) \geq -\varepsilon_n\|v - u_n\| \text{ for all } v \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N), \tag{6.13}$$

where $\{\varepsilon_n\}$ is a positive sequence such that $\varepsilon_n \to 0$. Since the sequence $\{u_n\}$ is bounded in $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$, one can find an element $u \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ weakly in $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$, and $u_n \to u$ strongly in $L^\infty(\mathbb{R}^N)$, due to Proposition 6.12.

Due to Proposition 2.3 for every $u, v \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ we have

$$E^\circ_{\lambda,\mu}(u; v) \leq \int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u\nabla v + |u|^{p-2}uv) + \lambda\Phi_1^\circ(u; v) + \mu(g \circ \Phi_2)^\circ(u; v). \tag{6.14}$$

Put $v = u$ in (6.13) and apply relation (6.14) for the pairs $(u, v) = (u_n, u-u_n)$ and $(u, v) = (u, u_n - u)$, we have that

$$\begin{aligned} I_n \quad \leq \quad & \varepsilon_n \|u - u_n\| - E^\circ_{\lambda,\mu}(u; u_n - u) + \lambda[\Phi_1^\circ(u_n; u - u_n) + \Phi_1^\circ(u; u_n - u)] \\ & + \mu[(g \circ \Phi_2)^\circ(u_n; u - u_n) + (g \circ \Phi_2)^\circ(u; u_n - u)], \end{aligned}$$

where

$$\begin{aligned} I_n \quad \overset{\text{not.}}{=} \quad & \int_{\mathbb{R}^N} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u)(\nabla u_n - \nabla u) \\ & + \int_{\mathbb{R}^N} (|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u). \end{aligned}$$

Since $\{u_n\}$ is bounded in $W^{1,p}_{\text{rad}}(\mathbb{R}^N)$, we have that $\lim_{n\to\infty} \varepsilon_n \|u - u_n\| = 0$. Fixing $z^* \in \partial E^\circ_{\lambda,\mu}(u)$ arbitrarily, we have $\langle z^*, u_n - u\rangle \leq E^\circ_{\lambda,\mu}(u; u_n - u)$. Since $u_n \rightharpoonup u$ weakly in $W^{1,p}_{\text{rad}}(\mathbb{R}^N)$, we have that $\liminf_{n\to\infty} E^\circ_{\lambda,\mu}(u; u_n - u) \geq 0$. The functions $\Phi_1^\circ(\cdot;\cdot)$ and $(g \circ \Phi_2)^\circ(\cdot;\cdot)$ are upper semicontinuous functions on $L^\infty(\mathbb{R}^N)$. Since $u_n \to u$ strongly in $L^\infty(\mathbb{R}^N)$, the upper limit of the last four terms is less or equal than 0 as $n \to \infty$, see Proposition 2.3 (f_4).
Consequently,

$$\limsup_{n\to\infty} I_n \leq 0. \tag{6.15}$$

Since $|t - s|^p \leq (|t|^{p-2}t - |s|^{p-2}s)(t - s)$ for every $t, s \in \mathbb{R}^m$ $(m \in \mathbb{N})$ we infer that $\|u_n - u\|^p \leq I_n$. The last inequality combined with (6.15) leads to the fact that $u_n \to u$ strongly in $W^{1,p}_{\text{rad}}(\mathbb{R}^N)$, as claimed.

It remains to prove relation (2.53) from Theorem 2.30. First, we construct the function $u_0 \in W^{1,p}_{\text{rad}}(\mathbb{R}^N)$ such that $\Phi_1(u_0) < 0$.

On account of $(\tilde{\alpha})$, one can fix $R > 0$ such that $\alpha_R = \text{essinf}_{|x|\leq R}\alpha(x) > 0$. For $\sigma \in]0, 1[$ define the function

$$w_\sigma(x) = \begin{cases} 0, & \text{if} \quad x \in \mathbb{R}^N \setminus B_N(0, R); \\ \tilde{t}, & \text{if} \quad x \in B_N(0, \sigma R); \\ \frac{\tilde{t}}{R(1-\sigma)}(R - |x|), & \text{if} \quad x \in B_N(0, R) \setminus B_N(0, \sigma R), \end{cases}$$

where $B_N(0, r)$ denotes the $N-$dimensional open ball with center 0 and radius $r > 0$, and $\tilde{t}$ comes from $(\tilde{\mathbf{F}}3)$. Since $\alpha \in L^\infty_{\text{loc}}(\mathbb{R}^N)$, then $M(\alpha, R) = \sup_{x\in B_N(0,R)} \alpha(x) < \infty$. A simple estimate shows that

$$-\Phi_1(w_\sigma) \geq \omega_N R^N[\alpha_R F(\tilde{t})\sigma^N - M(\alpha, R) \max_{|t|\leq|\tilde{t}|} |F(t)|(1 - \sigma^N)].$$

When $\sigma \to 1$, the right hand side is strictly positive; choosing σ_0 close enough to 1, for $u_0 = w_{\sigma_0}$ we have $\Phi_1(u_0) < 0$.

Let us define the function for every $t > 0$ by

$$\beta(t) = \inf\{\Phi_1(u) : u \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N),\ \frac{\|u\|^p}{p} < t\}.$$

We have that $\beta(t) \leq 0$, for $t > 0$, and Proposition 6.14 yields that

$$\lim_{t\to 0^+} \frac{\beta(t)}{t} = 0. \tag{6.16}$$

We consider the $u_0 \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$, for wchich $\Phi_1(u_0) < 0$. Therefore it is possible to choose a number $\eta > 0$ such that

$$0 < \eta < -\Phi_1(u_0)\left[\frac{\|u_0\|^p}{p}\right]^{-1}.$$

By (6.16) we get the existence of a number $t_0 \in (0, \frac{\|u_0\|^p}{p})$ such that $-\beta(t_0) < \eta t_0$. Thus

$$\beta(t_0) > \left[\frac{\|u_0\|^p}{p}\right]^{-1}\Phi_1(u_0)t_0. \tag{6.17}$$

Due to the choice of t_0 and using (6.17), we conclude that there exists $\rho_0 > 0$ such that

$$-\beta(t_0) < \rho_0 < -\Phi_1(u_0)\left[\frac{\|u_0\|^p}{p}\right]^{-1} t_0 < -\Phi_1(u_0). \tag{6.18}$$

Define now the function $\varphi : W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N) \times \mathbb{I} \to \mathbb{R}$ by

$$\varphi(u,\lambda) = \frac{\|u\|^p}{p} + \lambda\Phi_1(u) + \lambda\rho_0,$$

where $\mathbb{I} = [0, +\infty)$. We prove that the function φ satisfies the inequality

$$\sup_{\lambda\in\mathbb{I}} \inf_{u\in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N))} \varphi(u,\lambda) < \inf_{u\in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)} \sup_{\lambda\in\mathbb{I}} \varphi(u,\lambda). \tag{6.19}$$

The function

$$\mathbb{I} \ni \lambda \mapsto \inf_{u\in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)}\left[\frac{\|u\|^p}{p} + \lambda(\rho_0 + \Phi_1(u))\right]$$

is obviously upper semicontinuous on $\mathbb{I}$. It follows from (6.18) that

$$\lim_{\lambda\to+\infty} \inf_{u\in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)} \varphi(u,\lambda) \leq \lim_{\lambda\to+\infty}\left[\frac{\|u_0\|^p}{p} + \lambda(\rho_0 + \Phi_1(u_0))\right] = -\infty.$$

Thus we find an element $\overline{\lambda} \in \mathbb{I}$ such that

$$\sup_{\lambda\in\mathbb{I}} \inf_{u\in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)} \varphi(u,\lambda) = \inf_{u\in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)}\left[\frac{\|u\|^p}{p} + \overline{\lambda}(\rho_0 + \Phi_1(u))\right]. \tag{6.20}$$

Since $-\beta(t_0) < \rho_0$, it follows from the definition of β that for all $u \in W^{1,p}_{\text{rad}}(\mathbb{R}^N)$ with $\frac{\|u\|^p}{p} < t_0$ we have $-\Phi_1(u) < \rho_0$. Hence

$$t_0 \le \inf\{\frac{\|u\|^p}{p} : u \in W^{1,p}_{\text{rad}}(\mathbb{R}^N),\ -\Phi_1(u) \ge \rho_0\ \}. \tag{6.21}$$

On the other hand,

$$\begin{aligned} \inf_{u\in W^{1,p}_{\text{rad}}(\mathbb{R}^N)} \sup_{\lambda\in\mathbb{I}} \varphi(u,\lambda) &= \inf_{u\in W^{1,p}_{\text{rad}}(\mathbb{R}^N)} \left[\frac{\|u\|^p}{p} + \sup_{\lambda\in\mathbb{I}} (\lambda(\rho_0+\Phi_1(u)))\right] \\ &= \inf_{u\in W^{1,p}_{\text{rad}}(\mathbb{R}^N)} \left\{\frac{\|u\|^p}{p} : -\Phi_1(u) \ge \rho_0\right\}. \end{aligned}$$

Thus inequality (6.21) is equivalent to

$$t_0 \le \inf_{u\in W^{1,p}_{\text{rad}}(\mathbb{R}^N)} \sup_{\lambda\in\mathbb{I}} \varphi(u,\lambda). \tag{6.22}$$

We consider two cases. First, when $0 \le \overline{\lambda} < \dfrac{t_0}{\rho_0}$, then we have that

$$\inf_{u\in W^{1,p}_{\text{rad}}(\mathbb{R}^N)} \left[\frac{\|u\|^p}{p} + \overline{\lambda}(\rho_0+\Phi_1(u))\right] \le \varphi(0,\overline{\lambda}) = \overline{\lambda}\rho_0 < t_0.$$

Combining this inequality with (6.20) and (6.22) we obtain (6.19).

Now, if $\dfrac{t_0}{\rho_0} \le \overline{\lambda}$, then from (6.17) and (6.18), it follows that

$$\begin{aligned} \inf_{u\in W^{1,p}_{\text{rad}}(\mathbb{R}^N)} \left[\frac{\|u\|^p}{p} + \overline{\lambda}(\rho_0+\Phi_1(u))\right] &\le \frac{\|u_0\|^p}{p} + \overline{\lambda}(\rho_0+\Phi_1(u_0)) \\ &\le \frac{\|u_0\|^p}{p} + \frac{t_0}{\rho_0}(\rho_0+\Phi_1(u_0)) < t_0. \end{aligned}$$

It remains to apply again (6.20) and (6.22), which concludes the proof of (6.19).

Due to Theorem 2.30, there exist a non-empty open set $A \subset \Lambda$ and $r > 0$ with the property that for every $\lambda \in A$ there exists $\mu_0 \in]0, \lambda+1]$ such that, for each $\mu \in [0,\mu_0]$ the functional $\mathcal{E}^{\text{rad}}_{\lambda,\mu} = \frac{1}{p}\|\cdot\|^p + \lambda\Phi_1 + \mu\Phi_2$ defined on $W^{1,p}_{\text{rad}}(\mathbb{R}^N)$ has at least three critical points in $W^{1,p}_{\text{rad}}(\mathbb{R}^N)$ whose $\|\cdot\|$-norms are less than r. Applying Proposition 6.13, the critical points of $\mathcal{E}^{\text{rad}}_{\lambda,\mu}$ are also critical points of $\mathcal{E}_{\lambda,\mu}$, thus, radially weak solutions of problem $(\tilde{P}_{\lambda,\mu})$. Due to the embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$, if $\tilde{r} = c_\infty r$, then the L^∞-norms of these elements are less than $\tilde{r}$ which concludes our proof. □

The second problem studied in this subsection is the following differential inclusion problem:

$$\text{(DI)} \qquad \begin{cases} -\triangle_p u + |u|^{p-2}u \in \alpha(x)\partial F(u(x)), & x \in \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N), \end{cases}$$

where $2 \le N < p < +\infty$, $\alpha \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is radially symmetric, and ∂F stands for the generalized gradient of a locally Lipschitz function $F : \mathbb{R} \to \mathbb{R}$. By a solution of (DI) it will be understood an element $u \in W^{1,p}(\mathbb{R}^N)$ for which there corresponds a mapping $\mathbb{R}^N \ni x \mapsto \zeta_x$ with $\zeta_x \in \partial F(u(x))$ for almost every $x \in \mathbb{R}^N$ having the property that for every $v \in W^{1,p}(\mathbb{R}^N)$, the function $x \mapsto \alpha(x)\zeta_x v(x)$ belongs to $L^1(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u \nabla v + |u|^{p-2}uv)dx = \int_{\mathbb{R}^N} \alpha(x)\zeta_x v(x)dx. \tag{6.23}$$

Under suitable oscillatory assumptions on the potential F at zero or at infinity, we show the existence of infinitely many, radially symmetric solutions of (DI). These results appear in the paper of Kristály [30].

For $l = 0$ or $l = +\infty$, set

$$F_l := \limsup_{|\rho| \to l} \frac{F(\rho)}{|\rho|^p}. \tag{6.24}$$

Problem (DI) will be studied in the following four cases:

- $0 < F_l < +\infty$, whenever $l = 0$ or $l = +\infty$ and
- $F_l = +\infty$, whenever $l = 0$ or $l = +\infty$.

In the next in this subsection we assume that:

(H) • $F : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz, $F(0) = 0$, and $F(s) \ge 0$, $\forall s \in \mathbb{R}$;
• $\alpha \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is radially symmetric, and $\alpha(x) \ge 0$, $\forall x \in \mathbb{R}^N$.

Let $\mathcal{F} : L^\infty(\mathbb{R}^N) \to \mathbb{R}$ be a function defined by

$$\mathcal{F}(u) = \int_{\mathbb{R}^N} \alpha(x)F(u(x))dx.$$

Since F is continuous and $\alpha \in L^1(\mathbb{R}^N)$, we easily seen that $\mathcal{F}$ is well-defined. Moreover, if we fix a $u \in L^\infty(\mathbb{R}^N)$ arbitrarily, there exists $k_u \in L^1(\mathbb{R}^N)$ such that for every $x \in \mathbb{R}^N$ and $v_i \in \mathbb{R}$ with $|v_i - u(x)| < 1$, $(i \in \{1, 2\})$ one has

$$|\alpha(x)F(v_1) - \alpha(x)F(v_2)| \le k_u(x)|v_1 - v_2|.$$

Indeed, if we fix some small open intervals I_j $(j \in J)$, such that $F|_{I_j}$ is Lipschitz function (with Lipschitz constant $L_j > 0$) and $[-\|u\|_{L^\infty} - 1, \|u\|_{L^\infty} + 1] \subset \cup_{j\in J} I_j$, then we choose $k_u = \alpha \max_{j\in J} L_j$. (Here, without losing the generality, we supposed that $\mathrm{card}J < +\infty$.) Thus, we are in the position to apply Theorem 2.7.3 from [10, p. 80]; namely, $\mathcal{F}$ is a locally Lipschitz function on $L^\infty(\mathbb{R}^N)$ and for every closed subspace E of $L^\infty(\mathbb{R}^N)$ we have

$$\partial(\mathcal{F}|_E)(u) \subseteq \int_{\mathbb{R}^N} \alpha(x)\partial F(u(x))dx, \quad \text{for every } u \in E, \tag{6.25}$$

where $\mathcal{F}|_E$ stands for the restriction of $\mathcal{F}$ to E. The interpretation of (6.25) is as follows (see also [10]): For every $\zeta \in \partial(\mathcal{F}|_E)(u)$ there corresponds a mapping $\mathbb{R}^N \ni x \mapsto \zeta_x$ such that $\zeta_x \in \partial F(u(x))$ for almost every $x \in \mathbb{R}^N$ having the property that for every $v \in E$ the function $x \mapsto \alpha(x)\zeta_x v(x)$ belongs to $L^1(\mathbb{R}^N)$ and

$$\langle \zeta, v\rangle_E = \int_{\mathbb{R}^N} \alpha(x)\zeta_x v(x)dx.$$

Now, let $\mathcal{E} : W^{1,p}(\mathbb{R}^N) \to \mathbb{R}$ be the energy functional associated to our problem (DI), i.e., for every $u \in W^{1,p}(\mathbb{R}^N)$ set

$$\mathcal{E}(u) = \frac{1}{p}\|u\|^p_{W^{1,p}} - \mathcal{F}(u).$$

It is clear that $\mathcal{E}$ is locally Lipschitz on $W^{1,p}(\mathbb{R}^N)$ and we have

Proposition 6.16. *Any critical point $u \in W^{1,p}(\mathbb{R}^N)$ of $\mathcal{E}$ is a solution of* (DI).

Proof. Combining $0 \in \partial\mathcal{E}(u) = -\triangle_p u + |u|^{p-2}u - \partial(\mathcal{F}|_{W^{1,p}(\mathbb{R}^N)})(u)$ with the interpretation of (6.25), the desired requirement yields, see (6.23). □

Since α is radially symmetric, then $\mathcal{E}$ is $O(N)$-invariant, i.e. $\mathcal{E}(gu) = \mathcal{E}(u)$ for every $g \in O(N)$ and $u \in W^{1,p}(\mathbb{R}^N)$, we are in the position to apply the Principle of Symmetric Criticality for locally Lipschitz functions, see Remark 3.9. Therefore we have

Proposition 6.17. *Any critical point of $\mathcal{E}_r = \mathcal{E}|_{W^{1,p}_{\rm rad}(\mathbb{R}^N)}$ will be also a critical point of $\mathcal{E}$.*

Remark 6.18. In view of Propositions 6.16 and 6.17, it is enough to find critical points of $\mathcal{E}_r$ in order to guarantee solutions for (DI). This fact will be carried out by means of Theorem 2.31, setting

$$X := W^{1,p}_{\rm rad}(\mathbb{R}^N),\ \tilde{X} := L^\infty(\mathbb{R}^N),\ \Phi := -\mathcal{F},\ \text{ and } \Psi := \|\cdot\|^p_r, \tag{6.26}$$

where the notation $\|\cdot\|_r$ stands for the restriction of $\|\cdot\|_{W^{1,p}}$ into $W^{1,p}_{\rm rad}(\mathbb{R}^N)$. A few assumptions are already verified. Indeed, the embedding $X \hookrightarrow \tilde{X}$ is compact (cf. Theorem 6.12), $\Phi = -\mathcal{F}$ is locally Lipschitz, while $\Psi = \|\cdot\|^p_r$ is of class C^1 (thus, locally Lipschitz as well), coercive and weakly sequentially lower semicontinuous (see [7, Proposition III.5]). Moreover, $\mathcal{E}_r \equiv \Phi|_{W^{1,p}_{\rm rad}(\mathbb{R}^N)} + \frac{1}{p}\Psi$. According to (6.26), the function φ (defined in (2.57)) becomes

$$\varphi(\rho) = \inf_{\|u\|^p_r<\rho} \frac{\sup_{\|v\|^p_r\le\rho} \mathcal{F}(v) - \mathcal{F}(u)}{\rho - \|u\|^p_r}, \quad \rho > 0. \tag{6.27}$$

The investigation of the numbers γ and δ (defined in (2.58)), as well as the cases (A) and (B) from Theorem 2.31 constitute the objective of the next.

Theorem 6.19. (A. Kristály [30]; The case $0 < F_l < +\infty$) *Let $l = 0$ or $l = +\infty$, and let $2 \leq N < p < +\infty$. Let $F : \mathbb{R} \to \mathbb{R}$ and $\alpha : \mathbb{R}^N \to \mathbb{R}$ be two functions which satisfy the hypotheses (H) and $0 < F_l < +\infty$. Assume that $\|\alpha\|_{L^\infty} F_l > 2^N p^{-1}$ and there exists a number $\beta_l \in]2^N (pF_l)^{-1}, \|\alpha\|_{L^\infty}[$ such that*

$$\frac{2}{(2^{-N} p \beta_l F_l - 1)^{1/p}} < \sup\{r : \text{meas}(B_N(0,r) \setminus \alpha^{-1}(]\beta_l, +\infty[)) = 0\}. \tag{6.28}$$

Assume further that there are sequences $\{a_k\}$ and $\{b_k\}$ in $]0, +\infty[$ with $a_k < b_k$, $\lim_{k\to+\infty} b_k = l$, $\lim_{k\to+\infty} \frac{b_k}{a_k} = +\infty$ such that

$$\sup\{\text{sign}(s)\xi : \xi \in \partial F(s), |s| \in]a_k, b_k[\} \leq 0. \tag{6.29}$$

Then, problem (DI) *possesses a sequence $\{u_n\}$ of solutions which are radially symmetric and*

$$\lim_{n\to+\infty} \|u_n\|_{W^{1,p}} = l.$$

In addition, if $F(s) = 0$ for every $s \in]-\infty, 0[$, then the elements u_n are non-negative.

Proof. Since $\lim_{k\to+\infty} b_k = +\infty$, instead of the sequence $\{b_k\}$, we may consider a non-decreasing subsequence of it, denoted again by $\{b_k\}$. Fix an $s \in \mathbb{R}$ such that $|s| \in]a_k, b_k]$. By using Lebourg's mean value theorem (see [10, Theorem 2.3.7]), there exists $\theta \in]0, 1[$ and $\xi_\theta \in \partial F(\theta s + (1-\theta)\text{sign}(s)a_k)$ such that

$$\begin{aligned} F(s) - F(\text{sign}(s)a_k) &= \xi_\theta(s - \text{sign}(s)a_k) = \text{sign}(s)\xi_\theta(|s| - a_k) \\ &= \text{sign}(\theta s + (1-\theta)\text{sign}(s)a_k)\xi_\theta(|s| - a_k). \end{aligned}$$

According now to (6.29), we obtain that $F(s) \leq F(\text{sign}(s)a_k)$ for every $s \in \mathbb{R}$ complying with $|s| \in]a_k, b_k]$. In particular, we are led to $\max_{[-a_k,a_k]} F = \max_{[-b_k,b_k]} F$ for every $k \in \mathbb{N}$. Therefore, one can fix a $\overline{\rho}_k \in [-a_k, a_k]$ such that

$$F(\overline{\rho}_k) = \max_{[-a_k,a_k]} F = \max_{[-b_k,b_k]} F. \tag{6.30}$$

Moreover, since $\{b_k\}$ is non-decreasing, the sequence $\{|\overline{\rho}_k|\}$ can be chosen non-decreasingly as well. In view of (6.28) we can choose a number μ such that

$$\frac{2}{(2^{-N} p \beta_\infty F_\infty - 1)^{1/p}} < \mu < \tag{6.31}$$

$$< \sup\{r : \text{meas}(B_N(0,r) \setminus \alpha^{-1}(]\beta_\infty, +\infty[)) = 0\}.$$

In particular, one has

$$\alpha(x) > \beta_\infty, \quad \text{for a.e. } x \in B_N(0, \mu). \tag{6.32}$$

For every $k \in \mathbb{N}$ we define

$$u_k(x) = \begin{cases} 0, & \text{if} \quad x \in \mathbb{R}^N \setminus B_N(0,\mu); \\ \overline{\rho}_k, & \text{if} \quad x \in B_N(0,\frac{\mu}{2}); \\ \frac{2\overline{\rho}_k}{\mu}(\mu - |x|), & \text{if} \quad x \in B_N(0,\mu) \setminus B_N(0,\frac{\mu}{2}). \end{cases} \tag{6.33}$$

It is easy to see that u_k belongs to $W^{1,p}(\mathbb{R}^N)$ and it is radially symmetric. Thus, $u_k \in W^{1,p}_{\text{rad}}(\mathbb{R}^N)$. Let $\rho_k = (\frac{b_k}{c_\infty})^p$, where c_∞ is the embedding constant of $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$.

Claim 1. *There exists a $k_0 \in \mathbb{N}$ such that $\|u_k\|_r^p < \rho_k$, for every $k > k_0$.*

Since $\lim_{k\to+\infty} \frac{b_k}{a_k} = +\infty$, there exists a $k_0 \in \mathbb{N}$ such that

$$\frac{b_k}{a_k} > c_\infty(\mu^N \omega_N K(p,N,\mu))^{1/p}, \quad \text{for every } k > k_0, \tag{6.34}$$

where ω_N denotes the volume of the N-dimensional unit ball and

$$K(p,N,\mu) := \frac{2^p}{\mu^p}\left(1 - \frac{1}{2^N}\right) + 1. \tag{6.35}$$

Thus, for every $k > k_0$ one has

$$\begin{aligned} \|u_k\|_r^p &= \int_{\mathbb{R}^N} |\nabla u_k|^p dx + \int_{\mathbb{R}^N} |u_k|^p dx \\ &\leq \left(\frac{2|\overline{\rho}_k|}{\mu}\right)^p (\text{vol}B_N(0,\mu) - \text{vol}B_N(0,\frac{\mu}{2})) + |\overline{\rho}_k|^p \text{vol}B_N(0,\mu) \\ &= |\overline{\rho}_k|^p \mu^N \omega_N K(p,N,\mu) \leq a_k^p \mu^N \omega_N K(p,N,\mu) \\ &< (\frac{b_k}{c_\infty})^p = \rho_k, \end{aligned}$$

which proves Claim 1.

Now, let φ from (6.27) and $\gamma = \liminf_{\rho\to+\infty} \varphi(\rho)$ defined in (2.58).

Claim 2. $\gamma = 0$.

By definition, $\gamma \geq 0$. Suppose that $\gamma > 0$. Since $\lim_{k\to+\infty} \frac{\rho_k}{|\overline{\rho}_k|^p} = +\infty$, there is a number $k_1 \in \mathbb{N}$ such that for every $k > k_1$ we have

$$\frac{\rho_k}{|\overline{\rho}_k|^p} > \frac{2}{\gamma}(F_\infty + 1)(\|\alpha\|_{L^1} - \beta_\infty \overline{\mu}^N \omega_N) + \mu^N \omega_N K(p,N,\mu), \tag{6.36}$$

where $\overline{\mu}$ is an arbitrary fixed number complying with

$$0 < \overline{\mu} < \min\left\{ \left(\frac{\|\alpha\|_{L^1}}{\beta_\infty \omega_N}\right)^{1/N}, \frac{\mu}{2} \right\}. \tag{6.37}$$

Moreover, since $|\overline{\rho}_k| \to +\infty$ as $k \to +\infty$ (otherwise we would have $F_\infty = 0$), by the definition of F_∞, see (6.24), there exists a $k_2 \in \mathbb{N}$ such that

$$\frac{F(\overline{\rho}_k)}{|\overline{\rho}_k|^p} < F_\infty + 1, \quad \text{for every } k > k_2. \tag{6.38}$$

Now, let $v \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ arbitrarily fixed with $\|v\|_r^p \leq \rho_k$. Due to the continuous embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$, we have $\|v\|^p_{L^\infty} \leq c^p_\infty \rho_k = b^p_k$. Therefore, one has

$$\sup_{x\in\mathbb{R}^N} |v(x)| \leq b_k.$$

In view of (6.30), we obtain

$$F(v(x)) \leq \max_{[-b_k,b_k]} F = F(\overline{\rho}_k), \quad \text{for every } x \in \mathbb{R}^N. \tag{6.39}$$

Hence, for every $k > \max\{k_0, k_1, k_2\}$, one has

$$\begin{aligned}
\sup_{\|v\|_r^p\leq\rho_k} \mathcal{F}(v) \quad - \quad & \mathcal{F}(u_k) \\
= \quad & \sup_{\|v\|_r^p\leq\rho_k} \int_{\mathbb{R}^N} \alpha(x)F(v(x))dx - \int_{\mathbb{R}^N} \alpha(x)F(u_k(x))dx \\
\leq \quad & F(\overline{\rho}_k)\|\alpha\|_{L^1} - \int_{B_N(0,\overline{\mu})} \alpha(x)F(u_k(x))dx \\
\leq \quad & F(\overline{\rho}_k)(\|\alpha\|_{L^1} - \beta_\infty \overline{\mu}^N \omega_N) \\
\leq \quad & (F_\infty + 1)|\overline{\rho}_k|^p(\|\alpha\|_{L^1} - \beta_\infty \overline{\mu}^N \omega_N) \\
\leq \quad & \frac{\gamma}{2}(\rho_k - |\overline{\rho}_k|^p \mu^N \omega_N K(p, N, \mu)) \\
\leq \quad & \frac{\gamma}{2}(\rho_k - \|u_k\|_r^p).
\end{aligned}$$

Since $\|u_k\|_r^p < \rho_k$ (cf. Claim 1), and $\rho_k \to +\infty$ as $k \to +\infty$, we obtain

$$\gamma = \liminf_{\rho\to+\infty} \varphi(\rho) \leq \liminf_{k\to+\infty} \varphi(\rho_k) \leq \liminf_{k\to+\infty} \frac{\sup_{\|v\|_r^p\leq\rho_k} \mathcal{F}(v) - \mathcal{F}(u_k)}{\rho_k - \|u_k\|_r^p} \leq \frac{\gamma}{2},$$

contradiction. This proves Claim 2.

Claim 3. *$\mathcal{E}_r$ is not bounded below on $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$.*

By (6.31), we find a number ε_∞ such that

$$0 < \varepsilon_\infty < F_\infty - \frac{2^N}{p\beta_\infty}\left(\left(\frac{2}{\mu}\right)^p + 1\right). \tag{6.40}$$

In particular, for every $k \in \mathbb{N}$, $\sup_{|\rho|\geq k} \frac{F(\rho)}{|\rho|^p} > F_\infty - \varepsilon_\infty$. Therefore, we can fix $\tilde{\rho}_k$ with $|\tilde{\rho}_k| \geq k$ such that

$$\frac{F(\tilde{\rho}_k)}{|\tilde{\rho}_k|^p} > F_\infty - \varepsilon_\infty. \tag{6.41}$$

Now, define $w_k \in W^{1,p}_{\rm rad}(\mathbb{R}^N)$ in the same way as u_k, see (6.33), replacing $\overline{\rho}_k$ by $\tilde{\rho}_k$. We obtain

$$\begin{aligned}
\mathcal{E}_r(w_k) &= \frac{1}{p}\|w_k\|_r^p - \mathcal{F}(w_k) \\
&\leq \frac{1}{p}|\tilde{\rho}_k|^p \mu^N \omega_N K(p,N,\mu) - \int_{B_N(0,\frac{\mu}{2})} \alpha(x) F(w_k(x))dx \\
&\leq \frac{1}{p}|\tilde{\rho}_k|^p \mu^N \omega_N K(p,N,\mu) - (F_\infty - \varepsilon_\infty)|\tilde{\rho}_k|^p \beta_\infty \omega_N \left(\frac{\mu}{2}\right)^N \\
&= |\tilde{\rho}_k|^p \mu^N \omega_N \left(\frac{1}{p}K(p,N,\mu) - \frac{1}{2^N}(F_\infty - \varepsilon_\infty)\beta_\infty\right) < -\frac{1}{p}|\tilde{\rho}_k|^p \omega_N \left(\frac{2}{\mu}\right)^{p-N}.
\end{aligned}$$

Since $|\tilde{\rho}_k| \to +\infty$ as $k \to +\infty$, we obtain $\lim_{k\to+\infty} \mathcal{E}_r(w_k) = -\infty$, which ends the proof of Claim 3.

The case $0 < F_\infty < +\infty$. It is enough to apply Remark 6.18. Indeed, since $\gamma = 0$ (cf. Claim 2) and the function $\mathcal{E}_r \equiv -\mathcal{F}|_{W^{1,p}_{\rm rad}(\mathbb{R}^N)} + \frac{1}{p}\|\cdot\|_r^p$ is not bounded below (cf. Claim 3), the alternative (A1) from Theorem 2.31, applied to $\lambda = \frac{1}{p}$, is excluded. Thus, there exists a sequence $\{u_n\} \subset W^{1,p}_{\rm rad}(\mathbb{R}^N)$ of critical points of $\mathcal{E}_r$ with $\lim_{n\to+\infty}\|u_n\|_r = +\infty$.

Now, let us suppose that $F(s) = 0$ for every $s \in]-\infty, 0[$, and let u be a solution of (DI). Denote $S = \{x \in \mathbb{R}^N : \ u(x) < 0\}$, and assume that $S \neq \emptyset$. In virtue of Remark 6.11, the set S is open. Define $u_S : \mathbb{R}^N \to \mathbb{R}$ by $u_S = \min\{u, 0\}$. Applying (6.23) for $v := u_S \in W^{1,p}(\mathbb{R}^N)$ and taking into account that $\zeta_x \in \partial F(u(x)) = \{0\}$ for every $x \in S$, one has

$$0 = \int_{\mathbb{R}^N} (|\nabla u|^{p-2}\nabla u \nabla u_S + |u|^{p-2}u u_S)dx = \int_S (|\nabla u|^p + |u|^p)dx = \|u\|^p_{W^{1,p}(S)},$$

which contradicts the choice of the set S. This ends the proof in this case.

Remark 6.20. A closer inspection of the proof allows us to replace hypothesis (6.28) by a weaker, but a more technical condition. More specifically, it is enough to require that $p\|\alpha\|_{L^\infty} F_l > 1$, and instead of (6.28), put

$$\sup_M \left\{ N_{\beta_l} - \frac{1}{(1-\sigma)(p\beta_l F_l \sigma^N - 1)^{1/p}} \right\} > 0, \tag{6.42}$$

where

$$M = \{(\sigma, \beta_l) : \sigma \in](p\|\alpha\|_{L^\infty}F_l)^{-1/N}, 1[,\ \beta_l \in](pF_l\sigma^N)^{-1}, \|\alpha\|_{L^\infty}[\}$$

and

$$N_{\beta_l} = \sup\{r : \mathrm{meas}(B_N(0,r) \setminus \alpha^{-1}(]\beta_l, +\infty[)) = 0\}.$$

Now, in the construction of the functions w_k we replace the radius $\frac{\mu}{2}$ of the ball by $\sigma\mu$, where σ is chosen according to (6.42).

The case $0 < F_0 < +\infty$. The proof works similarly as in the case $0 < F_\infty < +\infty$ and we will show only the differences. The sequence $\{\rho_k\}$ defined as above, converges now to 0, while the same holds for $\{\overline{\rho}_k\}$. Instead of Claim 2, we can prove that $\delta = \liminf_{\rho \to 0^+} \varphi(\rho) = 0$. Since 0 is the unique global minimum of $\Psi = \|\cdot\|_r^p$, it would be enough to show that 0 is not a local minimum of $\mathcal{E}_r \equiv -\mathcal{F}|_{W^{1,p}_{\rm rad}(\mathbb{R}^N)} + \frac{1}{p}\|\cdot\|_r^p$, in order to exclude alternative (B1) from Theorem 2.31. To this end, we fix $\tilde{\rho}_k$ with $|\tilde{\rho}_k| \le \frac{1}{k}$ such that

$$\frac{F(\tilde{\rho}_k)}{|\tilde{\rho}_k|^p} > F_0 - \varepsilon_0,$$

where ε_0 is fixed in a similar manner as in (6.40), replacing β_∞, F_∞ by β_0, F_0, respectively. If we take w_k as in case $0 < F_\infty < +\infty$, then it is clear that $\{w_k\}$ strongly converges now to 0 in $W^{1,p}_{\rm rad}(\mathbb{R}^N)$, while $\mathcal{E}_r(w_k) < -\frac{1}{p}|\tilde{\rho}_k|^p\omega_N\,(2/\mu)^{p-N} < 0 = \mathcal{E}_r(0)$. Thus, 0 is not a local minimum of $\mathcal{E}_r$. So, there exists a sequence $\{u_n\} \subset W^{1,p}_{\rm rad}(\mathbb{R}^N)$ of critical points of $\mathcal{E}_r$ such that $\lim_{n\to+\infty} \|u_n\|_r = 0 = \inf_{W^{1,p}_{\rm rad}(\mathbb{R}^N)} \Psi$. This concludes completely the proof of Theorem 6.19.

In the next result we trait the case when the function F has oscillation at infinity. We have the following result.

Theorem 6.21. (A. Kristály [30]; The case $F_l = +\infty$) *Let $l = 0$ or $l = +\infty$, and let $2 \le N < p < +\infty$. Let $F : \mathbb{R} \to \mathbb{R}$ and $\alpha : \mathbb{R}^N \to \mathbb{R}$ be two functions which satisfy (H) and $F_l = +\infty$. Assume that $\|\alpha\|_{L^\infty} > 0$, and there exist $\mu > 0$ and $\beta_l \in]0, \|\alpha\|_{L^\infty}[$ such that*

$$\mathrm{meas}(B_N(0,\mu) \setminus \alpha^{-1}(]\beta_l, +\infty[)) = 0, \tag{6.43}$$

and there are sequences $\{a_k\}$ and $\{b_k\}$ in $]0, +\infty[$ with $a_k < b_k$, $\lim_{k\to+\infty} b_k = l$, $\lim_{k\to+\infty} \frac{b_k}{a_k} = +\infty$ such that

$$\sup\{\mathrm{sign}(s)\xi : \xi \in \partial F(s), |s| \in]a_k, b_k[\} \le 0,$$

and

$$\limsup_{k\to+\infty} \frac{\max_{[-a_k,a_k]} F}{b_k^p} < (pc_\infty^p \|\alpha\|_{L^1})^{-1}, \tag{6.44}$$

where c_∞ is the embedding constant of $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$. Then the conclusions of Theorem 6.19 hold.

Proof. **The case** $F_\infty = +\infty$. Due to (6.43),

$$\alpha(x) > \beta_\infty, \quad \text{for a.e. } x \in B_N(0,\mu). \tag{6.45}$$

Let $\overline{\rho}_k$ and ρ_k as in the proof of Theorem 6.19, as well as u_k, defined this time by means of $\mu > 0$ from (6.45).

Claim 1'. *There exists a $k_0 \in \mathbb{N}$ such that $\|u_k\|_r^p < \rho_k$, for every $k > k_0$.*

The proof is similarly as in the proof of Theorem 6.19.

CLAIM 2'. $\gamma < \frac{1}{p}$.

Note that $F(\overline{\rho}_k) = \max_{[-a_k,a_k]} F$, cf. (6.30). Since $|\overline{\rho}_k| \leq a_k$, then $\lim_{k\to+\infty} \frac{|\overline{\rho}_k|}{b_k} = 0$. Combining this fact with (6.44), and choosing $\varepsilon > 0$ sufficiently small, one has

$$\limsup_{k\to+\infty} \frac{F(\overline{\rho}_k) + |\overline{\rho}_k|^p \mu^N \omega_N p^{-1} \|\alpha\|_{L^1}^{-1} K(p,N,\mu)}{b_k^p} < ((p+\varepsilon)c_\infty^p \|\alpha\|_{L^1})^{-1},$$

where $K(p,N,\mu)$ is from (6.35). According to the above inequality, there exists $k_3 \in \mathbb{N}$ such that for every $k > k_3$ we readily have

$$\begin{aligned} F(\overline{\rho}_k)\|\alpha\|_{L^1} &\leq (p+\varepsilon)^{-1} c_\infty^{-p} b_k^p - p^{-1}|\overline{\rho}_k|^p \mu^N \omega_N K(p,N,\mu) \\ &\leq \frac{1}{p+\varepsilon}\left(\rho_k - \frac{p+\varepsilon}{p}\|u_k\|_r^p\right) < \frac{1}{p+\varepsilon}\left(\rho_k - \|u_k\|_r^p\right). \end{aligned}$$

Thus, for every $k > k_3$, one has

$$\sup_{\|v\|_r^p \leq \rho_k} \mathcal{F}(v) - \mathcal{F}(u_k) < F(\overline{\rho}_k)\|\alpha\|_{L^1} < \frac{1}{p+\varepsilon}\left(\rho_k - \|u_k\|_r^p\right).$$

Hence $\gamma \leq \frac{1}{p+\varepsilon} < \frac{1}{p}$, which concludes the proof of Claim 2'.

CLAIM 3'. *$\mathcal{E}_r$ is not bounded below on $W^{1,p}_{\rm rad}(\mathbb{R}^N)$.*

Since $F_\infty = +\infty$, for an arbitrarily large number $M > 0$, we can fix $\tilde{\rho}_k$ with $|\tilde{\rho}_k| \geq k$ such that

$$\frac{F(\tilde{\rho}_k)}{|\tilde{\rho}_k|^p} > M. \tag{6.46}$$

Define $w_k \in W^{1,p}_{\rm rad}(\mathbb{R}^N)$ as in (6.33), putting $\tilde{\rho}_k$ instead of $\overline{\rho}_k$. We obtain

$$\begin{aligned} \mathcal{E}_r(w_k) &= \frac{1}{p}\|w_k\|_r^p - \mathcal{F}(w_k) \\ &\leq \frac{1}{p}\mu^N \omega_N |\tilde{\rho}_k|^p K(p,N,\mu) - \int_{B_N(0,\frac{\mu}{2})} \alpha(x) F(w_k(x))dx \\ &\leq |\tilde{\rho}_k|^p \mu^N \omega_N \left(\frac{1}{p}K(p,N,\mu) - \frac{1}{2^N} M \beta_\infty\right). \end{aligned}$$

Since $|\tilde{\rho}_k| \to +\infty$ as $k \to +\infty$, and M is large enough we obtain that

$$\lim_{k\to+\infty} \mathcal{E}_r(w_k) = -\infty.$$

The proof of Claim 3' is concluded.

Proof concluded. Since $\gamma < \frac{1}{p}$ (cf. Claim 2'), we can apply Theorem 2.31 (A) for $\lambda = \frac{1}{p}$. The rest is the same as in Theorem 6.19.

The case $F_0 = +\infty$.

We follow the line of $F_\infty = +\infty$. The sequences $\{\rho_k\}$, $\{\overline{\rho}_k\}$ are defined as above; they converge to 0. Let $\mu > 0$ be as in (6.45), replacing β_∞ by β_0. Instead of

Claim 2', we may prove that $\delta = \liminf_{\rho \to 0^+} \varphi(\rho) < \frac{1}{p}$. Now, we are in the position to apply Theorem 2.31 (B) with $\lambda = \frac{1}{p}$. Since $F_0 = +\infty$, for an arbitrarily large number $M > 0$, we may choose $\tilde{\rho}_k$ with $|\tilde{\rho}_k| \le \frac{1}{k}$ such that

$$\frac{F(\tilde{\rho}_k)}{|\tilde{\rho}_k|^p} > M.$$

Define $w_k \in W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ by means of $\tilde{\rho}_k$ as above. It is clear that $\{w_k\}$ strongly converges to 0 in $W^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ while

$$\mathcal{E}_r(w_k) \le |\tilde{\rho}_k|^p \mu^N \omega_N \left(\frac{1}{p} K(p, N, \mu) - \frac{1}{2^N} M \beta_0 \right) < 0 = \mathcal{E}_r(0).$$

Consequently, in spite of the fact that 0 is the unique global minimum of $\Psi = \| \cdot \|_r^p$, it is not a local minimum of $\mathcal{E}_r$; thus, (B1) can be excluded. The rest is the same as in the proof of Theorem 6.19. This completes the proof of Theorem 6.21.

In the next we give some example. We suppose that $2 \le N < p < +\infty$.

Example 6.22. Let $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(s) = \frac{2^{N+p+3}}{p} |s|^p \max\{0, \sin \ln(\ln(|s| + 1) + 1)\},$$

and $\alpha : \mathbb{R}^N \to \mathbb{R}$ by

$$\alpha(x) = \frac{1}{(1 + |x|^N)^2}. \tag{6.47}$$

Then (DI) has an unbounded sequence of radially symmetric solutions.

Proof. The functions F and α clearly fulfill (H). Moreover, $F_\infty = \frac{2^{N+p+3}}{p}$. Since $\|\alpha\|_{L^\infty} = 1$, we may fix $\beta_\infty = 1/4$ which verifies (6.28). For every $k \in \mathbb{N}$ let

$$a_k = e^{e^{(2k-1)\pi} - 1} - 1 \quad \text{and} \quad b_k = e^{e^{2k\pi} - 1} - 1.$$

If $a_k \le |s| \le b_k$, then $(2k-1)\pi \le \ln(\ln(|s| + 1) + 1) \le 2k\pi$, thus $F(s) = 0$ for every $s \in \mathbb{R}$ complying with $a_k \le |s| \le b_k$. So, $\partial F(s) = \{0\}$ for every $|s| \in]a_k, b_k[$ and (6.29) is verified. Thus, all the assumptions of Theorem 6.19 are satisfied. $\square$

Example 6.23. Fix $\sigma \in \mathbb{R}$. Let $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(s) = \begin{cases} \frac{8^{N+1}}{p} s^{p-\sigma} \max\{0, \sin \ln \ln \frac{1}{s}\}, & s \in]0, e^{-1}[; \\ 0, & s \notin]0, e^{-1}[, \end{cases}$$

and let $\alpha : \mathbb{R}^N \to \mathbb{R}$ be as in (6.47). Then, for every $\sigma \in [0, \min\{p-1, p(1 - e^{-\pi})\}[$, (DI) admits a sequence of non-negative, radially symmetric solutions which strongly converges to 0 in $W^{1,p}(\mathbb{R}^N)$.

Proof. Since $\sigma < p-1$, (H) is verified. We distinguish two cases: $\sigma = 0$, and $\sigma \in]0, \min\{p-1, p(1-e^{-\pi})\}[$.

Case 1. $\sigma = 0$. We have $F_0 = \frac{8^{N+1}}{p}$. If we choose $\beta_0 = (1+2^N)^{-2}$, this clearly verifies (6.28). For every $k \in \mathbb{N}$ set

$$a_k = e^{-e^{2k\pi}} \quad \text{and} \quad b_k = e^{-e^{(2k-1)\pi}}. \tag{6.48}$$

For every $s \in [a_k, b_k]$, one has $(2k-1)\pi \leq \ln\ln\frac{1}{s} \leq 2k\pi$; thus $F(s) = 0$. So, $\partial F(s) = \{0\}$ for every $s \in]a_k, b_k[$ and (6.29) is verified. Now, we apply Theorem 6.19.

Case 2. $\sigma \in]0, \min\{p-1, p(1-e^{-\pi})\}[$. We have $F_0 = +\infty$. In order to verify (6.43), we fix for instance $\beta_0 = (1+2^N)^{-2}$ and $\mu = 2$. Take $\{a_k\}$ and $\{b_k\}$ in the same way as in (6.48). The inequality in (6.44) becomes obvious since

$$\limsup_{k\to+\infty} \frac{\max_{[-a_k, a_k]} F}{b_k^p} \leq \frac{8^{N+1}}{p} \limsup_{k\to+\infty} \frac{a_k^{p-\sigma}}{b_k^p} =$$

$$= \frac{8^{N+1}}{p} \lim_{k\to+\infty} e^{[p-e^{\pi}(p-\sigma)]e^{(2k-1)\pi}} = 0.$$

Therefore, we may apply Theorem 6.21. □

Example 6.24. Let $\{a_k\}$ and $\{b_k\}$ be two sequences such that $a_1 = 1$, $b_1 = 2$ and $a_k = k^k$, $b_k = k^{k+1}$ for every $k \geq 2$. Define, for every $s \in \mathbb{R}$ the function

$$f(s) = \begin{cases} \frac{b_{k+1}^p - b_k^p}{a_{k+1} - b_k}, & \text{if } s \in [b_k, a_{k+1}[; \\ 0, & \text{otherwise.} \end{cases}$$

Then the problem

$$\begin{cases} -\triangle_p u + |u|^{p-2}u \in \frac{\sigma}{(1+|x|^N)^2}[\underline{f}(u(x)), \overline{f}(u(x))], & x \in \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N), \end{cases}$$

possesses an unbounded sequence of non-negative, radially symmetric solutions whenever $0 < \sigma < \frac{N}{p}\left(\frac{p-N}{2p}\right)^p (\text{Area}S^{N-1})^{-1}$.

Proof. Let $F(s) = \int_0^s f(t)dt$. Since the function f is locally (essentially) bounded, F is locally Lipschitz. A more explicit expression of F is

$$F(s) = \begin{cases} b_k^p - b_1^p + \frac{b_{k+1}^p - b_k^p}{a_{k+1} - b_k}(s - b_k), & \text{if } s \in [b_k, a_{k+1}[; \\ b_k^p - b_1^p, & \text{if } s \in [a_k, b_k[; \\ 0, & \text{otherwise.} \end{cases}$$

An easy calculation shows, as we expect, that $\partial F(s) = [\underline{f}(s), \overline{f}(s)]$ for every $s \in \mathbb{R}$. Taking $\alpha(x) = \frac{\sigma}{(1+|x|^N)^2}$, (H) is verified, and $\|\alpha\|_{L^1} = \frac{\sigma}{N}\text{Area}S^{N-1}$. Moreover,

$$F_\infty = \limsup_{|s|\to+\infty} \frac{F(s)}{|s|^p} \geq \lim_{k\to+\infty} \frac{F(a_k)}{a_k^p} = \lim_{k\to+\infty} \frac{b_k^p - b_1^p}{a_k^p} = +\infty.$$

Choosing $\mu = 1$ and $\beta_\infty = \sigma/4$, (6.43) is verified, while (6.29) becomes trivial. Since $\max_{[-a_k,a_k]} F = F(a_k) = b_k^p - b_1^p$, relation (6.44) reduces to $pc_\infty^p \|\alpha\|_{L^1} < 1$ which is fulfilled due to the choice of σ and to Remark 6.10. It remains to apply Theorem 6.21. □

6.4. **An application to variational-hemivariational inequalities.** In this subsection we give two applications of the Principle of Symmetric Criticality for Motreanu-Panagiotopulos functionals. These results appear in the paper of Lisei and Varga [36].

First we formulate the problem. For this let $F : \mathbb{R}^L \times \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function, which is locally Lipschitz in the second variable (the real variable) and satisfies the following conditions:

($\overline{F}$1) $F(z,0) = 0$ for all $z \in \mathbb{R}^L \times \mathbb{R}^M$ and there exist $c_1 > 0$ and $r \in]p, p^\star[$ such that

$$|\xi| \leq c_1(|s|^{p-1} + |s|^{r-1}),\ \forall \xi \in \partial F(z,s),\ (z,s) \in \mathbb{R}^L \times \mathbb{R}^M \times \mathbb{R}.$$

We denoted by $\partial F(z,s)$ the generalized gradient of $F(z,\cdot)$ at the point $s \in \mathbb{R}$ and $p^\star = \dfrac{(L+M)p}{L+M-p}$ is the critical Sobolev exponent.

Let $a : \mathbb{R}^L \times \mathbb{R}^M \to \mathbb{R}$ $(L \geq 2)$ be a nonnegative continuous function satisfying the following assumptions:

(A_1) $a(x,y) \geq a_0 > 0$ if $|(x,y)| \geq R$ for a large $R > 0$;

(A_2) $a(x,y) \to +\infty$, when $|y| \to +\infty$ uniformly for $x \in \mathbb{R}^L$;

(A_3) $a(x,y) = a(x',y)$ for all $x, x' \in \mathbb{R}^L$ with $|x| = |x'|$ and all $y \in \mathbb{R}^M$.

Consider the following subspaces of $W^{1,p}(\mathbb{R}^L \times \mathbb{R}^M)$

$$\tilde{E} = \{u \in W^{1,p}(\mathbb{R}^L \times \mathbb{R}^M) : u(x,y) = u(x',y)\ \forall\ x, x' \in \mathbb{R}^L, |x| = |x'|, \forall y \in \mathbb{R}^M\},$$

$$E = \{u \in W^{1,p}(\mathbb{R}^L \times \mathbb{R}^M) : \int\limits_{\mathbb{R}^{L+M}} a(z)|u(z)|^p dz < \infty\},$$

$$E_a = \tilde{E} \cap E = \{u \in \tilde{E} : \int\limits_{\mathbb{R}^{L+M}} a(z)|u(z)|^p dz < \infty\},$$

endowed with the norm

$$\|u\|^p = \int\limits_{\mathbb{R}^{L+M}} |\nabla u(z)|^p dz + \int\limits_{\mathbb{R}^{L+M}} a(z)|u(z)|^p dz$$

and the closed convex cone $\mathcal{K} = \{v \in E\ :\ v \geq 0 \text{ a.e. in } \mathbb{R}^L \times \mathbb{R}^M\}$.

The aim of this subsection is to study the following eigenvalue problem (P_λ): *For* $\lambda > 0$ *find* $u \in \mathcal{K}$ *such that*

$$\int_{\mathbb{R}^{L+M}} |\nabla u(z)|^{p-2}\nabla u(z)(\nabla v(z) - \nabla u(z))dz + \int_{\mathbb{R}^{L+M}} a(z)u^{p-1}(z)(v(z)-u(z))dz$$
$$+\lambda \int_{\mathbb{R}^{L+M}} F^0(z,u(z);v(z)-u(z))dz \geq 0$$

for all $v \in \mathcal{K}$, where $F^0(z,s;t)$ is the generalized directional derivative of $F(z,\cdot)$ at the point s in the direction t.

Let $\mathcal{I}_\lambda : E \to]-\infty,+\infty]$ be defined by

$$\mathcal{I}_\lambda(u) = \frac{1}{p}||u||^p - \lambda\mathcal{F}(u) + \psi_{\mathcal{K}}(u),$$

where $\psi_{\mathcal{K}}(u)$ denotes the indicator function of the closed convex cone $\mathcal{K}$,i.e.

$$\psi_{\mathcal{K}}(u) = \begin{cases} 0, & \text{if } x \in \mathcal{K} \\ +\infty, & \text{otherwise.} \end{cases}$$

Clearly $\psi_{\mathcal{K}}$ is convex and lower-semicontinuous on E.

Now we rewrite problem (P_λ) by using the duality map. By Theorem 3.5 from [1] it follows that E is a separable, reflexive and uniform convex Banach space. We denote by $E^\star$ its dual. Let $A : E \to E^\star$ the duality mapping corresponding to the weight function $\varphi : [0,+\infty[\to [0,+\infty[$ defined by $\varphi(t) = t^{p-1}$, where $p \in]1,+\infty[$. It is well known that the duality mapping J satisfies the following conditions:

$$||Au||_\star = \varphi(||u||) \text{ and } \langle Au, u\rangle = ||Au||_\star ||u|| \ \text{ for all } \ u \in E.$$

Moreover, the functional $\chi : E \to \mathbb{R}$ defined by $\chi(u) = \frac{1}{p}||u||^p$ is convex and Gateaux differentiable on E, and $d\chi = A$. The problem (P_λ) can be reformulated in the following way: *For* $\lambda > 0$ *find* $u \in \mathcal{K}$ *such that*

$$\langle Au, v-u\rangle + \lambda \int_{\mathbb{R}^{L+M}} F^0(z,u(z);v(z)-u(z))dx \geq 0$$

for every $v \in \mathcal{K}$.

Lemma 6.25. *Fix* $\lambda > 0$ *arbitrary. Every critical point* $u \in E$ *of the functional* $\mathcal{I}_\lambda$ *is a solution of the problem* (P_λ).

Proof. Since $u \in E$ is a critical point of the functional $\mathcal{I}_\lambda$, one has

$$\langle Au, v-u\rangle + \lambda(-\mathcal{F})^0(u; v-u) + \psi_{\mathcal{K}}(v) - \psi_{\mathcal{K}}(u) \geq 0$$

for every $v \in E$. From Proposition 4.5 we obtain

$$\langle Au, v-u\rangle + \lambda \int_{\mathbb{R}^{L+M}} F^0(z,u(z);u(z)-v(z))dz + \psi_{\mathcal{K}}(v) - \psi_{\mathcal{K}}(u) \geq 0$$

for every $v \in E$.

Therefore $u \in \mathcal{K}$ and for every $v \in \mathcal{K}$ we have

$$\langle Au, v-u\rangle + \lambda \int_{\mathbb{R}^{L+M}} F^0(z, u(z); u(z)-v(z))dz \geq 0. \qquad \square$$

Let $a : \mathbb{R}^L \times \mathbb{R}^M \to \mathbb{R}$ $(L \geq 2)$ be a function, which satisfy the assumptions (A_1), (A_2), (A_3). We consider the following subspaces of $W^{1,p}(\mathbb{R}^L \times \mathbb{R}^M)$

$$\tilde{E} = \{u \in W^{1,p}(\mathbb{R}^L \times \mathbb{R}^M) : u(x,y) = u(x',y) \ \forall\ x, x' \in \mathbb{R}^L, |x| = |x'|, \forall y \in \mathbb{R}^M\},$$

$$E = \{u \in W^{1,p}(\mathbb{R}^L \times \mathbb{R}^M) : \int\limits_{\mathbb{R}^{L+M}} a(z)|u(z)|^p dz < \infty\},$$

$$E_a = \tilde{E} \cap E = \{u \in \tilde{E} : \int\limits_{\mathbb{R}^{L+M}} a(z)|u(z)|^p dz < \infty\}$$

endowed with the norm

$$\|u\|^p = \int\limits_{\mathbb{R}^{L+M}} |\nabla u(z)|^p dz + \int\limits_{\mathbb{R}^{L+M}} a(z)|u(z)|^p dz.$$

The next result is proved by de Morais Filho, Souto, Marcos Do [42] and is a very useful tool in our investigations.

Theorem 6.26. *If (A_1), (A_2) and (A_3) hold, then the Banach space E_a is continuously embedded in $L^s(\mathbb{R}^L \times \mathbb{R}^M)$, if $p \leq s \leq p^*$, and compactly embedded if $p < s < p^*$.*

We have,

$$\|u\|_s \leq C(s)\|u\| \quad \text{for each } u \in E_a,$$

where $\|\cdot\|_s$ is the norm in $L^s(\mathbb{R}^L \times \mathbb{R}^M)$ and $C(s) > 0$ is the embedding constant.

Let

$$G = \left\{ g : E \to E : g(v) = v \circ \begin{pmatrix} R & 0 \\ 0 & Id_{\mathbb{R}^M} \end{pmatrix}, R \in O(\mathbb{R}^L) \right\},$$

where $O(\mathbb{R}^L)$ is the set of all rotations on $\mathbb{R}^L$ and $Id_{\mathbb{R}^M}$ denotes the $M \times M$ identity matrix. The elements of G leave $\mathbb{R}^{L+M}$ invariant, i.e. $g(\mathbb{R}^{L+M}) = \mathbb{R}^{L+M}$ for all $g \in G$.

The action of G over E is defined by

$$(gu)(z) = u(g^{-1}z), \quad g \in G, \quad u \in E, \quad \text{a.e. } z \in \mathbb{R}^{L+M}.$$

As usual we shall write gu in place of $\pi(g)u$.

A function u defined on $\mathbb{R}^{L+M}$ is said to be G-invariant, if

$$u(gz) = u(z), \ \ \forall \ \ g \in G, \ \text{a.e. } z \in \mathbb{R}^{L+M}.$$

Then $u \in E$ is G-invariant if and only if $u \in \Sigma$, where

$$\Sigma := E_a = \tilde{E} \cap E.$$

We observe that the norm

$$\|u\| = \left\{ \int_{\mathbb{R}^{L+M}} (|\nabla u(z)|^p + a(z)|u(z)|^p) dz \right\}^{\frac{1}{p}}$$

is G-invariant.

In order to study our problem we give the assumptions on the nonlinear function F. We assume that $F : \mathbb{R}^L \times \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, which is locally Lipschitz in the second variable, satisfying condition **($\overline{F}$1)** and moreover:

($\overline{F}$2) $\displaystyle\lim_{s \to 0} \frac{\max\{|\xi| : \xi \in \partial F(z,s)\}}{|s|^{p-1}} = 0$ uniformly for every $z \in \mathbb{R}^{L+M}$.

($\overline{F}$3) There exists $\nu > p$ such that

$$\nu F(z,s) + F^0(z,s;-s) \leq 0, \ \forall (z,s) \in \mathbb{R}^{L+M} \times \mathbb{R}.$$

($\overline{F}$4) There exists $r > 0$ such that

$$\inf\{F(z,s) : (z,|s|) \in \mathbb{R}^{L+M} \times [r, \infty)\} > 0.$$

Remark 6.27. a) If $F : \mathbb{R}^{L+M} \times \mathbb{R} \to \mathbb{R}$ satisfies **($\overline{F}$1)** and **($\overline{F}$2)**, then for every $\varepsilon > 0$, there exists $c(\varepsilon) > 0$ such that

i) $|\xi| \leq \varepsilon |s|^{p-1} + c(\varepsilon)|s|^{r-1}, \ \forall\, \xi \in \partial F(z,s), \ (z,s) \in \mathbb{R}^{L+M} \times \mathbb{R};$

ii) $|F(z,s)| \leq \varepsilon |s|^p + c(\varepsilon)|s|^r, \ \forall\, (z,s) \in \mathbb{R}^{L+M} \times \mathbb{R}.$

b) If $F : \mathbb{R}^{L+M} \times \mathbb{R} \to \mathbb{R}$ satisfies **($\overline{F}$1)**, **($\overline{F}$3)** and **($\overline{F}$4)**, then there exist $c_2, c_3 > 0$ and $\nu \in]p, p^\star[$ such that

$$F(z,s) \geq c_2|s|^\nu - c_3|s|^p.$$

To study the existence of the solutions of problem (P_λ), it is sufficient to prove the existence of critical points of the functional $\mathcal{I}_\lambda$ (see Lemma 6.25).

We have the following result, which appear in the paper of Lisei-Varga [36].

Theorem 6.28. (Lisei-Varga [36]) *Let $F : \mathbb{R}^L \times \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}$ be a function, which satisfies **($\overline{F}$1)**-**($\overline{F}$4)** and $F(\cdot, s)$ is G-invariant for every $s \in \mathbb{R}$. Then for every $\lambda > 0$ problem (P_λ) has a nontrivial positive solution.*

Before to prove this result we introduce some notations and we prove some auxiliary results. We have that the cone $\mathcal{K}$ is G-invariant, it follows that $\psi_{\mathcal{K}}$ is G-invariant. Taking into account that the action of G is linear and isometric on E, we deduce that the function $\chi(u) = \dfrac{1}{p}||u||^p$ is G-invariant. The function $\mathcal{F}$ is also G-invariant, because $F(\cdot, s)$ is G-invariant for every $s \in \mathbb{R}$. If we apply Theorem 3.8, it is sufficient to prove that the functional $\mathcal{I}_\Sigma := \mathcal{I}_\lambda\Big|_\Sigma$ has critical points, which implies that the functional $\mathcal{I}_\lambda$ has critical points, which are solutions for problem (P_λ). We introduce the following notations:

$$||\cdot||_\Sigma = ||\cdot||\Big|_\Sigma,\ \mathcal{F}_\Sigma = \mathcal{F}\Big|_\Sigma,\ \psi_\Sigma = \psi_\mathcal{K}\Big|_\Sigma$$

and the restricted duality map $A_\Sigma : \Sigma \to \Sigma^*$ with $A_\Sigma = A\Big|_\Sigma$. Therefore we have

$$\mathcal{I}_\Sigma(u) = \frac{1}{p}||u||_\Sigma^p - \lambda\mathcal{F}_\Sigma(u) + \psi_\Sigma(u).$$

In the next we verify that the conditions of Theorem 3.4 are satisfied by the functional $\mathcal{I}_\Sigma$.

Proposition 6.29. *If $F : \mathbb{R}^L \times \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}$ verifies the conditions **$(\overline{F}1)$-$(\overline{F}3)$** and $F(\cdot, s), s \in \mathbb{R}$ is G- invariant, then $\mathcal{I}_\Sigma$ satisfies the (PS) condition, for every $\lambda > 0$.*

Proof. Let $\lambda > 0$ and $c \in \mathbb{R}$ be some fixed numbers and let $(u_n) \subset \Sigma$ be a sequence such that

$$\mathcal{I}_\Sigma(u_n) = \frac{1}{p}||u_n||_\Sigma^p - \lambda\mathcal{F}_\Sigma(u_n) + \psi_\Sigma(u_n) \to c \tag{6.49}$$

and for every $v \in \Sigma$ we have

$$\langle A_\Sigma u_n, v-u_n\rangle+\lambda\int_{\mathbb{R}^{L+M}} F^0(z, u_n(z); u_n(z)-v(z))dz+\psi_\Sigma(v)-\psi_\Sigma(u_n) \geq -\varepsilon_n||v-u_n||_\Sigma, \tag{6.50}$$

for a sequence (ε_n) in $[0, +\infty[$ with $\varepsilon_n \to 0$.

By (6.49) one concludes that $(u_n) \subset \mathcal{K} \cap \Sigma$. Setting $v = 2u_n$ in (6.50), we obtain

$$\langle A_\Sigma u_n, u_n\rangle + \lambda \int_{\mathbb{R}^{L+M}} F^0(z, u_n(z); -u_n(z))dz \geq -\varepsilon_n||u_n||_\Sigma. \tag{6.51}$$

By (6.49) one has for large $n \in \mathbb{N}$ that

$$c + 1 \geq \frac{1}{p}||u_n||_\Sigma^p - \lambda\mathcal{F}_\Sigma(u_n). \tag{6.52}$$

We multiply inequality (6.51) with ν^{-1} and use Proposition 4.5 to obtain

$$\varepsilon_n\frac{||u_n||_\Sigma}{\nu} \geq -\frac{\langle A_\Sigma u_n, u_n\rangle}{\nu} - \frac{\lambda}{\nu}\int_{\mathbb{R}^{L+M}} F^0(z, u_n(z); -u_n(z))dz. \tag{6.53}$$

Adding the inequalities (6.52) and (6.53), and using (F3) we get

$$\begin{aligned} c + 1 + \frac{\varepsilon_n}{\nu}||u_n||_\Sigma &\geq \left(\frac{1}{p} - \frac{1}{\nu}\right)||u_n||_\Sigma^p \\ &\quad - \lambda\int_{\mathbb{R}^{L+M}}[F(z, u_n(z)) + \frac{1}{\nu}F^0(z, u_n(z); -u_n(z))]dz \\ &\geq \left(\frac{1}{p} - \frac{1}{\nu}\right)||u_n||_\Sigma^p. \end{aligned}$$

From this, we get that the sequence $(u_n) \subset \mathcal{K} \cap \Sigma$ is bounded. Because E is reflexive, it follows that Σ is reflexive too and there exists an element $u \in \Sigma$ such that $u_n \rightharpoonup u$ weakly. Since $\mathcal{K} \cap \Sigma$ is closed and convex, we get $u \in \mathcal{K} \cap \Sigma$. Moreover, from (6.50) with $v = u$ we obtain

$$\langle A_\Sigma u_n, u - u_n\rangle + \lambda \int_{\mathbb{R}^{L+M}} F^0(z, u_n(z); u_n(z) - u(z))dz \geq -\varepsilon_n ||u_n - u||_\Sigma. \tag{6.54}$$

From this we get

$$\begin{aligned}
&\langle A_\Sigma u_n, u_n - u\rangle \leq \lambda \int_{\mathbb{R}^{L+M}} F^0(z, u_n(z); u_n(z) - u(z))dz + \varepsilon_n ||u_n - u||_\Sigma \\
&\leq \lambda \int_{\mathbb{R}^{L+M}} \max\{\xi_n(z)(u_n(z) - u(z)) \ : \ \xi_n(z) \in \partial F(z, u_n(z))\}dz + \varepsilon_n ||u_n - u||_\Sigma \\
&\leq \lambda \int_{\mathbb{R}^{L+M}} \Big(\varepsilon |u_n(z)|^{p-1} + c(\varepsilon)|u_n(z)|^{r-1}\Big)|u_n(z) - u(z)|dz + \varepsilon_n ||u_n - u||_\Sigma.
\end{aligned}$$

Hence, by Hölder's inequality and the fact that the inclusion $\Sigma \hookrightarrow L^p(\mathbb{R}^{L+M})$ is continuous (see Theorem 6.26), we obtain

$$\langle A_\Sigma u_n, u_n - u\rangle \leq \lambda\varepsilon C(p)||u_n - u||_\Sigma ||u_n||_p^{p-1} + \lambda c(\varepsilon)||u_n - u||_r ||u_n||_r^{r-1} + \varepsilon_n ||u_n - u||_\Sigma.$$

Moreover, the inclusion $\Sigma \hookrightarrow L^r(\mathbb{R}^{L+M})$ is compact for $r \in]p, p^*[$ (see Theorem 6.26), therefore $||u_n - u||_r \to 0$ as $n \to +\infty$. For $\varepsilon \to 0^+$ and $n \to +\infty$ we obtain that $\limsup_{n\to+\infty} \langle A_\Sigma u_n, u_n - u\rangle \leq 0$. Finally, since the duality operator J_Σ has the (S_+) property we obtain $u_n \to u$ in $\mathcal{K}$, because $\mathcal{K}$ is closed. $\square$

Proposition 6.30. *If $F : \mathbb{R}^L \times \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}$ verifies* **($\overline{F}$1)-($\overline{F}$4)** *and $F(\cdot, s)$ is G-invariant for every $s \in \mathbb{R}$, then for every $\lambda > 0$ the following assertions are true:*

i) *there exist constants $\alpha_\lambda > 0$ and $\rho_\lambda > 0$ such that $\mathcal{I}_\Sigma(u) \geq \alpha_\lambda$ for all $||u||_\Sigma = \rho_\lambda$;*
ii) *there exists $e_\lambda \in \mathcal{K}$ with $||e_\lambda|| > \rho_\lambda$ and $\mathcal{I}_\Sigma(e_\lambda) \leq 0$.*

Proof. From Remark 6.27 and from the fact that the embedding $\Sigma \hookrightarrow L^l(\mathbb{R}^{L+M})$ is continuous for $l \in [p, p^\star]$, it follows that

$$\mathcal{F}_\Sigma(u) \leq \varepsilon C^p(p)||u||_\Sigma^p + c(\varepsilon)C^r(r)||u||_\Sigma^r,$$

for every $u \in \Sigma$. It is suffices to restrict our attention to elements u which belong to $\mathcal{K} \cap \Sigma$, otherwise $\mathcal{I}_\Sigma(u)$ will be $+\infty$, i.e. i) holds trivially.

Let $\lambda > 0$ be arbitrary. We choose $\varepsilon \in]0, \frac{1}{p\lambda C^p(p)}[$ and for $u \in \mathcal{K} \cap \Sigma$ we have

$$\mathcal{I}_\Sigma(u) = \frac{1}{p}||u||_\Sigma^p - \lambda\mathcal{F}_\Sigma(u) \geq \left(\frac{1}{p} - \lambda\varepsilon C^p(p)\right)||u||_\Sigma^p - \lambda c(\varepsilon)C^r(r)||u||_\Sigma^r.$$

We denote by $M = \frac{1}{p} - \lambda\varepsilon C^p(p)$ and $N = \lambda c(\varepsilon)C^r(r)$ and we consider the function $g : \mathbb{R}_+ \to \mathbb{R}$ given by $g(t) = Mt^p - Nt^r$. The function g attains its global maximum in the point $t_\lambda = \left(\frac{pM}{rN}\right)^{\frac{1}{r-p}}$. If we take $\rho_\lambda = t_\lambda$ and $\alpha_\lambda \in]0, g(t_\lambda)[$, the condition i) is fulfilled.

To prove ii) from b) Remark 6.27 we observe that for every $u \in \mathcal{K} \cap \Sigma$ we have

$$\mathcal{I}_\Sigma(u) \leq \frac{1}{p}||u||_\Sigma^p + \lambda c_3 C^p(p)||u||_\Sigma^p - \lambda c_2||u||_\nu^\nu.$$

If we fix an element $v \in (\mathcal{K} \cap \Sigma) \setminus \{0\}$ and in place of u we put tv, then we have

$$\mathcal{I}_\Sigma(tv) \leq \left(\frac{1}{p} + \lambda c_3 C^p(p)\right)||v||_\Sigma^p t^p - \lambda c_2||v||_\nu^\nu t^\nu.$$

From this we see that if t is large enough, then $||tv||_\Sigma > \rho_\lambda$ and $\mathcal{I}_\Sigma(tv) < 0$. If we take $e_\lambda = tv$ we obtain the desired results. □

Proof of Theorem 6.28. Now we prove that the conditions of Theorem 3.4 are satisfied by the functional $\mathcal{I}_\Sigma$. Because $F(z, 0) = 0$, it follows that

$$\mathcal{I}_\Sigma(0) = \int_{\mathbb{R}^{L+M}} F(z, 0)dz = 0.$$

From Proposition 6.29 we get that $\mathcal{I}_\Sigma$ satisfies the (PS) condition. Proposition 6.30 implies that $\mathcal{I}_\Sigma$ satisfies the conditions (i) and (ii) from Theorem 3.4, hence the number

$$c_\lambda = \inf_{f\in\Gamma} \sup_{t\in[0,1]} I_\Sigma(f(t)),$$

where

$$\Gamma_\lambda = \{f \in C([0, 1], \Sigma) \;:\; f(0) = 0,\; f(1) = e_\lambda\;\},$$

is a critical value of $\mathcal{I}_\Sigma$ with $c_\lambda \geq \alpha_\lambda$. □

In the next we replace (F3) and (F4) with the following two conditions

($\overline{F}'$3) There exist $q \in]0, p[, \nu \in [p, p^\star], \alpha \in L^{\frac{\nu}{\nu-q}}(\mathbb{R}^{L+M}), \beta \in L^1(\mathbb{R}^{L+M})$ such that

$$F(z, s) \leq \alpha(z)|s|^q + \beta(z)$$

for all $s \in \mathbb{R}$ and a.e. $z \in \mathbb{R}^{L+M}$;

($\overline{F}'$4) There exists $u_0 \in \mathcal{K}$ such that $\int_{\mathbb{R}^{L+M}} F(z, u_0(z))dz > 0$.

We have the following result.

Theorem 6.31. (Lisei-Varga [36]) *Let $F : \mathbb{R}^L \times \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}$ be a function which satisfies ($\overline{F}$1),($\overline{F}$2), ($\overline{F}'$3), ($\overline{F}'$4) and $F(\cdot, s)$ is G-invariant for all $s \in \mathbb{R}$. Then there exists an open interval $\Lambda_0 \subset \Lambda$ such that for each $\lambda \in \Lambda_0$ problem (P_λ) has at least three distinct solutions which are axially symmetric.*

To prove Theorem 6.31 we combine Theorem 3.5 with Theorem 3.8. First we consider the functional $f : E \times \Lambda \to]-\infty, +\infty]$ given by $f(u, \lambda) = I_1(u) + \lambda I_2(u)$, where

$$I_1(u) = \frac{1}{p}||u||^p + \psi_{\mathcal{K}}(u), \quad I_2(u) = -\mathcal{F}(u) = -\int_{\mathbb{R}^{L+M}} F(z, u(z))dz.$$

As in Lemma 6.25 we have that every critical point of the function $f = I_1 + \lambda I_2$ is a solution of problem (P_λ). Using Theorem 3.8 it is sufficient to prove that the functional $f_\Sigma = \Big(I_1 + \lambda I_2\Big)\Big|_\Sigma$ satisfies conditions from Theorem 3.5, where we choose $h_1, \Psi_1, h_2 : \Sigma \to \mathbb{R}$

$$h_1(u) = \frac{1}{p}||u||_\Sigma^p, \ \Psi_1(u) = \psi_\Sigma(u), \ h_2(u) = -\mathcal{F}_\Sigma(u) = -\int_{\mathbb{R}^{L+M}} F(z, u(z))dz, \ u \in \Sigma,$$

and take

$$I_1 = h_1 + \Psi_1, \ I_2 = h_2.$$

First we prove that (a_1) holds.

Proposition 6.32. *If $F : \mathbb{R}^L \times \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}$ verifies the conditions* **($\overline{F}$1)** *and* **($\overline{F}$2)***, then h_1 is weakly sequentially lower semicontinuous and h_2 is weakly sequentially continuous.*

Proof. The weakly sequentially lower semicontinuity of $h_1 = \frac{1}{p}||\cdot||_\Sigma^p$ is standard (every convex lower semicontinuous function is sequentially lower semicontinuous, see e.g. [7]).

In order to prove the weakly sequentially continuity of h_2 we assume that (u_n) is a sequence in Σ such that $u_n \rightharpoonup u$ (in Σ). We will prove that $\mathcal{F}_\Sigma(u_n) \to \mathcal{F}_\Sigma(u)$.

By Lebourg's Mean Value Theorem (see [10]) it follows that there exist $\theta_n \in [0, 1]$ and $v_n \in \partial\mathcal{F}_\Sigma(u + \theta_n(u_n - u))$ such that

$$\mathcal{F}_\Sigma(u_n) - \mathcal{F}_\Sigma(u) = \langle v_n, u_n - u\rangle.$$

We denote $w_n = u + \theta_n(u_n - u)$. Using the definition of $\mathcal{F}_\Sigma^0$, Proposition 4.5 it follows that

$$\begin{aligned} \mathcal{F}_\Sigma(u_n) - \mathcal{F}_\Sigma(u) \quad &\leq \quad (\mathcal{F}_\Sigma)^0(w_n; u_n - u) \leq \int\limits_{\mathbb{R}^{L+M}} F^\circ(z, w_n(z); u_n(z) - u(z))dz \\ &= \int\limits_{\mathbb{R}^{L+M}} \max\Big\{\langle v(z), u_n(z) - u(z)\rangle : v \in \partial F(z, w_n(z))\Big\}. \end{aligned}$$

Now we use Remark 6.27 to get

$$\mathcal{F}_\Sigma(u_n) - \mathcal{F}_\Sigma(u) \leq \int\limits_{\mathbb{R}^{L+M}} \Big(\varepsilon|w_n(z)|^{p-1} + c(\varepsilon)|w_n(z)|^{r-1}\Big)|u_n(z) - u(z)|dz.$$

We use Hölder's inequality and the fact that the inclusion $\Sigma \hookrightarrow L^p(\mathbb{R}^{L+M})$ is continuous (see Theorem 6.26) to obtain

$$\mathcal{F}_\Sigma(u_n) - \mathcal{F}_\Sigma(u) \leq \varepsilon C(p)\|u_n - u\|_\Sigma \|w_n\|_p^{p-1} + c(\varepsilon)C(r)\|u_n - u\|_r \|w_n\|_r^{r-1}. \quad (6.55)$$

Now we use the same ideas as before for $-\mathcal{F}_\Sigma$ and find the existence of $\tau_n \in [0,1]$ and $\hat{v}_n \in \partial(-\mathcal{F}_\Sigma)(u + \tau_n(u_n - u))$ such that

$$\mathcal{F}_\Sigma(u) - \mathcal{F}_\Sigma(u_n) = \langle \hat{v}_n, u_n - u\rangle.$$

We denote $\hat{w}_n = u + \tau_n(u_n - u)$. Using the definition of $-\mathcal{F}_\Sigma^0$, and properties of the generalized gradient (see [10]), it follows that

$$\mathcal{F}_\Sigma(u) - \mathcal{F}_\Sigma(u_n) \quad \leq \quad (-\mathcal{F}_\Sigma)^0(\hat{w}_n; u_n - u) = (\mathcal{F}_\Sigma)^0(\hat{w}_n; u - u_n).$$

Analogously to (6.55) we get

$$\mathcal{F}_\Sigma(u) - \mathcal{F}_\Sigma(u_n) \leq \varepsilon C(p)\|u_n - u\|_\Sigma \|\hat{w}_n\|_p^{p-1} + c(\varepsilon)C(r)\|u_n - u\|_r \|\hat{w}_n\|_r^{r-1}. \quad (6.56)$$

Using (6.55) and (6.56) we have

$$\begin{aligned} |\mathcal{F}_\Sigma(u_n) - \mathcal{F}_\Sigma(u)| \quad &\leq \quad \varepsilon C(p)\|u_n - u\|_\Sigma(\|w_n\|_p^{p-1} \qquad (6.57)\\ &\quad + \|\hat{w}_n\|_p^{p-1}) + c(\varepsilon)C(r)\|u_n - u\|_r(\|w_n\|_r^{r-1} + \|\hat{w}_n\|_r^{r-1}). \end{aligned}$$

The inclusion $\Sigma \hookrightarrow L^r(\mathbb{R}^{L+M})$ is compact for $r \in]p, p^*[$ (see Theorem 6.26), then we get that $||u_n - u||_r \to 0$ as $n \to +\infty$, while the sequences (w_n) and $(\hat{w}_n)$ are bounded in the $\|\cdot\|_p$ and $\|\cdot\|_r$ norms. Then in (6.57) we get $\mathcal{F}_\Sigma(u_n) \to \mathcal{F}_\Sigma(u)$. Hence h_2 is weakly sequentially continuous. $\square$

Proof of Theorem 6.31. For this let $u \in \mathcal{K} \cap \Sigma$, from condition $(\overline{F}'3)$ and from the fact that the embedding $\Sigma \hookrightarrow L^\nu(\mathbb{R}^{L+M})$ is continuous and $q < p$ it follows that

$$\begin{aligned} f_\Sigma(u, \lambda) \quad &\geq \quad \frac{1}{p}||u||_\Sigma^p - \lambda \int_{\mathbb{R}^{L+M}} \alpha(z)|u(z)|^q dz - \lambda \int_{\mathbb{R}^{L+M}} \beta(z) dz \\ &\geq \quad \frac{1}{p}||u||_\Sigma^p - \lambda||\alpha||_{\frac{\nu}{\nu-q}} ||u||_\nu^q - \lambda||\beta||_1 \\ &\geq \quad \frac{1}{p}||u||_\Sigma^p - \lambda||\alpha||_{\frac{\nu}{\nu-q}} C^q(q)||u||_\Sigma^q - \lambda||\beta||_1. \end{aligned}$$

Therefore, if $||u||_\Sigma \to +\infty$, we have $f_\Sigma(u, \lambda) \to +\infty$. Let $(u_n) \subset \mathcal{K} \cap \Sigma$ be a sequence such that

$$f_\Sigma(u_n, \lambda) \to c \quad (6.58)$$

and for every $v \in \Sigma$ we have

$$\langle A_\Sigma u_n, v - u_n\rangle + \lambda \int_{\mathbb{R}^{L+M}} F^0(z, u_n(z); u_n(z) - v(z)) dz + \psi_\Sigma(v) - \psi_\Sigma(u_n) \geq -\varepsilon_n ||v - u_n||_\Sigma, \quad (6.59)$$

for a sequence (ε_n) in $[0,+\infty[$ with $\varepsilon_n \to 0$. From (6.58) follows that the sequence (u_n) is bounded in $\mathcal{K}\cap\Sigma$ and as in Proposition 6.29 we get that there exists an element $u \in \mathcal{K}\cap\Sigma$ such that $u_n \to u$. Let us define the function

$$g(t) = \sup\left\{\mathcal{F}_\Sigma(u) : \ \frac{1}{p}||u||_\Sigma^p \le t\right\}.$$

Using ii) from Remark 6.27 and the fact that the inclusion $\Sigma \hookrightarrow L^l(\mathbb{R}^{L+M}), l \in [p, p^\star]$ is continuous, it follows that

$$g(t) \le \varepsilon C^p(p)t + c(\varepsilon)C^r(r)t^{\frac{r}{p}}. \tag{6.60}$$

On the other hand $g(t) \ge 0$ for each $t > 0$, then from the above relation we get

$$\lim_{t\to 0^+}\frac{g(t)}{t} = 0. \tag{6.61}$$

By $(\overline{F}'4)$ it is clear that $u_0 \neq 0$ (since $\mathcal{F}(0) = 0$). Therefore it is possible to choose a number η such that

$$0 < \eta < \mathcal{F}_\Sigma(u_0)\left[\frac{1}{p}||u_0||_\Sigma^p\right]^{-1}.$$

From $\lim_{t\to 0^+}\frac{g(t)}{t} = 0$ it follows the existence of a number $t_0 \in \left]0, \frac{1}{p}||u_0||_\Sigma^p\right[$ such that $g(t_0) < \eta t_0$. Thus

$$g(t_0) < \left[\frac{1}{p}||u_0||_\Sigma^p\right]^{-1}\mathcal{F}_\Sigma(u_0)t_0.$$

Let $\rho_0 > 0$ such that

$$g(t_0) < \rho_0 < \left[\frac{1}{p}||u_0||_\Sigma^p\right]^{-1}\mathcal{F}_\Sigma(u_0)t_0. \tag{6.62}$$

Due to the choice of t_0 and (6.62) we have

$$\rho_0 < \mathcal{F}_\Sigma(u_0). \tag{6.63}$$

Define $h : \Lambda = [0,+\infty[\to \mathbb{R}$ by $h(\lambda) = \rho_0\lambda$. We prove that the function h satisfies the inequality

$$\sup_{\lambda\in\Lambda}\inf_{u\in\mathcal{K}\cap\Sigma}(f_\Sigma(u,\lambda) + h(\lambda)) < \inf_{u\in\mathcal{K}\cap\Sigma}\sup_{\lambda\in\Lambda}(f_\Sigma(u,\lambda) + h(\lambda)).$$

The function

$$\Lambda \ni \lambda \mapsto \inf_{u\in\mathcal{K}\cap\Sigma}\left[\frac{1}{p}||u||_\Sigma^p + \lambda(\rho_0 - \mathcal{F}_\Sigma(u))\right]$$

is obviously upper semicontinuous on Λ.

From (6.63) it follows that

$$\lim_{\lambda\to+\infty}\inf_{u\in\mathcal{K}\cap\Sigma}[f_\Sigma(u,\lambda)+\rho_0\lambda] \le \lim_{\lambda\to+\infty}\left[\frac{1}{p}||u_0||_\Sigma^p+\lambda(\rho_0-\mathcal{F}_\Sigma(u_0))\right]=-\infty. \quad (6.64)$$

Thus we find an element $\overline{\lambda}\in\Lambda$ such that

$$\sup_{\lambda\in\Lambda}\inf_{u\in\mathcal{K}\cap\Sigma}(f_\Sigma(u,\lambda)+\rho_0\lambda)=\inf_{u\in\mathcal{K}\cap\Sigma}\left[\frac{1}{p}||u||_\Sigma^p+\overline{\lambda}(\rho_0-\mathcal{F}_\Sigma(u))\right]. \quad (6.65)$$

From $g(t_0)<\rho_0$ it follows that for all $u\in\Sigma$ with $\frac{1}{p}||u||_\Sigma^p\le t_0$, we have $\mathcal{F}_\Sigma(u)<\rho_0$. Hence

$$t_0\le\inf\left\{\frac{1}{p}||u||_\Sigma^p\ :\ \mathcal{F}_\Sigma(u)\ge\rho_0\right\}. \quad (6.66)$$

On the other hand,

$$\begin{aligned}\inf_{u\in\mathcal{K}\cap\Sigma}\sup_{\lambda\in\Lambda}(f_\Sigma(u,\lambda)+\rho_0\lambda) &= \inf_{u\in\mathcal{K}\cap\Sigma}\left[\frac{1}{p}||u||_\Sigma^p+\sup_{\lambda\in\Lambda}(\lambda(\rho_0-\mathcal{F}_\Sigma(u)))\right]\\ &= \inf\left\{\frac{1}{p}||u||_\Sigma^p\ :\ \mathcal{F}_\Sigma(u)\ge\rho_0\right\}.\end{aligned}$$

Thus (6.66) is equivalent with

$$t_0\le\inf_{u\in\mathcal{K}\cap\Sigma}\sup_{\lambda\in\Lambda}[f_\Sigma(u,\lambda)+\rho_0\lambda]. \quad (6.67)$$

There are two distinct cases:

(I) If $0\le\overline{\lambda}<\dfrac{t_0}{\rho_0}$, we have

$$\inf_{u\in\mathcal{K}\cap\Sigma}\left[\frac{1}{p}||u||_\Sigma^p+\overline{\lambda}(\rho_0-\mathcal{F}_\Sigma(u))\right]\le f_\Sigma(0,\overline{\lambda})=\overline{\lambda}\rho_0<t_0.$$

Combining the above inequality with (6.65) and (6.67) we obtain the inequality from (a_2) Theorem 3.5.

(II) If $\dfrac{t_0}{\rho_0}\le\overline{\lambda}$, then from $\rho_0<\mathcal{F}_\Sigma(u_0)$ and (6.62) it follows

$$\begin{aligned}\inf_{u\in\mathcal{K}\cap\Sigma}\left[\frac{1}{p}||u||_\Sigma^p+\overline{\lambda}(\rho_0-\mathcal{F}_\Sigma(u))\right] &\le \frac{1}{p}||u_0||_\Sigma^p+\overline{\lambda}(\rho_0-\mathcal{F}_\Sigma(u_0))\\ &\le \frac{1}{p}||u_0||_\Sigma^p+\frac{t_0}{\rho_0}(\rho_0-\mathcal{F}_\Sigma(u_0))<t_0.\end{aligned}$$

Theorem 3.5 implies that there exists an open interval $\Lambda_0\subset\Lambda$, such that for each $\lambda\in\Lambda_0$, the function $f_\Sigma(\cdot,\lambda)$ has at least three critical points in $\mathcal{K}\cap\Sigma$. Therefore, problem (P_λ) has at least three distinct solutions for every $\lambda\in\Lambda_0$. This ends the proof. $\square$

We conclude this subsection with two examples for which Theorem 6.28 and 6.31 can be applied.

Example 6.33. Let $k \in \mathbb{R}, k > 1$. We define the sequence of real numbers (A_n) by $A_0 = 0$, and

$$A_n = \frac{1}{1^k} + \frac{1}{2^k} + \frac{1}{3^k} + \cdots + \frac{1}{n^k}, \quad n \geq 1.$$

Let $r > p > 2$. We consider the functions $f, F : \mathbb{R} \to \mathbb{R}$ given respectively by

$$f(s) = s|s|^{p-2}(|s|^{r-p} + A_n) \text{ for } s \in]-n-1, -n] \cup [n, n+1[, \ n \in \mathbb{N},$$

$$F(u) = \int_0^u f(s)ds \text{ for } u \in]-n-1, -n] \cup [n, n+1[, \ n \in \mathbb{N}.$$

Clearly F satisfies $(\overline{F}\mathbf{1})$, $(\overline{F}\mathbf{2})$, $(\overline{F}\mathbf{3})$ and $(\overline{F}\mathbf{4})$, hence owing to Theorem 6.28 problem (P_λ) has a nontrivial positive solution.

Example 6.34. Let $A : \mathbb{R}^L \to \mathbb{R}$ be a continuous, nonnegative, not identically zero, axially symmetric function with compact support in $\mathbb{R}^L$. We consider $F : \mathbb{R}^L \times \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}$ defined by

$$F((x, y), s) = A(x) \min\{s^r, |s|^q\} \quad \text{ for } (x, y) \in \mathbb{R}^L \times \mathbb{R}^M, \ s \in \mathbb{R},$$

where $r \in \left]p, \dfrac{(L+M)p}{L+M-p}\right[$ is an odd number and $q \in]0, p[$. The function F satisfies the assumptions $(\overline{F}\mathbf{1})$, $(\overline{F}\mathbf{2})$, $(\overline{F}'\mathbf{3})$ and $(\overline{F}'\mathbf{4})$ and $F(\cdot, s)$ is G-invariant for all $s \in \mathbb{R}$. Theorem 6.31 implies that there exists an open interval $\Lambda_0 \subset \Lambda$ such that for each $\lambda \in \Lambda_0$ problem (P_λ) has at least three distinct solutions which are axially symmetric.

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