STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LV**, Number 2, June 2010

ON A NEW SEQUENCE SPACE DEFINED BY MUSIELAK-ORLICZ FUNCTIONS

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Abstract. In this paper we define a new sequence space $m(\mathcal{M}, \phi, p)$, which is a generalization of $m(\phi, p)$ (B. C. Tripathy and M. Sen [12]) by Musielak-Orlicz functions. We study some of the properties of this space.

1. **Introduction**

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by

$$M(x + y) \leq M(x) + M(y)$$

then this function is called a modular function, defined and discussed by Nakano [10] and Musielak [7] and others. It is well known that if M is a convex functions and $M(0) = 0$, then $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$ (see [1], [2], [9]).

Lindendstrauss and Tzafriri [5] used the idea of Orlicz function to construct the sequence space

$$\ell_M = \left\{ x = (x_k) : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\}.$$

The space ℓ_M with the norm

$$||x|| = \inf\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\}$$

becomes a Banach space which is called an Orlicz sequence space. For $M(x) = x^p$, $1 \leq p < \infty$, the space ℓ_M coincides with the classical sequence space l_p.

Received by the editors: 05.01.2009.
2000 *Mathematics Subject Classification.* 40A05, 46A45.
Key words and phrases. Symmetric space, normal space, completeness, Banach space, Orlicz-function, Musielak-Orlicz function.

A sequence $\mathcal{M} = (M_k)$ of Orlicz functions is called a Musielak-Orlicz function [See [3], [4], [6], [7]). In addition, a **Musielak-Orlicz function** $N = (N_k)$ is called a complementary function of a Musielak-Orlicz function $\mathcal{M}$ if

$$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\},\ k = 1, 2, \cdots$$

For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $l_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows:

$$l_{\mathcal{M}} := \{x \in s : I_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0\},$$

$$h_{\mathcal{M}} := \{x \in s : I_{\mathcal{M}}(cx) < \infty \text{ for all } c > 0\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k),\ x = (x_k) \in l_{\mathcal{M}}.$$

We consider $l_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf\{k > 0 : I_{\mathcal{M}}(\frac{x}{k}) \leq 1\},$$

or equipped with the Orlicz norm

$$||x||^0 = \inf\{\frac{1}{k}(1 + I_{\mathcal{M}}(kx)) : k > 0\}.$$

If $x = (x_n)$ is a sequence, then $S(x)$ denotes the set of all permutation of the elements of (x_n). A sequence space E is said to be symmetric if $S(x) \subset E$ for all $x \in E$. A sequence space E is said to be solid if $(y_n) \in E$ whenever $(x_n) \in E$ and $|y_n| \leq |x_n|$ for all $n \in \mathbb{N}$.

A BK-space is a Banach sequence space E in which the coordinate maps are continuous, i.e. if $(x_k^{(n)})_k \in E$, then

$$||(x_k^{(n)}) - (x_k)|| \to 0 \ as\ n \to \infty$$

$$\Rightarrow |(x_k^{(n)}) - (x_k)| \to\ 0 \ as\ n \to \infty, \quad \text{for each fixed k.}$$

Let $\mathcal{C}$ denote the space whose elements are finite sets of distinct positive integers. Given any element σ of $\mathcal{C}$, we denote by $c(\sigma)$ the sequence $\{c_n(\sigma)\}$ which is such that $c_n(\sigma) = 1$ if $n \in \sigma, c_n(\sigma) = 0$ otherwise. Further, let

$$\mathcal{C}_s = \left\{\sigma \in \mathcal{C} : \sum_{n=1}^{\infty} c_n(\sigma) \leq s\right\} \text{(cf.[8])},$$

be the set of those σ whose support has cardinality at most s. Throughout the paper ϕ_n denotes a non-decreasing sequence of positive numbers such that $n\phi_{n+1} \leq (n+1)\phi_n$ for all $n \in \mathbb{N}$.

The space $m(\phi)$ is defined as follows (Sargent [11]):

$$m(\phi) := \left\{ x = (x_k) \in \omega : \sup_{s \geq 1} \sup_{\sigma \in \mathcal{C}_s} \frac{1}{\phi_s} \sum_{k \in \sigma} |x_k| < \infty \right\}.$$

The space $m(\phi, p)$ is defined as follows (B.C. Tripathy and M. Sen [12]):
For $1 \leq p < \infty$,

$$m(\phi, p) := \left\{ x = (x_k) \in \omega : \sup_{s \geq 1} \sup_{\sigma \in \mathcal{C}_s} \frac{1}{\phi_s} \left\{ \sum_{k \in \sigma} |x_k|^p \right\}^{1/p} < \infty \right\}.$$

In this paper we introduce the space $m(\mathcal{M}, \phi, p)$ as follows:

Let $\mathcal{M} = (M_k)$ be a Musielak-Orlicz function. We define the following sequence space

$$m(\mathcal{M}, \phi, p) := \left\{ x = (x_k) \in \omega : \sup_{s \geq 1} \sup_{\sigma \in \mathcal{C}_s} \frac{1}{\phi_s} \left\{ \sum_{k \in \sigma} \left[M_k \left(\frac{|x_k|}{\rho} \right) \right]^p \right\}^{1/p} < \infty, \text{ for some } \rho > 0 \right\}.$$

It is clear that if $M_k(x) = x$ then $m(\mathcal{M}, \phi, p) = m(\phi, p)$.

Throughout ω, l^p, l^1, l^∞ denote the spaces of all p-absolutely summable, absolutely summable and bounded sequences respectively. $\mathbb{N}$ and $\mathbb{C}$ denotes the set of all natural numbers and complex numbers, respectively.

2. **Main results**

Theorem 2.1. *The space $m(\mathcal{M}, \phi, p)$ is complete.*

Proof. Let $\{x^{(n)}\}$ be a Cauchy sequence in $m(\mathcal{M}, \phi, p)$. Then

$$\sup_{s \geq 1} \sup_{\sigma \in \mathcal{C}_s} \frac{1}{\phi_s} \left\{ \sum_{i \in \sigma} \left[M_i \left(\frac{|x_i|}{\rho} \right) \right]^p \right\}^{1/p} < \infty,$$

for some $\rho > 0$ and for all n $n = 1, 2, 3, \cdots$).

For each $\epsilon > 0$, there exists a positive integer n_0 such that

$$||x^{(m)} - x^{(n)}||_{m(\mathcal{M}, \phi, p)} < \epsilon, \text{ for all } m, n \geq n_0.$$

This implies that

$$\sup_{s \geq 1} \sup_{\sigma \in \mathcal{C}_s} \frac{1}{\phi_s} \left\{ \sum_{i \in \sigma} \left[M_i \left(\frac{|x_i^{(m)} - x_i^{(n)}|}{\rho} \right) \right]^p \right\}^{1/p} < \epsilon, \tag{2.1}$$

for some $\rho > 0$ and for all $m, n \geq n_0$.

Hence

$$|x_i^{(m)} - x_i^{(n)}| < \epsilon \phi_1 \text{ for all } m, n \geq n_0 \textit{ and } \text{for all } i \in \mathbb{N},$$

showing that for each fixed i $(1 \leq i < \infty)$, the sequence $\{x_i^{(n)}\}$ is a Cauchy sequence in $\mathbb{C}$.

Let $x_i^{(n)} \to x_i$ as $n \to \infty$. We define $x = (x_1, x_2, x_3, \cdots)$. We need to show that $x \in m(\mathcal{M}, \phi, p)$ and $x^{(n)} \to x$.

From (2.1) we get, for each fixed s

$$\sum_{i\in\sigma}\left[M_i\left(\frac{|x_i^{(m)} - x_i^{(n)}|}{\rho}\right)\right]^p < \epsilon^p {\phi_s}^p \ , \text{ for some } \rho > 0, \text{ for all } \ m, n \geq n_0 \text{ and } \sigma \in \mathcal{C}_s.$$

Taking $n \to \infty$ we get

$$\sum_{i\in\sigma}\left[M_i\left(\frac{|x_i^{(m)} - x_i^{(n)}|}{\rho}\right)\right]^p < \epsilon^p {\phi_s}^p \ , \text{ for some } \rho > 0, \text{ for all } \ m, n \geq n_0 \text{ and } \sigma \in \mathcal{C}_s.$$

This implies that

$$\sup_{s\geq 1}\sup_{\sigma\in\mathcal{C}_s}\frac{1}{\phi_s}\left\{\sum_{i\in\sigma}\left[M_i\left(\frac{|x_i^{(m)} - x_i|}{\rho}\right)\right]^p\right\}^{1/p} < \epsilon, \tag{2.2}$$

for some $\rho > 0$ and for all $m, n \geq n_0$.

$$\Rightarrow x^{(n)} - x \in m(\mathcal{M}, \phi, p), \text{ for all } n \geq n_0.$$

Hence $x = x^{(n_0)} + x - x^{(n_0)} \in m(\mathcal{M}, \phi, p)$ as $m(\mathcal{M}, \phi, p)$ is a linear space.

From (2.2)

$$||x^{(n)} - x||_{m(\mathcal{M},\phi,p)} < \epsilon, \text{ for all } \ n \geq n_0,$$

which implies that

$$||x^{(n)} - x||_{m(\mathcal{M},\phi,p)} \to 0, \text{ as } n \to \infty.$$

Hence $m(\mathcal{M}, \phi, p)$ $(1 \leq p < \infty)$ is a Banach space.

Theorem 2.2. *The space $m(\mathcal{M}, \phi, p)$ is a BK-space.*

Proof. Suppose that

$$||x^{(n)} - x||_{m(\mathcal{M},\phi,p)} \to 0 \ as \ n \to \infty.$$

For each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$||x^{(n)} - x|| < \epsilon \text{ for all } \ n \geq n_0.$$

This implies that

$$\sup_{s\geq 1}\sup_{\sigma\in\mathcal{C}_s}\frac{1}{\phi_s}\left\{\sum_{k\in\sigma}\left[M_k\left(\frac{|x_k^{(n)} - x_k|}{\rho}\right)\right]^p\right\}^{1/p} < \epsilon, \text{ for some } \rho > 0 \ \text{ and for all } \ n \geq n_0.$$

Consequently

$$|x_k^{(n)} - x_k| < \epsilon\phi_1, \text{ for all } n \geq n_0 \text{ and for all } k.$$

So $|x_k^{(n)} - x_k| \to 0$ *as* $n \to \infty$ and the proof is complete. □

Proposition 2.3. *1. The space* $m(\mathcal{M}, \phi, p)$ *is a symmetric space. If* $x \in m(\mathcal{M}, \phi, p)$ *and* $v \in S(x)$*, then* $||v||_{m(\mathcal{M},\phi,p)} = ||x||_{m(\mathcal{M},\phi,p)}$.

2. The space $m(\mathcal{M}, \phi, p)$ *is a normal space.*

Proposition 2.4. $m(\phi) \subseteq m(\mathcal{M}, \phi, p)$.

Proof. Suppose that $x \in m(\phi)$. Then

$$||x||_{m(\phi)} = \sup_{s\geq 1} \sup_{\sigma \in \mathcal{C}_s} \frac{1}{\phi_s} \left\{ \sum_{n\in\sigma} |x_n| \right\} = K < \infty.$$

Hence for each fixed s,

$$\sum_{n\in\sigma} |x_n| \leq K\phi_s, \ \sigma \in \mathcal{C}_s.$$

This implies that

$$\left\{ \sum_{n\in\sigma} \left[M_n \left(\frac{|x_n|}{\rho} \right) \right]^p \right\}^{1/p} \leq K\phi_s, \ \sigma \in \mathcal{C}_s, \text{ for some } \rho > 0,$$

so that

$$\sup_{s\geq 1} \sup_{\sigma\in\mathcal{C}_s} \left[\frac{1}{\phi_s} \left\{ \sum_{n\in\sigma} \left[M_n \left(\frac{|x_n|}{\rho} \right) \right]^p \right\}^{1/p} \right] \leq K, \text{ for some } \rho > 0.$$

Thus $x \in m(\mathcal{M}, \phi, p)$ and this completes the proof. □

Proposition 2.5. $m(\mathcal{M}, \phi, p) \subseteq m(\mathcal{M}, \psi, p)$ *if and only if* $\sup_{s\geq 1}(\frac{\phi_s}{\psi_s}) < \infty$.

*Proof.*Let $\sup_{s\geq 1}(\frac{\phi_s}{\psi_s}) = K < \infty$. Then $\phi_s \leq K\psi_s$. Now if $(x_k) \in m(\mathcal{M}, \phi, p)$, then

$$\sup_{s\geq 1} \sup_{\sigma\in\mathcal{C}_s} \left[\frac{1}{\phi_s} \left\{ \sum_{n\in\sigma} \left[M_n \left(\frac{|x_n|}{\rho} \right) \right]^p \right\}^{1/p} \right] < \infty, \text{ for some } \rho > 0 \ .$$

This implies that

$$\sup_{s\geq 1} \sup_{\sigma\in\mathcal{C}_s} \left[\frac{1}{K\psi_s} \left\{ \sum_{n\in\sigma} \left[M_n \left(\frac{|x_n|}{\rho} \right) \right]^p \right\}^{1/p} \right] < \infty, \text{ for some } \rho > 0,$$

so that

$$||x||_{m(\mathcal{M},\psi,p)} < \infty.$$

Hence $m(\mathcal{M}, \phi, p) \subseteq m(\mathcal{M}, \psi, p)$.

Conversely, suppose that $m(\mathcal{M},\phi,p)\subseteq m(\mathcal{M},\psi,p)$. We need to show that

$$\sup_{s\geq 1}(\frac{\phi_s}{\psi_s})=\sup_{s\geq 1}(\eta_s)<\infty.$$

Let $\sup_{s\geq 1}(\eta_s)=\infty$. Then there exists a subsequence (η_{s_i}) of (η_s) such that

$$\lim_{i\to\infty}(\eta_{s_i})=\infty.$$

Then for $(x_k)\in m(\mathcal{M},\phi,p)$ we have

$$\sup_{s\geq 1}\sup_{\sigma\in\mathcal{C}_s}\left[\frac{1}{\psi_s}\left\{\sum_{n\in\sigma}\left[M_n\left(\frac{|x_n|}{\rho}\right)\right]^p\right\}^{1/p}\right]$$

$$\geq \sup_{s_i\geq 1}\sup_{\sigma\in\mathcal{C}_{s_i}}\left[\psi_{s_i}\frac{1}{\phi_{s_i}}\left\{\sum_{n\in\sigma}\left[M_n\left(\frac{|x_n|}{\rho}\right)\right]^p\right\}^{1/p}\right]=\infty,$$

for some $\rho>0$.

This implies that $(x_k)\notin m(\mathcal{M},\psi,p)$, a contradiction which completes the proof. □

Theorem 2.6. *$l^p\subseteq m(\mathcal{M},\phi,p)\subset l^\infty$.*

Proof. Since $m(\mathcal{M},\phi,p)=l^p$ for $M_k(x)=x$ and $\phi_n=1$, for all $n\in\mathbb{N}$, it follows that $l^p\subseteq m(\mathcal{M},\phi,p)$.

Next, let $x\in m(\mathcal{M},\phi,p)$. Then

$$\sup_{s\geq 1}\sup_{\sigma\in\mathcal{C}_s}\left[\frac{1}{\phi_s}\left\{\sum_{n\in\sigma}\left[M_n\left(\frac{|x_n|}{\rho}\right)\right]^p\right\}^{1/p}\right]=K<\infty,\ \text{for some } \rho>0.$$

This implies that

$$|x_n|\leq K\phi_1,\ \text{for all } n\in\mathbb{N},$$

so that $x\in l^\infty$. Thus $m(\mathcal{M},\phi,p)\subset l^\infty$.

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