STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LV**, Number 2, June 2010

ON THE REGULARITY OF SOLUTIONS OF A BOUNDARY VALUE PROBLEM USING DECOMPOSITION AND LOCALIZATION TECHNIQUES IN A CORNER

HAMID BENSERIDI AND MOURAD DILMI

Abstract. The subject of this work is the study of the singular behaviour of solutions for the Lamé system with Dirichlet, mixed and Neumann conditions in a bounded domain. A technique of localization of the problem in a corner is presented. The method is an adaptation of that of Kondratiev [8] extended to the weighted Sobolev spaces. This method have been considered by many authors.

1. **Introduction**

Questions of existence and uniqueness have been considered in Grisvard [6] for the Lamé system in the classical framework of weighted Sobolev spaces with weight in a polygon. The Sobolev spaces with double weight have been introduced in Dauge [3] for the Stokes system in a polygon.

In [1], Benseridi and Dilmi have used the complex Fourier transform with respect to the first variable in an infinite sector for a class of double weighted Sobolev spaces, to study problems of existence, unicity, regularity, and singularity of solutions of the Lamé system.

In their paper, Benseridi and Merouani [2], have studied some transmission problems related to the Lamé system in a polyhedron for a class of double weighted Sobolev spaces. They have given an explicit description of singularities of the variational solutions for the homogeneous case, by the same they have shown that the singular behaviour of the solutions is governed by a sequence of transcendental equations.

Here, we give an extension for some results previously obtained by the above mentioned authors. This paper is organized as follows: In section 1 we give some

Received by the editors: 16.02.2009.
2000 *Mathematics Subject Classification.* 35B40, 35B65, 35C20.
Key words and phrases. Corner, decomposition, eigenvalue, elasticity system, localization, singular functions, weighted Sobolev space.

basic tools and properties related to the weighted Sobolev spaces which will be useful for the next. Section 2 is concerned with the notations and the formulation of our problem (P_1), while section 3, we study the regularity of the weak solution of the mixed problem (P_1) by using the technique of localization in a corner and this is done by means of the weighted Sobolev spaces. The solution is expressed as a some of a regular and a singular part. Finally, we state our main result by giving an explicit calculus of the singular functions that appear in the singular part of the solutions of the three problems (Dirichlet, Neumann and mixed). To do this, we compute the eigenvalues and the corresponding eigenvectors.

2. **Overview on the weighted Sobolev spaces**

In this section we give some basic tools and properties related to the weighted Sobolev spaces which will be useful in the next.

In what follows Ω is an infinite plane-sector of an opening ω

$$\Omega = \{(x, y) : x + iy = re^{i\theta}\ ,\ r > 0,\ 0 < \theta < \omega\}.$$

B is the strip defined by: $B = \mathbb{R}\times]0\,,\omega[$, θ_0, θ_∞ are two reals: $\theta_0 \leq \theta_\infty$.

Definition 2.1. *Let Ω be an open bounded set of $\mathbb{R}^n$ with closure $\overline{\Omega}$, and boundary Γ. Let $\rho \in C^\infty(\mathbb{R}^n)$, $\rho > 0$ on Ω, $\rho = 0$ on Γ and* gradiant(ρ) *is nonnull on Γ. For a positive integer l, α and p two real numbers such that $p > 1$, $W_\alpha^{l,p}(\Omega)$ is the Banach space of the distributions u on Ω such that $\rho^\alpha D^\beta u \in L^p(\Omega)$, for $|\beta| \leq l$, equipped with the norm*

$$\|u\|_{W_\alpha^{l,p}(\Omega)} = \left[\sum_{|\beta|\leq l} \left\|\rho^\alpha D^\beta u\right\|_{L^p(\Omega)}^p\right]^{\frac{1}{p}}.$$

Definition 2.2. *Let $s \in \mathbb{N}$, we define the space $V^s(B)$ by*

$$V^s(B) = \{u \in L^2(B)\ /\ (1+\xi^2)^{\frac{k}{2}} u \in L^2(\mathbb{R}\ H^{s-k}(]0,\omega[)\ ),\ k = 0, ..., s\}.$$

$V^s(B)$ is a Hilbert space for which the scalar product is given by

$$\langle u, v\rangle = \sum_{k=0}^{s} \int_B\int (1+\xi^2)^k \left|D_\theta^{s-k} u\right| \left|D_\theta^{s-k} v\right| d\theta d\xi.$$

Lemma 2.3. ([5, 8]). *Let $\eta_1, \eta_2 \in \mathbb{R}$ such that, $\eta_1 \leq \eta_2$. If $f \in L^2_{\eta_1,\eta_2}(B)$, then*

1) $\forall\eta \in [\eta_1, \eta_2]$, $e^{\eta\, t} f \in L^2(B)$, and $\left\|e^{\eta\, t} f\right\|_{L^2(B)} \leq \|f\|_{L^2_{\eta_1,\eta_2}(B)}$;

2) $\forall\eta \in]\eta_1, \eta_2[$, $e^{\eta\, t} f \in L^1(B)$, and $\left\|e^{\eta\, t} f\right\|_{L^1(B)} \leq c\|f\|_{L^2_{\eta_1,\eta_2}(B)}$.

Definition 2.4. *We denote by T the partial Fourier transform with respect to the first variable on B, then*

$$T\,(f)\,(\varphi,\theta) = T\left(e^{\eta\, t} f\right)(\xi,\theta),\ \textit{with}\ \varphi = \xi + i\eta,$$

where $T\,(e^{\eta\, t} f)$ denotes the real Fourier transform of $e^{\eta\, t} f$ with respect to the first variable.

Clearly f admits a complex Fourier transform, if and only if, $e^{\eta\, t} f$ admits a real Fourier transform.

For simplicity we write: $T\,(f)\,(\varphi,\theta) = \widehat{f}\,(\varphi,\theta)$.

Property 2.5. *Let $f \in H^s_{\eta_1,\eta_2}(B)$, then, for every $k, j \in \mathbb{N}$ such that $k + j \leq s$, we have*

$$T\left(\frac{\partial^{k+j}\, f}{\partial\, t^k\, \partial\, \theta^j}\right) = (i\varphi)^k\ \frac{\partial^j}{\partial \theta^j} T\,(f)\,(\varphi,\theta),\ (i^2 = -1),$$

for every φ in $\mathbb{C}$ and $\mathrm{Im}\varphi \in [\eta_1, \eta_2]$.

3. **Notations and formulation of the problem**

Ω denotes an homogeneous body, elastic and isotrope, occupying a bounded domain of $\mathbb{R}^2$ with a polygonal rectilignal boundary $\Gamma = \underset{j\in J}{\cup} \Gamma_j$, where Γ_j are open piecewise lines. $\{J_1, J_2\}$ is a partition of J, s_j will be the origin of Γ_{j+1}, and s_{j+1} its extremity according to the usual orientation.

The opening of the angle formed by Γ_j and Γ_{j+1} towards the interior of Ω will be denoted ω_j, with $0 < \omega_j < 2\pi$ for all $j \in J$. Ω then defined is consequently an open bounded domain with Lipschitz boundary. All results on this kind of domain are valid here.

It is more convenient to work at the origin with polar coordinates. Therefore by a translation first and then by a rotation, we can bring back s_j, Γ_j, Γ_{j+1} to O, OX, O_ω (ω is the angle formed by OX and O_ω towards the interior of Ω).

Our interest is to study the properties of regularity for a weak solution of the following mixed problem (Dirichlet-Neumann)

$$\begin{cases} \mu\Delta u + (\lambda+\mu)\,\nabla\,(\mathrm{div} u) = f \ \text{ in } \Omega \\ u = 0 \qquad\qquad \text{on } \underset{j\in J_1}{\cup} \Gamma_j \\ \sigma\,(u)\,.\tau = 0 \qquad\quad \text{on } \underset{j\in J_2}{\cup} \Gamma_j \end{cases} \qquad (P_1)$$

where λ and μ are the elasticity coefficients with $\lambda > 0$ and $\lambda + \mu \geq 0$, (u), (f) designate respectively the displacement vector and the density of external powers. σ denote the stress tensor with $\sigma = (\sigma_{hk})$, h, $k = 1, 2$. The σ_{hk} elements are given by the Hooke's law

$$\sigma_{hk}(u) = 2\mu\varepsilon_{hk}(u) + \lambda\,\mathrm{div}(u)\delta_{hk},$$

where $\varepsilon_{hk}(u) = \frac{1}{2}(\partial_k u_h + \partial_h u_k)$ the symmetric deformation velocity tensor. τ is the normal vector.

Definition 3.1. *We denote by* V *the closure of the set*

$$\left\{ v \in C^\infty\left(\overline{\Omega}\right)^2, \ v/_{\Gamma_j} = 0 \ \text{ for every } j \in J_1 \right\} \ \text{ in } H^1\left(\Omega\right)^2.$$

In order to define a weak solution, we introduce a symmetric bilinear form on V^2 *by considering the scalar product of system* (P_1). *More explicitly*

$$\begin{array}{ccl} l & : & V^2 \longrightarrow \mathbb{R} \\ (u,v) & \longmapsto & l\left(u,v\right) = -\displaystyle\int_\Omega \sum_{h=1}^{2}\sum_{k=1}^{2}\sigma_{hk}\left(u\right)\varepsilon_{hk}\left(v\right)dx. \end{array}$$

Definition 3.2. *The function* $u \in V$ *is a weak solution for problem* (P_1) *if*

$$l\left(u,v\right) = \int_\Omega \sum_{h=1}^{2} f_h v_h dx, \ \forall v \in V.$$

There is no particular problem to apply the variational method for the resolution of (P_1) because the Korn inequality is still valid in a polygon, moreover it is known that there exists a unique weak solution $u \in V$, if the bilinear form is bounded in V^2 and coercive. These conditions are verified if $mes\,(\underset{j\in J_1}{\cup}\Gamma_j) > 0$. When it is the Neumann problem ($J_1 = \emptyset$), we suppose that the necessary condition of existence is verified, the orthogonality of the rigid displacements data, i.e. $\int_\Omega \sum_{h=1}^{2} f_h v_h dx = 0$, for every v of the form $v(x,y) = (a + cy, b - cx)$, with a, b, c arbitrary reals.

4. **Localization of the problem in a corner**

The analysis of the existence, the unicity and the regularity for the boundary value problem (P_1) is more developed when the domain Ω is sufficiently smooth. Many results has been obtained by many authors. The principal regularity is in the interior of the domain Ω and on $\Gamma / \underset{j\in J}{\cup} V_j$, where V_j is a closed neighbourhood of a vertex s_j. (s_j), $j \in J$, are called singular points.

In the sequel, we only envisage the singular behaviour of the solution of (P_1) in a neighbourhood of a singular point, then we transpose the results to the weak solution; for this aim we consider the function $\rho(r)$ such that

$$\begin{aligned} 0 &\leq \rho(r) \leq 1,\ \rho(r) \in C^{\infty}(]0,\omega[) \\ \rho(r) &= \begin{cases} 1, & \text{if} \quad 0 \leq r \leq \delta \\ 0, & \text{if} \quad r \geq 2\delta \end{cases}, \end{aligned}$$

where δ is the smallest positive real for which no singular point of Γ is in the circle $\{x : |x| \leq 3\delta\}$. We denote by $\mathbb{K}$ an infinite plane sector.

We set $w = \rho u$, the problem (P_1) will be

$$\begin{cases} \mu \triangle w + (\lambda + \mu) \bigtriangledown (\mathrm{div} w) = F \ \text{in } \mathbb{K} & (4.1), \\ w = 0, \text{on } \Gamma_0 \cup \Gamma_\omega & (4.2), \\ \qquad\qquad \text{or} & \\ \begin{cases} w = 0 \ \text{on } \Gamma_0 \ \text{and } \sigma(w).\tau = 0, \ \text{on } \Gamma_\omega & (4.3), \\ \sigma(w).\tau = 0 \ \text{on } \Gamma_0 \cup \Gamma_\omega & (4.4), \end{cases} \end{cases} \qquad (P_2)$$

where F is depending on f, ρ and u.

Since we are interested with the solution in a neighbourhood of the vertex $(\rho = 1)$, we can suppose for simplicity, that F is an arbitrary given data (which does not depend on the solution u). Under this hypothesis, we have $F = \rho f$.

5. **The regularity in the weighted Sobolev spaces**

This section is concerned with the decomposition of the solution in a regular part and a singular part. We denote by $A(D_x)$ the differential operator for system (4.1)

$$A(D_x) = \begin{pmatrix} (\lambda + 2\mu)\frac{\partial^2}{\partial x^2} + \mu\frac{\partial^2}{\partial y^2} & (\lambda + \mu)\frac{\partial^2}{\partial x \partial y} \\ (\lambda + \mu)\frac{\partial^2}{\partial x \partial y} & (\lambda + 2\mu)\frac{\partial^2}{\partial y^2} + \mu\frac{\partial^2}{\partial x^2} \end{pmatrix},$$

and $B(D_x)$ the boundary operator (4.2), (4.3), (4.4).

For (4.4) we have

$$B(D_x) = \begin{pmatrix} (2\mu + \lambda)\tau_1\frac{\partial}{\partial x} + \mu\tau_2\frac{\partial}{\partial y} & \lambda\tau_1\frac{\partial}{\partial y} + \mu\tau_2\frac{\partial}{\partial x} \\ \mu\tau_1\frac{\partial}{\partial y} + \lambda\tau_2\frac{\partial}{\partial x} & \mu\tau_1\frac{\partial}{\partial x} + (2\mu + \lambda)\tau_2\frac{\partial}{\partial y} \end{pmatrix}.$$

Let $a(D_x) = [A(D_x), B(D_x)]$ be the operator defined by

$$a(D_x) : \ H^2_{\beta,\beta}(\mathbb{K})^2 \longrightarrow L^2_{\beta,\beta}(\mathbb{K})^2 \times H^{2-m-\frac{1}{2}}_{\beta,\beta}(\Gamma_0)^2 \times H^{2-m-\frac{1}{2}}_{\beta,\beta}(\Gamma_\omega)^2,$$

where m represents the order of the trace operator, $m = 0$ for the Dirichlet condition and $m = 1$ for the Neumann condition.

By passing to the polar coordinates, and we apply the complex Fourier transform with respect to the first variable, the boundary value problem $a\,(D_X)\,w = F$ will be

$$a(z, D_\theta)\widehat{\widehat{w}} = \widehat{\widehat{F}},$$

where

$$a(z, D_\theta) = [A(z, D_\theta), B(z, D_\theta)] : \; H^2\,(]0, \omega[)^2 \longrightarrow L^2\,(]0, \omega[)^2 \times \mathbb{C}^2 \times \mathbb{C}^2,$$

with $A(z, D_\theta) = (A_{hk})$, $i^2 = -1$ and

$$\begin{aligned}
A_{11} &= -\mu z^2 + (\lambda+\mu)\left(\left(\frac{-z^2}{2} - iz\right)\cos 2\theta - \frac{-z^2}{2}\right) + \\
&\quad (\lambda+\mu)\,(1-iz)\sin 2\theta \frac{d}{d\theta} + \left(\mu + (\lambda+\mu)\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right)\right)\frac{d^2}{d\theta^2}. \\
A_{12} &= A_{21} = (\lambda+\mu)\left(\left(\frac{-z^2}{2} - iz\right)\sin 2\theta + (iz-1)\cos 2\theta \frac{d}{d\theta} - \frac{1}{2}\frac{d^2}{d\theta^2}\right). \\
A_{22} &= -\mu z^2 + (\lambda+\mu)\left(\left(\frac{z^2}{2} + iz\right)\cos 2\theta - \frac{z^2}{2}\right) + \\
&\quad (\lambda+\mu)\,(iz-1)\sin 2\theta \frac{d}{d\theta} + (\lambda+\mu)\left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right)\frac{d^2}{d\theta^2}.
\end{aligned}$$

$B(z, D_\theta)$ is the boundary operator.

For condition (4.2) and $\theta = 0$, we get

$$B(z, D_\theta) = \begin{pmatrix} \frac{d}{d\theta} & iz \\ 2\nu iz & 2(1-\nu)\frac{d}{d\theta} \end{pmatrix},$$

where

$$\nu = \frac{\lambda}{2\,(\lambda+\mu)}.$$

Definition 5.1. *The complex number $z = z_0$ is called an eigenvalue of $a(z, D_\theta)$ if there exists a nontrivial solution $e^0\,(z_0, \theta) \in H^2\,(]0, \omega[)^2$ for the equation*

$$a(z, D_\theta)e\,(z, \theta)\,|_{z=z_0} = 0.$$

$e^0\,(z_0, \theta)$ is called the eigenvector of $a(z, D_\theta)$ corresponding to z_0. The function $e^1\,(z_0, \theta)$ is an associated vector to z_0 if

$$-i\frac{da(z, D_\theta)}{dz}\,|_{z=z_0}\, e^0\,(z_0, \theta) + a(z_0, D_\theta)e^1\,(z_0, \theta) = 0.$$

Theorem 5.2. *The operator $A(D_x)$ is an isomorphism if and only if $a(z, D_\theta)$ has no eigenvalue with imaginary part $\beta - 1$.*

Theorem 5.3. *Let θ_0, θ_∞ two reals such that $\theta_0 \leq \theta_\infty$. We suppose that the operator $a(z, D_\theta)$ has no eigenvalue on the lines $\mathbb{R} + i(\theta_0 - 1)$, $\mathbb{R} + i(\theta_\infty - 1)$, then for every $F \in L^2_{\theta_0,\theta_\infty}(\mathbb{K})^2$ the solution $w \in H^2_{\theta_\infty,\theta_\infty}(\mathbb{K})^2$ of problem (P_2) is written in the following form*

$$\overline{w}(r,\theta) = \sum_{l=1}^{N}\sum_{\sigma=1}^{I_l}\sum_{k=0}^{\delta_{\sigma\, l}} C_{\sigma k l}\overline{u}_{k,l}^{(\sigma)}(r,\theta) + \overline{V}(r,\theta),$$

where $V \in H^2_{\theta_0,\theta_0}(\mathbb{K})^2$, $z_1, z_2, .., z_N$ are the eigenvalues of $a(z, D_\theta)$ such that $\theta_0 - 1 \leq$ $\mathrm{Im} z_l \leq \theta_\infty - 1$,

$$I_l = \dim\left(span\left\{e_1^0(z_0,\theta),\ e_2^0(z_0,\theta),...\right\}\right),$$

$$\delta_{\sigma\, l} = \begin{cases} 1, \text{ if an associated vector exists for } z_l \text{ and } e^0(z_0,\theta), \\ 0, \text{ otherwise.} \end{cases}$$

$C_{\sigma k l}$ are constants,

$$\overline{u}_{k,l}^{(\sigma)}(r,\theta) = r^{iz_l}\sum_{s=0}^{k}(\log r)^s e_\sigma^{k-s}(z_l,\theta)$$

are called singular functions.

Proof. See A. M. Sandig, U. Richter, R. Sandig [10]. □

We consider a weak solution $u \in V$ of problem (P_1).

Lemma 5.4. *Let $f \in L^2_{1+\varepsilon,1+\varepsilon}(\Omega)^2$, where ε is a small positive real, then*

$$\rho u \in H^2_{1+\varepsilon,1+\varepsilon}(\mathbb{K})^2.$$

Proof. We consider a sequence of domains Ω_h, $h = 1, 2, ...$ where $\Omega_h = \Omega \cap R_h$, with

$$R_h = \left\{x : \frac{\delta}{2^{h+1}} \leq |x| \leq \frac{\delta}{2^h}\right\}.$$

For $\widehat{\delta} = 2\delta$, we consider the function $\widehat{\rho}(r)$ such that

$$\begin{aligned} \widehat{\rho}(r) &\in C^\infty(]0,\infty[), 0 \leq \widehat{\rho}(r) \leq 1 \\ \widehat{\rho}(r) &= \begin{cases} 1 & if \quad 0 \leq r \leq \widehat{\delta} \\ 0 & if \quad r \geq 2\widehat{\delta} \end{cases} \end{aligned}$$

we have

$$\underset{h}{\cup}\Omega_h = \mathbb{K}_0 \subset \mathbb{K}.$$

The standard theorems of regularity give, for $|\gamma| = 2$

$$\iint_{\Omega_h} |D^\gamma u|^2\, dx \leq c\left[\iint_{\Omega_{h-1}\cup\, \Omega_h \cup\, \Omega_{h+1}} |f|^2\, dx + \iint_{\Omega_{h-1}\cup\, \Omega_h\cup\, \Omega_{h+1}} r^{-4}\, |u|^2\, dx\right]. \qquad (5.1)$$

Multiplying by $(\frac{\widehat{\delta}}{2^h})^{2(1+\varepsilon)}$, we obtain

$$\begin{aligned}\iint\limits_{\Omega_h} r^{2(1+\varepsilon)} \left|D^\gamma u\right|^2 dx \quad &\leq \quad c(\iint\limits_{\Omega_{h-1}\cup\ \Omega_h\cup\ \Omega_{h+1}} r^{2(1+\varepsilon)} \left|f\right|^2 dx + \\ &+ \iint\limits_{\Omega_{h-1}\cup\ \Omega_h\cup\ \Omega_{h+1}} r^{2(-1+\varepsilon)} \left|u\right|^2 dx)\end{aligned}$$

and by summing with respect to h, from 0 to ∞, the inequality (5.1) becomes

$$\iint\limits_{\mathbb{K}_0} r^{2(1+\varepsilon)} \left|D^\gamma u\right|^2 dx \leq c \left[\iint\limits_{\mathbb{K}_0} r^{2(1+\varepsilon)} \left|f\right|^2 dx + \iint\limits_{\mathbb{K}_0} r^{2(-1+\varepsilon)} \left|u\right|^2 dx \right].$$

Clearly

$$\iint\limits_{\mathbb{K}_0} r^{2(-1+\varepsilon)} \left|u\right|^2 dx = \iint\limits_{\mathbb{K}_0} r^{2(-1+\varepsilon)} \left|\rho u\right|^2 dx.$$

Now, passing to polar coordinates and using Hardy inequality, we get

$$\int\limits_0^\infty \left|f(t)\right|^2 t^{(\varepsilon'-2)} dt \leq \left(\frac{2}{\left|\varepsilon'-1\right|}\right)^2 \int\limits_0^\infty \left|f'(t)\right|^2 t^{\varepsilon'} dt,$$

for $\varepsilon' > 1$ and $\lim\limits_{t\to\infty} f(t) = 0$. We get for $\overline{u}(r,\theta) = u(x,y)$

$$\begin{aligned}\iint\limits_{\mathbb{K}_0} r^{-2+2\varepsilon+1} \left|\widehat{\rho\overline{u}}\right|^2 dr d\theta \ &\leq \int\limits_0^\omega \int\limits_0^\infty r^{-2+2\varepsilon+1} \left|\widehat{\rho\overline{u}}\right|^2 dr d\theta \\ \leq \int\limits_0^\omega (\frac{2}{2\varepsilon})^2 \int\limits_0^\infty r^{2\varepsilon} \left|\frac{\partial}{\partial r}\widehat{\rho\overline{u}}\right|^2 r dr d\theta \ &\leq \ c \iint\limits_{\Omega\ \cap\ \operatorname{sup} p\ \widehat{\eta}} r^{2\varepsilon} \left|u\right|^2 dx \\ +c \iint\limits_{\Omega\ \cap\ \operatorname{sup} p\ \widehat{\eta}} r^{2\varepsilon} (\left|\text{gradiant} u_1\right|^2 + \left|\text{gradiant} u_2\right|^2) dx &\leq c \left\|u\right\|_{H^1(\Omega)^2}.\end{aligned}$$

For $|\gamma| = 2$ we have

$$\iint\limits_{\mathbb{K}_0} r^{2(1+\varepsilon)} \left|D^\gamma \rho u\right|^2 dx \leq c \sum_{|\gamma'|\leq\ 2} \iint\limits_{\mathbb{K}_0} r^{2(1+\varepsilon)} \left|D^{\gamma'} \rho u\right|^2 dx,$$

therefore, $\rho u \in H^2_{1+\varepsilon\,,1+\varepsilon}(\mathbb{K})^2$. □

We can now give the following theorem.

Theorem 5.5. ([10]) *Let $u \in V$ a weak solution of problem (P_1). Let ε a real positive small number such that the operator $a(z, D_\theta)$ has no eigenvalue with imaginary part ε or (-1). We suppose that $\rho f \in L^2(\mathbb{K})^2$, then*

$$\rho^2 \overline{u}(r,\theta) = \rho \sum_{l=1}^{N} \sum_{\sigma=1}^{I_l} \sum_{k=0}^{\delta_{\sigma l}} C_{\sigma k l} \overline{u}_{k,l}^{(\sigma)}(r,\theta) + \rho \overline{V}(r,\theta),$$

where $\rho V \in H^2(\mathbb{K})^2$.

6. Computation of the singular functions

Our goal is to compute the functions $\overline{u}_{k,l}^{(\sigma)}(r,\theta)$ for the three problems (Dirichlet, Neumann, and mixed). To do this, we have to know the eigenvalues z_l of $a(z, D_\theta)$, the corresponding eigenvectors, and the associated vectors.

6.1. Dirichlet problem.

Lemma 6.1. *If z_l is an eigenvalue of $a(z, D_\theta)$, for the angle ω, $\omega \notin \{\pi, 2\pi\}$, we get $I_l = 1$ and*

$$e_1^0(z_l,\theta) = C_3(z_l) Y_3(z_l,\theta) + C_4(z_l) Y_4(z_l,\theta),$$

where

$$Y_3(z_l,\theta) = \begin{pmatrix} -\cosh(z_l\theta) + \cosh(z_l+2i)\theta \\ (1 - \frac{2(3-4\nu)}{iz_l}) i \sinh(z_l\theta) - i\sinh(z_l+2i)\theta \end{pmatrix}.$$

$$Y_4(z_l,\theta) = \begin{pmatrix} -(1 + \frac{2(3-4\nu)}{iz_l}) i \sinh(z_l\theta) + i\sinh(z_l+2i)\theta \\ -\cosh(z_l\theta) + \cosh(z_l+2i)\theta \end{pmatrix}.$$

$$C_3(z_l) = -\cosh(z_l\omega) + \cosh(z_l+2i)\omega.$$

$$C_4(z_l) = -\left(1 - \frac{2(3-4\nu)}{iz_l}) i \sinh(z_l\omega\right) + i\sinh(z_l+2i)\omega.$$

For $\omega = \pi$ or $\omega = 2\pi$ we have

$$z_l = -il \text{ or } z_l = -\frac{il}{2},\ l = 1, 2, ...$$

For the two cases $I_l = 2$ and $e_1^0(z_l,\theta) = Y_3(z_l,\theta)$, $e_2^0(z_l,\theta) = Y_4(z_l,\theta)$ are two eigenvectors linearly independent.

Proof. Note that the eigenvectors of $a(z, D_\theta)$ are the zeros of the transcendental function $D_1(z)$ defined by

$$D_1(z) = 4\sin^2\omega + \left(\frac{2(3-4\nu)}{iz}\right)^2 \sinh^2(z\omega).$$

The general solution for the equation $A(z,D_\theta)e\,(z,\theta)=0$ is given by

$$\begin{aligned} e\,(z,\theta) &= C_1(z)\begin{pmatrix}\cosh(z\theta)\\ -i\sinh(z\theta)\end{pmatrix}+C_2(z)\begin{pmatrix} i\sinh(z\theta)\\ \cosh(z\theta)\end{pmatrix}+\\ & C_3(z)\begin{pmatrix}\cosh(z+2i)\theta\\ -i\sinh(z+2i)\theta-\frac{2(3-4\nu)}{iz}i\sinh(z\theta)\end{pmatrix}+\\ & C_4(z)\begin{pmatrix} i\sinh(z+2i)\theta\\ \cosh(z+2i)\theta+\frac{2(3-4\nu)}{iz}\cosh(z\theta)\end{pmatrix}.\end{aligned}$$

The condition $B(z_l,D_\theta)e\,(z_l,\theta)=0$ for $\theta=0$ shows that $C_1(z_l)=-C_3(z_l)$ and $C_2(z_l)=-C_4(z_l)(1+\frac{2(3-4\nu)}{iz_l})$. From the condition $B(z_l,D_\theta)e\,(z_l,\theta)=0$ for $\theta=\omega$, it comes that $M(z_l,\omega)C(z_l)=0$, where

$$\begin{aligned} M(z_l,\omega) &= \begin{pmatrix} -\cosh(z_l\omega) & -(1+\frac{2(3-4\nu)}{iz_l})i\sinh(z_l\omega)\\ -\frac{2(3-4\nu)}{iz_l}\,i\sinh(z_l\omega) & -\cosh(z_l\omega)\end{pmatrix}+\\ & \begin{pmatrix}\cosh(z_l+2i)\omega & i\sinh(z_l+2i)\omega\\ 0 & i\cosh(z_l+2i)\omega\end{pmatrix},\\ C(z) &= \begin{pmatrix} C_3(z_l)\\ C_4(z_l)\end{pmatrix}.\end{aligned}$$

The determinant of the matrix $M(z_l,\omega)$ is equal to $D_1(z_l)$ which is null, we can then choose $C_3(z_l),C_4(z_l)$ such that

$$C_4(z_l)=-(1-\frac{2(3-4\nu)}{iz_l})i\sinh(z_l\omega)+i\sinh(z_l+2i)\omega,$$

$$C_3(z_l)=-\cosh(z_l\omega)+\cosh(z_l+2i)\omega.$$

Replacing $C_3(z_l),C_4(z_l)$ by their values in the expression of solution $e\,(z,\theta)$, we obtain

$$e_1^0(z_l,\theta)=C_3(z_l)Y_3(z_l,\theta)+C_4(z_l)Y_4(z_l,\theta).$$

Now if $\omega=\pi$ or $\omega=2\pi$, the rank of the matrix of the system which results from the boundary condition $B(z_l,D_\theta)e\,(z_l,\theta)=0$ for $\theta=0$, $\theta=\omega$, is equal to 2, consequently $I_l=2$, we can choose $C_3(z_l)=0,C_4(z_l)=1$ or $C_3(z_l)=1,C_4(z_l)=0$, which proves that $e_1^0(z_l,\theta),e_2^0(z_l,\theta)$ are the linearly independant eigenvectors. □

Remark 6.2. *For $z=0$ we have $D_1(0)=2-2\cos 2\omega-4(3-4\nu)^2\omega^2$, then $D_1(0)$ is null if and only if $\omega=0$, consequently $z=0$ is not an eigenvalue of $a(z,D_\theta)$.*

In the sequel, we are going to study the correlation between the order of multiplicity of an eigenvalue z_l of the operator $a(z,D_\theta)$ and the existence of an associated vector. For this, we denote by $m(z_l)$ the order of multiplicity of z_l.

The two following propositions are similar to [10].

Proposition 6.3. *Denote by $m(z_l)$ the order of multiplicity of (z_l), then*

$$m(z_l) = \sum_{\sigma=1}^{I_l} (\delta_{\sigma l} + 1) \geq I_l.$$

Proposition 6.4. *If $m(z_l) = 2$ and $I_l = 1$, then there exists an associated unique vector and if $\omega = \pi$ or $\omega = 2\pi$ if there is any associated vector.*

Lemma 6.5. *Suppose that*

$$(H) \left\{ \begin{array}{c} \tanh(z_l\omega) = \omega z_l, \\ \left(\dfrac{\sin\omega}{\omega}\right)^2 = \left[(3-4\nu)\cosh(z_l\omega)\right]^2, \\ \sinh(z_l\omega)\,\cosh(z_l\omega) \quad \text{is nonnull.} \end{array} \right.$$

Then $m(z_l) = 2$ and moreover the associated vector to the eigenvalue z_l is

$$e_1^1(z_l,\theta) = -i\frac{de_1^0(z,\theta)}{dz}\Big|_{z=z_l},$$

$e_1^0(z_l,\theta)$ is the eigenvector as defined in Lemma 6.1 by substituting z_l by z.

Proof. The hypothesis (H) is verified if and only if

$$D_1(z_l) = 0 \text{ and } D_1'(z_l) = 0.$$

Then $m(z_l) = 2$; which insures the existence of an associated vector.
We know that the associated vectors are the solutions of the equation

$$-i\frac{da(z,D_\theta)}{dz}\Big|_{z=z_l} e_1^0(z_l,\theta) + a(z_l,D_\theta)e_1^1(z_l,\theta) = 0 \tag{6.1}$$

i.e., for $z = z_l$

$$-i\frac{dA(z,D_\theta)}{dz} e_1^0(z,\theta) + A(z,D_\theta)e_1^1(z,\theta) = 0 \tag{6.2}$$

and

$$-i\frac{dB(z,\theta)}{dz} + B(z,D_\theta)e_1^1(z,\theta) = 0 \text{ for } \theta = 0,\ \theta = \omega.$$

$A(z,D_\theta)e_1^0(z,\theta) = 0$, for all z in a neighbourhood of z_l, then

$$\frac{d}{dz}\left[A(z,D_\theta)e_1^0(z,\theta)\right] = 0.$$

But

$$\frac{d}{dz}\left[A(z,D_\theta)e_1^0(z,\theta)\right] = A(z,D_\theta)\frac{de_1^0(z,\theta)}{dz} + \frac{dA(z,D_\theta)}{dz}e_1^0(z,\theta). \tag{6.3}$$

We multiply (6.2) by i, and then we compare it with (6.3), we find

$$e_1^1(z_l,\theta) = -i\frac{de_1^0(z,\theta)}{dz}\Big|_{z=z_l}.$$

In a similar way we prove that $\theta = 0$

$$-i\frac{dB(z,D_\theta)}{dz}\Big|_{z=z_l} + B(z_l,D_\theta)e_1^1(z_l,\theta) = 0.$$

For $\theta = \omega$, we have

$$B(z,D_\theta)e_1^0(z,\theta)=\begin{pmatrix} D_1(z_l)\\ 0\end{pmatrix},$$

then

$$\frac{dB(z,D_\theta)e_1^0(z,\theta)}{dz}\Big|_{z=z_l}=\begin{pmatrix} D_1'(z_l)\\ 0\end{pmatrix}.$$

On the other hand

$$\begin{aligned}\frac{dB(z,D_\theta)e_1^0(z,\theta)}{dz}\Big|_{z=z_l} &= \frac{dB(z,D_\theta)}{dz}\Big|_{z=z_l}e_1^0(z_l,\theta)+B(z_l,D_\theta)\frac{de_1^0(z_l,\theta)}{dz}\Big|_{z=z_l}\\ &= i\left[-i\frac{dB(z,D_\theta)}{dz}\Big|_{z=z_l}+B(z_l,D_\theta)e_1^1(z_l,\theta)\right]=0.\end{aligned}$$

The proof is complete. □

The following theorem gives a summary for the results concerning the singular functions.

Theorem 6.6. *The singular functions of the weak solution $u\in V$ of the Dirichlet problem are given as follows:*

(1) *If $\omega\notin\{\pi,2\pi\}$and z_l is a simple nonnull zero of $D_1(z)$, then there exists a unique singular function*

$$\overline{u}_{0,l}^{(1)}(r,\theta)=r^{iz_l}e_1^0(z_l,\theta).$$

(2) *If $\omega\notin\{\pi,2\pi\}$ and z_l is a nonnull double zero of $D_1(z)$, then there exist two singular functions*

$$\begin{aligned}\overline{u}_{0,l}^{(1)}(r,\theta) &= r^{iz_l}e_1^0(z_l,\theta),\\ \overline{u}_{1,l}^{(1)}(r,\theta) &= r^{iz_l}\left[\,e_1^1(z_l,\theta)+(\log r)e_1^0(z_l,\theta)\right].\end{aligned}$$

(3) *If $\omega=\pi$, then $z_l=-il$, $l=1,2,...$*

$$\begin{aligned}\overline{u}_{0,l}^{(1)}(r,\theta) &= r^l e_1^0(z_l,\theta),\\ \overline{u}_{0,l}^{(2)}(r,\theta) &= r^l e_2^0(z_l,\theta).\end{aligned}$$

(4) *If $\omega=2\pi$, then $z_l=-\frac{il}{2}$, $l=1,2,...$*

$$\begin{aligned}\overline{u}_{0,l}^{(1)}(r,\theta) &= r^{\frac{l}{2}}e_1^0(z_l,\theta),\\ \overline{u}_{0,l}^{(2)}(r,\theta) &= r^{\frac{l}{2}}e_2^0(z_l,\theta).\end{aligned}$$

Remark 6.7. *Notice that in* (1) *and* (2)

$$e_1^0(z_l,\theta)=C_3(z_l)Y_3(z_l,\theta)+C_4(z_l)Y_4(z_l,\theta),$$

and in (3) *and* (4)

$$e_1^0(z_l,\theta)=Y_3(z_l,\theta),\ e_2^0(z_l,\theta)=Y_4(z_l,\theta).$$

6.2. **The mixed problem.** We have previously found the expression for the general solution of equation $A(z, D_\theta)e(z,\theta) = 0$.

The Dirichlet-Neumann condition $B(z, D_\theta)e(z,\theta) = 0$ for $\theta = 0$ and $\theta = \omega$ gives a system of Cramer of order 4 with determinant

$$D_2(z) = -16\mu\left[z^2\sin^2\omega + 4(1-\nu)^2 + (3-4\nu)\sinh^2(z\omega)\right],\ \forall\ z,\ z \neq 0.$$

Therefore the eigenvalues of the operator $a(z, D_\theta)$ are the zeros of the transcendental equation

$$z^2\sin^2\omega + 4(1-\nu)^2 + (3-4\nu)\sinh^2(z\omega) = 0.$$

Remark 6.8. *If $z = 0$, then the determinant $D_2(0)$ is given by*

$$D_2(0) = 4(1-(3-4\nu)^2)(\lambda+2\mu).$$

In this case there is no eigenvalue.

Lemma 6.9. *If z_l is a zero of $D_2(z)$ then $I_l = 1$ and*

$$e_1^0(z_l,\theta) = C_3(z_l)Y_3(z_l,\theta) + C_4(z_l)Y_4(z_l,\theta)$$

is an eigenvector, where

$$\begin{aligned}
Y_3(z_l,\theta) &= \begin{pmatrix} \left(\frac{-4(1-\nu)}{iz_l} - 1\right)\cosh(z_l\omega) + \cosh(z_l+2i)\omega \\ \left(\frac{-2(1-2\nu)}{iz_l} + 1\right)i\sinh(z_l\omega) - i\sinh(z_l+2i)\omega \end{pmatrix}, \\
Y_4(z_l,\theta) &= \begin{pmatrix} \left(\frac{-2(1-2\nu)}{iz_l} - 1\right)i\sinh(z_l\omega) + i\sinh(z_l+2i)\omega \\ \left(\frac{4(1-\nu)}{iz_l} - 1\right)\cosh(z_l\omega) + \cosh(z_l+2i)\omega \end{pmatrix}, \\
C_3(z_l) &= \left(\frac{4(1-\nu)}{iz_l} - 1\right)\cosh(z_l\omega) + \cosh(z_l+2i)\omega, \\
C_4(z_l) &= \left(\frac{2(1-2\nu)}{iz_l} - 1\right)i\sinh(z_l\omega) + i\sinh(z_l+2i)\omega.
\end{aligned}$$

Proof. The rank of the matrix of $D_2(z)$ is equal to 3, consequently I_l=1. We use the same idea for the proof as in Lemma 6.1. We consider the general solution $e(z_l,\theta)$ of equation $A(z_l, D_\theta)e(z_l,\theta) = 0$, and we determine the constants $C_1(z_l)$, $C_2(z_l)$, $C_3(z_l)$ and $C_4(z_l)$ such that they verify the boundary condition $B(z_l, D_\theta)e(z_l,\theta) = 0$ for $\theta = 0$ and $\theta = \omega$.
Finally, we obtain the result $e_1^0(z_l,\theta) = C_3(z_l)Y_3(z_l,\theta) + C_4(z_l)Y_4(z_l,\theta)$.

Now we seek for the associated vectors.

Lemma 6.10. *Let z_l be a zero of $D_2(z)$.*
(1) If $m(z_l) = 2$, then the associated vectors exist.

(2) The equalities

$$
\begin{aligned}
(4\nu-3)\frac{\sinh(z_l\omega)\cosh(z_l\omega)}{z_l\omega} &= \frac{\sin^2\omega}{\omega^2},\\
(z_l\omega)\ \sinh(z_l\omega)\cosh(z_l\omega) &= \sinh^2(z_l\omega)+\frac{4(1-\nu)^2}{(3-4\nu)},
\end{aligned}
$$

are necessary and sufficient so that $m(z_l)=2$.

(3) The associated vectors are given by

$$e_1^1(z_l,\theta)=-i\frac{de_1^0(z,\theta)}{dz}\Big|_{z=z_l},$$

where $e_1^0(z,\theta)$ is the eigenvector defined in the previous lemma by replacing z_l by z.

Proof. (1) From proposition (1).

(2) The two equations are verified if and only if $D_2(z_l)=D_2'(z_l)=0$.

(3) The proof is similar to that of Lemma 6.5.

The following theorem is similar to Theorem 6.6 for the mixed problem.

Theorem 6.11. *The singular functions of the weak solution $u\in V$ for the mixed problem have the following forms*

(1) If z_l is a simple zero of $D_2(z)$, then there exists only one singular function

$$\overline{u}_{0,l}^{(1)}(r,\theta)=r^{iz_l}e_1^0(z_l,\theta).$$

(2) If z_l is a double zero of $D_2(z)$, then there exist two singular functions

$$
\begin{aligned}
\overline{u}_{0,l}^{(1)}(r,\theta) &= r^{iz_l}e_1^0(z_l,\theta),\\
\overline{u}_{1,l}^{(1)}(r,\theta) &= r^{iz_l}\left[\ e_1^1(z_l,\theta)+(\log r)e_1^0(z_l,\theta)\right],
\end{aligned}
$$

where $e_1^0(z_l,\theta)$ is the eigenvector defined in Lemma 6.9 and $e_1^1(z_l,\theta)$ is the associated vector defined in Lemma 6.10.

6.3. **Neumann problem.** In this case we consider the boundary conditions of Neumann $B(z,D_\theta)e(z,\theta)=0$. These conditions give a system of four equations with determinant is

$$D_3(z)=-32\mu^2z^2\left[-z^2\sin^2\omega+\sinh^2(z\omega)\right],\ \forall\ z,\ z\neq 0.$$

Therefore the boundary problem have a nontrivial solution, if and only if $D_3(z)=0$, consequently the eigenvalues of the operator $a(z,D_\theta)$ are the zeros of $D_3(z)$.

Remark 6.12. *For $z=0$, we have $D_3(0)=0$ for all ω.*

Lemma 6.13. *(1) We suppose that z_l is a zero of $D_3(z)$, $z_l\notin\{0,-i\}$, $\omega\notin\{\pi,2\pi\}$, then $I_l=1$ and*

$$e_1^0(z_l,\theta)=C_3(z_l)Y_3(z_l,\theta)+C_4(z_l)Y_4(z_l,\theta)$$

is an eigenvector, where

$$Y_3(z_l,\theta) = \begin{pmatrix} \left(\frac{-4(1-\nu)}{iz_l} - 1\right)\cosh(z_l\omega) + \cosh(z_l+2i)\omega \\ \left(\frac{-2(1-2\nu)}{iz_l} + 1\right) i\sinh(z_l\omega) - i\sinh(z_l+2i)\omega \end{pmatrix},$$

$$\begin{aligned} Y_4(z_l,\theta) &= \begin{pmatrix} \left(\frac{-2(1-2\nu)}{iz_l} - 1\right) i\sinh(z_l\omega) + i\sinh(z_l+2i)\omega \\ \left(\frac{4(1-\nu)}{iz_l} - 1\right)\cosh(z_l\omega) + \cosh(z_l+2i)\omega \end{pmatrix}, \\ C_3(z_l) &= \mu(z_l+2i)\sinh(z_l\omega) - z_l\sinh(z_l+2i)\omega, \\ C_4(z_l) &= iz_l\left[\cosh(z_l\omega) - \cosh(z_l+2i)\omega\right]. \end{aligned}$$

(2) The number $z_l = (-i)$ is an eigenvalue of $a(z, D_\theta)$ for all $\omega \in]0, 2\pi]$ and its eigenvector is

$$e_1^0(-i,\theta) = \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}.$$

(3) If $\omega = \pi$ (resp. $\omega = 2\pi$), then $z_l = -il$ (resp. $z_l = \frac{-il}{2}$), $l = 1, 2, ...,$ and $I_l = 2$. The eigenvectors are

(a) If $z_l = -i$

$$e_1^0(-i,\theta) = \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix},\ e_2^0(-i,\theta) = Y_3(-i,\theta).$$

(b) If z_l is different from $-i$

$$e_1^0(z_l,\theta) = Y_4(z_l,\theta)\ ,\ e_2^0(z_l,\theta) = Y_3(z_l,\theta).$$

Proof. (1) We consider the general solution $e(z,\theta)$ for the equation

$$A(z, D_\theta)e(z,\theta) = 0,$$

then using conditions $B(z_l, D_\theta)e(z_l,\theta) = 0$, for $\theta = 0$ and $\theta = \omega$, we obtain (1).
(2) It is easy to check that $z_l = (-i)$ is a zero of $D_3(z)$ for all $\omega \in]0, 2\pi[$.

The boundary conditions $B(z_l, D_\theta)e(z_l,\theta) = 0$ for $\theta = 0$, $\theta = \omega$ give the following system

$$\begin{pmatrix} -4\sin\omega & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{6.4}$$

We choose $C_3 = 0, C_4 = \frac{-1}{4(1-\nu)}$, we obtain

$$e_1^0(-i,\theta) = \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix}.$$

(3) When $\omega = \pi$ or $\omega = 2\pi$, the matrix in (6.4) will be null, we can then choose $C_3 = 1, C_4 = 0$ or $C_3 = 0, C_4 = 1$ thus (a). The proof of part (b) is similar to the last part in Lemma 6.1.

The following lemma illustrates the correlation between the order of multiplicity of an eigenvalue and the existence of an associated vector.

Lemma 6.14. *(1) If z_l is different from $-i$, $I_l = 1$ and $m(z_l) = 2$, then there exists an associated vector.*
(2) The conditions

$$\begin{aligned}\tanh(z_l\omega) &= \tanh(z_l\omega),\\ \cosh^2(z_l\omega) &= \frac{\sin^2\omega}{\omega^2},\ z_l \neq -i,\end{aligned}$$

are necessary and sufficient to $m(z_l) = 2$, and in this case the associated vectors are given by

$$e_1^1(z_l,\theta) = -i\frac{de_1^0(z,\theta)}{dz}\Big|_{z=z_l},$$

where $e_1^0(z,\theta)$ is the eigenvector defined in part (1) of the previous lemma by replacing z_l by z.

To prove this lemma, it suffices to compare with Lemma 6.5 and Lemma 6.10.

To close this section, we give a similar theorem as in 6.11 which corresponds to the Neumann case.

Theorem 6.15. *The singular functions of the weak solution $u \in V/\mathrm{Im}$ of the Neumann problem are*
(1) If $\omega \notin \{\pi, 2\pi\}$ and z_l is a simple zero of $D_3(z)$, then there exists a unique singular function

$$\overline{u}_{0,l}^{(1)}(r,\theta) = r^{iz_l}e_1^0(z_l,\theta).$$

(2) If $\omega \notin \{\pi, 2\pi\}$ and z_l is a double zero of $D_3(z)$, then there exist two singular functions

$$\begin{aligned}\overline{u}_{0,l}^{(1)}(r,\theta) &= r^{iz_l}e_1^0(z_l,\theta),\\ \overline{u}_{1,l}^{(1)}(r,\theta) &= r^{iz_l}\left[\ e_1^1(z_l,\theta) + (\log r)e_1^0(z_l,\theta)\right].\end{aligned}$$

(3) If $\omega = \pi$, then $z_l = -il$, $l = 1, 2, ...,$ and

$$\begin{aligned}\overline{u}_{0,l}^{(1)}(r,\theta) &= r^l e_1^0(-il,\theta),\\ \overline{u}_{0,l}^{(2)}(r,\theta) &= r^l e_2^0(-il,\theta).\end{aligned}$$

(4) If $\omega = 2\pi$, *then* $z_l = \frac{-il}{2}$, $l = 1, 2, ...$, *and*

$$\overline{u}_{0,l}^{(1)}(r,\theta) = r^{\frac{l}{2}} e_1^0 \left(\frac{-il}{2}, \theta \right),$$

$$\overline{u}_{0,l}^{(2)}(r,\theta) = r^{\frac{l}{2}} e_2^0 \left(\frac{-il}{2}, \theta \right).$$

Remark 6.16. *Note that*

- *In (1) and (2),* $e_1^0(z_l,\theta) = C_3(z_l)Y_3(z_l,\theta) + C_4(z_l)Y_4(z_l,\theta)$.
- *In (3) and (4),* $e_1^0(z_l,\theta) = Y_4(z_l,\theta)$, $e_2^0(z_l,\theta) = Y_3(z_l,\theta)$.
- $\mathrm{Im} = span\{(0,\ 1),\ (0,\ 1),\ (-x_2,\ x_1)\}$.

Aknowledgment. This work is dedicated to the memory of our colleague N. Mosbah.

References

[1] H. Benseridi, M. Dilmi, *Boundary value problems in plan sector with corners for a class of Sobolev spaces of double weight*, J. Appl. Funct. Anal., **3**(2008), no. 2, 233-242.

[2] H. Benseridi, B. Merouani, *Quelques problèmes de transmission liés au système de Lamé dans un polyèdre pour une classe d'espaces de Sobolev à doubles poids*, Rev. Roum. Sci. Techn.-Méc. Appl., **48**(2003), no. 1-6, 21-34.

[3] M. Dauge, *Etude de l'opérateur de Stokes dans un polygone: régularité, singularité et théorèmes d'indices*, Thèse de Doctorat de 3ème cycle, Université de Nantes, 1980.

[4] M. Dilmi, H. Benseridi, *Problème de contact sans frottement-Dirichlet pour les équations de Laplace et de Lamé dans un polygone*, Anal. Univ. Oradea, **XIV**(2007), 221-236.

[5] M. Dilmi, H. Benseridi, B. Merouani, *Nonlinear and oblique boundary value problems for the Lamé equations*, Applied Mathematical Sciences, **1**(2007), no. 51, 2517-2528.

[6] P. Grisvard, *Boundary value problems in plan polygons*, E.D.F., Bulletin de la Direction des études et Recherche, série C, Mathématique, **1**(1986), 21-59.

[7] J. P. Guillement, *Problème biharmonique dans un secteur plan*, Séminaire d'Analyse, Nantes (1978-1979), 143-160.

[8] V. A. Kondratiev, *Problèmes aux limites pour les équations elliptiques dans des domaines avec poids coniques ou anguleux*, Trudy Moskov. Mat. Obse., **16**(1967).

[9] N. Mosbah, *Etude du système de Lamé dans un polygone pour une classe d'espaces de Sobolev à double Poids*, Thèse de Magister, Université de Ferhat Abbas Sétif, (1998).

[10] A. M. Sandig, U. Richter, R. Sandig, *The regularity of boundary value problems for the Lamé equations in a polygonal domain*, Rostock. Math. Kolloq., **36**(1989), 21-50.

Univ. Ferhat Abbas, Faculty of Sciences, Department of Mathematics
Sétif, 19000, Algeria
E-mail address: m_benseridi@yahoo.fr

Univ. M. Boudiaf, Faculty of Sciences, Department of Mathematics
M'sila, 28000, Algeria
E-mail address: mouraddil@yahoo.fr