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ON THE $\delta(\varepsilon)$-STABLE OF COMPOSED RANDOM VARIABLES

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Abstract. Let ξ be a random variable (r.v.) with the characteristic function $\varphi(t)$ and ν be a r.v. with the generating function $a(z)$, ν is independent of ξ. It is known (see [1]) that the composed r.v. η of ξ and ν (denote by $\eta =< \nu, \xi >$) is the r.v. having the characteristic function $\psi(t) = a[\varphi(t)]$. The r.v. ν is called to be the first component of η and ξ is called to be the second component of η. In this paper, we shall investigate the changes of the distribution function of the composed r.v. η if we have the small changes of the distribution function of the first component ν or the second component ξ of η.

1. Introduction

Let ξ be a random variable (r.v.) with the characteristic function $\varphi(t)$ and the distribution function $F(x)$. Let ν be a r.v. independent of ξ and has the generating function $a(z)$ with the distribution function $A(x)$. It is known (see [1]) that the composed r.v. of ν and ξ is denote by

$$\eta =< \nu, \xi > \tag{1.1}$$

and has the characteristic function

$$\psi(t) = a[\varphi(t)]. \tag{1.2}$$

The r.v. ν is called to be the first component and the r.v. ξ is called to be the second component of the r.v. η.

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Example 1.1. Let us consider the integer valued nonegative r.v.:

$$\eta = \sum_{k=1}^{\nu} \xi_k \tag{1.3}$$

where $\xi_1, \xi_2, ...$ are i.i.d random variables have the same the distribution function with r.v. ξ, ν is a positive value r.v., independent of all ξ_k $(k = 1, 2, ...)$, η is composed r.v. of ν and ξ and $\eta =< \nu, \xi >$.

In many practical problems, we always meet this composed random variable (special in queuing theory - see [7]) where ν is assumed having Poisson law and ξ has the Exponential law. But, in practice, we also know best that ν has only a distribution function which arrives at Poisson law or ξ has a distribution function which arrives at Exponential law. Our question is the following: If we have the small changes of the distribution function of ν or ξ, whether the distribution function of $\eta =< \nu, \xi >$ shall has the small changes or not?

The composed r.v. η is called to be stable if the small changes in the distribution function of ν or ξ lead to the small changes in the distribution function of η.

More detail we have the following definitions:

Definition 1.1. Suppose that $\Psi(x)$ and $\psi(t)$ are the distribution function and characteristic function of η , $A_\varepsilon(x)$ and $a_\varepsilon(z)$ are the distribution function and the generating function of ν_ε such that

$$\rho(A; A_\varepsilon) = \sup_{x \in R} |A(x) - A_\varepsilon(x)| < \varepsilon$$

(for some sufficiently small positive number ε).

Put $\Psi^1_\varepsilon(x)$ be the distribution of the composed r.v. $< \nu_\varepsilon; \xi >$. The composed r.v. η is called to be $\delta_1(\varepsilon)$-stable on the first component with metric $\rho(.,.)$ if and only if

$$\rho(\Psi; \Psi^1_\varepsilon) \leqslant \delta_1(\varepsilon) \qquad (\delta_1(\varepsilon) \to 0 \quad when \quad \varepsilon \to 0).$$

Definition 1.2. Suppose that $F_\varepsilon(x)$ and $\varphi_\varepsilon(t)$ are the distribution function and the characteristic function of ξ_ε such that $\rho(F_\varepsilon; F) < \varepsilon$ (for some sufficiently small positive number ε) and $\Psi_\varepsilon^2(x)$ is distribution function with the characteristic function $\psi_\varepsilon^2(t)$ of the composed r.v. $< \nu; \xi_\varepsilon >$.

The composed r.v. η is called to be $\delta_2(\varepsilon)$-stable on the second component with metric $\rho(.,.)$ if and only if $\rho(\Psi; \Psi_\varepsilon^2) \leqslant \delta_2(\varepsilon) \qquad (\delta_2(\varepsilon) \to 0 \quad when \quad \varepsilon \to 0)$.

Remark 1.1. In some following stability theorems, metric $\rho(.,.)$ may be changed by metric $\lambda_0(.,.)$ (See [6])

$$\lambda_0(\Psi; \Psi_\varepsilon^2) = \sup_{t \in R} |\psi(t) - \psi_\varepsilon^2(t)|.$$

2. **Stability Theorems**

Theorem 2.1. *If the first component of the composed r.v. η has the generating function $a(z)$ which satisfies the following condition:*

$$|a(z_1) - a(z_2)| \leqslant K|z_2 - z_1|, \tag{2.1}$$

for all complex numbers $z_1, z_2, |z_1| \leqslant 1, |z_2| \leqslant 1$ and K is a constant, then η shall be $K\varepsilon$-stable on the second component with metric $\lambda_0(.,.)$.

Proof. According to the hypothesis $\lambda_0(F, F_\varepsilon) < \varepsilon$,

$$|\varphi(t) - \varphi_\varepsilon(t)| < \varepsilon, \quad \forall t$$

so that

$$|\psi(t) - \psi_\varepsilon^2(t)| = |a[\varphi(t)] - a[\varphi_\varepsilon(t)]| \leqslant K|\varphi(t) - \varphi_\varepsilon(t)| \leqslant K\varepsilon$$

for all t. That means

$$\lambda_0(\Psi; \Psi_\varepsilon^2) \leqslant K\varepsilon. \tag{2.2}$$

Example 2.1. If ν is the r.v. having the Poisson law with parameter $\lambda > 0$ and $\varphi_1(t)$ is the characteristic function of the r.v. ξ having exponential law with parameter $\theta > 0$ then the composed r.v. $\eta =< \nu; \xi >$ shall be $e^{4\lambda}\varepsilon$-stable on the second component with metric $\lambda_0(.,.)$ (where $e^{4\lambda}\varepsilon$ is a constant).

Example 2.2. If ν is r.v. having the binomial distribution function with the parameters p, n and ξ has the exponential distribution function with parameter $\theta > 0$ then $\eta =< \nu; \xi >$ shall be $np(1+2p)^{n-1}\varepsilon$-stable on the second component with metric $\lambda_0(.,.)$ (where $np(1+2p)^{n-1}\varepsilon$ is a constant).

Example 2.3. If ν is r.v. having the geometric distribution function with the parameters p $(p = 1 - q)$ and ξ has the exponential distribution function then $\eta =< \nu; \xi >$ shall be $\frac{q}{p}\varepsilon$-stable on the second component with metric $\lambda_0(.,.)$ (where $\frac{q}{p}\varepsilon$ is a constant).

All above examples are immediate from Theorem 2.1 since the corresponding generating functions clearly satisfy the condition (2.1). Indeed, for instance, to show Example 2.3, let $a_3(z)$ be the generating function of geometric law, i.e.:

$$a_3(z) = p[1 - qz]^{-1}.$$

For any complex numbers z_1, z_2 satisfying $|z_1| \leqslant 1, |z_2| \leqslant 1$; we have the following estimation:

$$|a_3(z_1) - a_3(z_2)| = |\frac{p}{1 - qz_1} - \frac{p}{1 - qz_2}| \leqslant \frac{pq|z_1 - z_2|}{|1 - qz_1||1 - qz_2|}$$

Notice that

$$|1 - qz_1| \geq |1 - q|z_1|| \geq 1 - q, \quad for \quad all \quad |z_1| \leqslant 1$$

$$|1 - qz_2| \geq |1 - q|z_2|| \geq 1 - q, \quad for \quad all \quad |z_2| \leqslant 1$$

It follows that

$$|a_3(z_1) - a_3(z_2)| \leqslant \frac{pq|z_1 - z_2|}{(1 - q)^2}.$$

Thus $a_3(z)$ satisfies the condition (2.1) with the constant $K = \frac{pq}{(1 - q)^2}$.

Theorem 2.2. (See [2]) *Suppose $\eta =< \nu, \xi >$, ν has the distribution function $A(x)$ such that*

$$\mu_A^\alpha = \int_0^{+\infty} z^\alpha dA(z) < +\infty, \quad \forall \alpha : 0 < \alpha < 1$$

and ξ has the stable law with the characteristic function:

$$\varphi(t) = \exp\{i\mu t - c|t|^\alpha(1 + i\beta\frac{t}{|t|}\omega(|t|; \alpha))\}, \tag{2.3}$$

where c, μ, α, β are real numbers, $c \geq 0, |\beta| \leqslant 1$ and

$$1 < \alpha_1 \leqslant \alpha \leqslant 2; \quad \omega(|t|; \alpha) = tg\frac{\alpha\pi}{2}. \tag{2.4}$$

For every $\varepsilon-$ sufficiently small positive number is given, such that

$$\varepsilon < (\frac{\pi}{3c_2})^3, c_1 = (c + |\beta||tg\frac{\alpha_1\pi}{2}| + |\mu|) \tag{2.5}$$

η shall be $K_1\varepsilon^{1/6}$-stable on the first component with metric $\rho(.,.)$.

Theorem 2.3. *Assume that ν has any distribution function $A(z)$ which has moment $\mu_A = \int_0^\infty zdA(z) < +\infty$, ξ has the stable law with the characteristic function satisfying condition (2.3), (2.4). Then, the composed r.v. $\eta =< \nu, \xi >$ shall be $K_2(\varepsilon)^{1/8}$-stable on the second component with metric $\rho(.,.)$ for some ε is sufficiently small number satisfying condition (2.5).*

Lemma 2.1. Let a be a complex number, $a = \rho e^{i\theta}$, such that

$$|\theta| \leqslant \frac{\pi}{3}, 0 \leqslant \rho \leqslant 1. \tag{2.6}$$

Then we always have following estimation

$$|a^t - 1| \leqslant \frac{\sqrt{14}t|a-1|}{(1-|a-1|)} \quad \text{for every} \quad t > 0,\ t \in R. \tag{2.7}$$

Proof. Since $a = \rho(cos\theta + isin\theta)$, it follows that

$$|a^t - 1|^2 = (\rho^t cost\theta - 1)^2 + (\rho^t sint\theta)^2. \tag{2.8}$$

We also have $(\rho^t cost\theta - 1) = (\rho^t - 1)cost\theta + (cost\theta - 1)$.

Notice that $|1 - cosx| \leqslant |x|$ for all $x \in R$, thus

$$|\rho^t cost\theta - 1| \leqslant |\rho^t - 1| + |t\theta|.$$

On the other hand, since $|sinu| \leqslant |u|$ for all u$\in R$, from (2.8) we shall have

$$|a^t - 1|^2 \leqslant 2|\rho^t - 1|^2 + 2t^2\theta^2 + \rho^{2t}(t\theta)^2. \tag{2.9}$$

We can see $|a-1|^2 = (\rho cos\theta - 1)^2 + \rho^2 sin^2\theta$. It follows that

$$|\rho sin\theta| \leqslant |a - 1|. \tag{2.10}$$

Further more

$$||a| - 1| \leqslant |a - 1| \Rightarrow |\rho - 1| \leqslant |a - 1|.$$

Hence

$$|\rho - 1| \geq -|a-1| \Rightarrow \rho \geq 1 - |a-1|. \tag{2.11}$$

From (2.10) we obtain $|sin\theta| \leqslant \dfrac{|a-1|}{\rho} \leqslant \dfrac{|a-1|}{1-|a-1|}$.

For every $\theta, |\theta| \leqslant \dfrac{\pi}{3}$, we always have inequality: $|sin\theta| \geq \dfrac{|\theta|}{2}$. So, from (2.)

$$|\theta| \leqslant \frac{2|a-1|}{1-|a-1|}.$$

From (2.9) and (2.11)

$$|a^t - 1|^2 \leqslant 2|\rho^t - 1|^2 + \frac{8t^2|a-1|^2}{(1-|a-1|)^2} + 4\frac{\rho^{2t}t^2|a-1|^2}{(1-|a-1|)^2}. \tag{2.12}$$

For all $t \geq 0$, the following inequality holds

$$1 - \rho^t \leqslant \frac{t(1-\rho)}{\rho}.$$

Notice $|1-\rho| = |1-|a|| \leqslant |a-1|$. We have

$$|1-\rho^t| \leqslant \frac{t|a-1|}{\rho}. \tag{2.13}$$

Hence by (2.12) and (2.13), we shall get: $|a^t - 1|^2 \leqslant \dfrac{14t^2|a-1|^2}{(1-|a-1|)^2}$.

Proof of theorem 2.3. At first, we shall estimate $|\psi(t) - \psi_\varepsilon(t)|$ for all $t, |t| \leqslant T(\varepsilon)$ where $T(\varepsilon) \to \infty$ when $\varepsilon \to 0$. At last, using Esseen's inequality (see [4]) we shall have the conclusion. Throughout the proof, we shall denote by $c_1, c_2, .., c_{14}, c_{15}$ are constants independent of ε. At first, we have:

$$|\psi(t) - \psi_\varepsilon(t)| = |a[\varphi(t)] - a[\varphi_\varepsilon(t)]| = |\int_0^{+\infty} [\varphi^z(t) - \varphi_\varepsilon^z(t)]dA(z)|$$

$$\leqslant |\int_1^{+\infty} [\varphi^z(t) - \varphi_\varepsilon^z(t)]dA(z)| + |\int_0^1 [\varphi^z(t) - \varphi_\varepsilon^z(t)]dA(z)| = J_1 + J_2. \tag{2.14}$$

Consider J_1: Using the Lagrange-formula of the function $[\varphi(t)]^z$ (for $|z| \geq 1$), we get

$$|\varphi^z(t) - \varphi_\varepsilon^z(t)| = z|\tilde{\varphi}(t)|^{z-1}|\varphi(t) - \varphi_\varepsilon(t)|, \tag{2.15}$$

where $\tilde{\varphi}(t)$ is a complex number satisfying the condition $|\tilde{\varphi}(t)| \leqslant \max\{|\varphi(t)|; |\varphi_\varepsilon(t)|\}$.

Notice that:

$$|\tilde{\varphi}(t)|^{z-1} \leqslant |\tilde{\varphi}(t)| \leqslant 1 \quad for\ \ all \quad z : 2 \leqslant z < +\infty$$

and

$$|\tilde{\varphi}(t)|^{z-1} \leqslant |\tilde{\varphi}(t)|^0 = 1 \quad for\ \ all \quad z : 1 \leqslant z < 2,$$

$$i.e., \quad |\tilde{\varphi}(t)|^{z-1} \leqslant 1 \quad for\ \ all \quad z : 1 \leqslant z < +\infty. \tag{2.16}$$

We shall have

$$|\varphi(t) - \varphi_\varepsilon(t)| = |\int_{-\infty}^{+\infty} e^{itx} d[F(x) - F_\varepsilon(x)]|.$$

For some $N = N(\varepsilon)$ (it also be chosen later), we also have the following estimation:

$$|\varphi(t) - \varphi_\varepsilon(t)| = |\int_{-N}^{+N} e^{itx} d[F(x) - F_\varepsilon(x)]| + 2\int_{N}^{+\infty} d[F(x) + F_\varepsilon(x)]$$

$$\leqslant |[F(x) - F_\varepsilon(x)]||_{-N}^{N} + |\int_{-N}^{N} [F(x) - F_\varepsilon(x)] d(e^{itx})| + 2\int_{N}^{+\infty} |\frac{x}{N}| d[F(x) + F_\varepsilon(x)]$$

$$\leqslant 2\varepsilon + \int_{-N}^{N} \varepsilon |t| dx + 2.\frac{\mu_F + \mu_{F_\varepsilon}}{N(\varepsilon)}.$$

(where $\mu_F = \int_{-\infty}^{+\infty} |x| dF(x) < +\infty$ and $\mu_{F_\varepsilon} = \int_{-\infty}^{+\infty} |x| dF_\varepsilon(x) < +\infty$). Now, for all t, $|t| \leqslant T(\varepsilon)$ (where $T(\varepsilon) \to \infty$ when $\varepsilon \to 0$, $T(\varepsilon)$ will be chosen later) we always have

$$|\varphi(t) - \varphi_\varepsilon(t)| \leqslant 2\varepsilon + 2N(\varepsilon)T(\varepsilon)\varepsilon + 2.\frac{\mu_F + \mu_{F_\varepsilon}}{N(\varepsilon)}. \tag{2.17}$$

Now, consider J_2. Using the Lagrange-formula of the function $[\varphi(t)]^z$ for all $z, 0 \leqslant z \leqslant 1$ at $\varphi_\varepsilon(t)$, we get

$$|\varphi^z(t) - \varphi_\varepsilon^z(t)| = \frac{z}{|\tilde{\tilde{\varphi}}(t)|^{1-z}} |\varphi(t) - \varphi_\varepsilon(t)|. \tag{2.18}$$

For every ε- satisfying condition (2.5) we shall choose $T(\varepsilon)$ such that

$$\min\{|\varphi(t)|; |\varphi_\varepsilon(t)|\} \geq c_4 \varepsilon^{1/2} \geq |\varphi(t) - \varphi_\varepsilon(t)| \text{ for all } t,\ |t| \leqslant T(\varepsilon),$$

(where c_4 is a constant independent of ε).

Because $\varphi(t)$ is the characteristic function of stable law satisfying condition (2.3), so we have the following estimations:

$$|\ln \varphi(t)| \leqslant |\mu||t| + |t|^\alpha (c + c|\beta||tg\frac{\alpha_1 \pi}{2}|) \leqslant |\mu||t| + c_2|t|^\alpha \leqslant T^\alpha(\varepsilon).$$

Thus , $\qquad |\varphi(t)| = |e^{\ln \varphi(t)}| \geq e^{-|\ln \varphi(t)|} \geq e^{-c_2 T^\alpha(\varepsilon)}.$

If we choose:

$$T(\varepsilon) = [\frac{1}{c_2} \ln \frac{1}{\varepsilon^{1/8}}]^{\frac{1}{\alpha}} \qquad (T(\varepsilon) \to \infty \ \ when \ \ \varepsilon \to 0). \tag{2.19}$$

Then $c_2 T^\alpha(\varepsilon) \leqslant \ln \frac{1}{\varepsilon^{1/8}}$ (for all $\alpha > 1$) and $|\varphi(t)| \geq e^{-c_2 T^\alpha(\varepsilon)} \geq \varepsilon^{1/8}$. Now we shall choose $N(\varepsilon) = \varepsilon^{-1/2}$ ($N(\varepsilon) \to +\infty$) when $\varepsilon \to 0$. Thus

$$2\varepsilon T(\varepsilon) N(\varepsilon) \leqslant \frac{2}{c_2^{1/\alpha}} \ln \frac{1}{\varepsilon^{1/8}} . \varepsilon^{1/2} \leqslant c_3 \varepsilon^{3/8}. \tag{2.20}$$

Put

$$c_0(\varepsilon) = 2\varepsilon + 2\varepsilon T(\varepsilon) N(\varepsilon) + 2\Big(\frac{\mu_F + \mu_{F_\varepsilon}}{N(\varepsilon)}\Big).$$

We shall have the estimation

$$c_0(\varepsilon) \leqslant 2\varepsilon + c_3 \varepsilon^{3/8} + 2(\mu_F + \mu_{F_\varepsilon})\varepsilon^{1/2} \leqslant c_4 \varepsilon^{1/2}. \tag{2.21}$$

That means, the condition:

$$c_4 \varepsilon^{1/2} \geq |\varphi(t) - \varphi_\varepsilon(t)| \tag{2.22}$$

shall be satisfied for every t, $|t| \leqslant T(\varepsilon)$.

Notice that, from $|\varphi(t) - \varphi_\varepsilon(t)| \leqslant c_4 \varepsilon^{1/2}$ we always have

$$||\varphi(t)| - |\varphi_\varepsilon(t)|| \leqslant |\varphi(t) - \varphi_\varepsilon(t)| \quad (See[5]).$$

So

$$|\varphi(t)| - |\varphi_\varepsilon(t)| \leqslant |\varphi(t) - \varphi_\varepsilon(t)| \leqslant c_4 \varepsilon^{1/2}$$

and

$$|\varphi_\varepsilon(t)| \geq |\varphi(t)| - c_4 \varepsilon^{1/2} \geq \varepsilon^{1/8} - c_4 \varepsilon^{1/2} \geq c_5 \varepsilon^{1/8}.$$

That also means, the estimation $\min\{|\varphi(t)|; |\varphi_\varepsilon(t)|\} \geq c_4 \varepsilon^{1/2}$ shall be satisfied.

On the other hand, for every complex number z_3 which belong to the interval joining z_1 and z_2 we have only two cases:

1) $|z_3| \geq \min\{|z_1|; |z_2|\}$
2) $|z_3| \geq \sqrt{\max\{|z_1|^2; |z_2|^2\} - \frac{|z_1 - z_2|^2}{2}}$.

Therefore

$$\tilde{\tilde{\varphi}}(t) \geq \min\{|\varphi(t)|; |\varphi_\varepsilon(t)|\} \geq c_5 \varepsilon^{1/8}$$

or

$$|\tilde{\tilde{\varphi}}(t)| \geq \sqrt{c_5^2\varepsilon^{2/8} - \frac{c_4\varepsilon^{2/4}}{2}} \geq c_6\varepsilon^{1/8}$$

i.e., $|\tilde{\tilde{\varphi}}(t)| \geq c_6\varepsilon^{1/8}$ in both above cases. Besides that, we always have,

$$|\tilde{\tilde{\varphi}}(t)|^{1-z} \geq |\tilde{\tilde{\varphi}}(t)| \qquad \text{for all complex number } z, 0 \leqslant |z| \leqslant 1. \tag{2.23}$$

Taking into account (2.18), (2.20), (2.23) we shall get

$$J_2 = |\int_0^1 |\varphi^z(t) - \varphi_\varepsilon^z(t)|dA(z)| \leqslant \int_0^1 |z|\frac{|\varphi(t) - \varphi_\varepsilon(t)|}{|\tilde{\tilde{\varphi}}(t)|}dA(z) \leqslant c_7\varepsilon^{3/8}. \tag{2.24}$$

Combine (2.14), (2.16), (2.17), (2.24) we can see that

$$J_1 + J_2 \leqslant \mu_A c_0(\varepsilon) + c_7\varepsilon^{3/8} \leqslant c_8\varepsilon^{3/8}. \tag{2.25}$$

Thus, for all t, $|t| \leqslant T(\varepsilon)$ (which is chosen from (2.19)) we always have the estimation

$$|\psi(t) - \psi_\varepsilon(t)| \leqslant c_8\varepsilon^{3/8}. \tag{2.26}$$

Now we shall choose $\delta = \delta(\varepsilon)$ be a positive number ($\delta(\varepsilon) \to 0$ when $\varepsilon \to 0$) such that

$$\max\{|\arg[\varphi(t)]|; |\arg[\varphi_\varepsilon(t)]|\} \leqslant \frac{\pi}{3}, \qquad \forall t, |t| \leqslant \delta(\varepsilon). \tag{2.27}$$

We always have

$$\int_{-T(\varepsilon)}^{T(\varepsilon)} |\frac{\psi(t) - \psi_\varepsilon(t)}{t}|dt \leqslant \int_{-\delta(\varepsilon)}^{\delta(\varepsilon)} |\frac{\psi(t) - \psi_\varepsilon(t)}{t}|dt + \int_{\delta(\varepsilon) \leqslant |t| \leqslant T(\varepsilon)} |\frac{\psi(t) - \psi_\varepsilon(t)}{t}|dt.$$

Consider $|\psi(t) - \psi_\varepsilon(t)|$ on $|t| \leqslant \delta(\varepsilon)$, we have

$$|\psi(t) - \psi_\varepsilon(t)| \leqslant \int_0^{+\infty} |\varphi^z(t) - 1|dA(z) + \int_0^{+\infty} |\varphi_\varepsilon^z(t) - 1|dA(z). \tag{2.28}$$

In $|t| \leqslant \delta(\varepsilon)$, with $\delta(\varepsilon)$ is chosen from the condition (2.27), the condition (2.6) of Lemma 2.1 shall be satisfied (with $a = \varphi(t)$), we shall use Lemma 2.1 and we have the following estimations

$$|\varphi^z(t) - 1| \leqslant \frac{\sqrt{14}z|\varphi(t) - 1|}{(1 - |\varphi(t) - 1|)}$$

and,

$$|\varphi_\varepsilon^z(t) - 1| \leqslant \frac{\sqrt{14}z|\varphi_\varepsilon(t) - 1|}{(1 - |\varphi_\varepsilon(t) - 1|)}. \tag{2.29}$$

for all complex numbers z.

Notice that, for all t:

$$|e^{itx}-1| = |(\cos tx-1)^2+\sin^2 tx| = 2\sin\frac{tx}{2} \leqslant |t||x|. \tag{2.30}$$

In $|t| \leqslant \delta(\varepsilon)$ with $\delta(\varepsilon)\to 0$ when $\varepsilon\to 0$, so we always have

$$|\varphi(t)-1| \leqslant \frac{1}{2}, \quad |\varphi_\varepsilon(t)-1| \leqslant \frac{1}{2},$$

and therefore, from (2.7)

$$\int_0^{+\infty} |\varphi_\varepsilon^z(t)-1|dA(z) \leqslant \int_0^{+\infty} \frac{\sqrt{14}|\varphi(t)-1|}{(1-\frac{1}{2})}dA(z) \leqslant c_9|t|.$$

Similarly,

$$\int_0^{+\infty} |\varphi_\varepsilon^z(t)-1|dA(z) \leqslant c_{10}|t|.$$

That means

$$\int_{-\delta(\varepsilon)}^{\delta(\varepsilon)} |\frac{\psi(t)-\psi_\varepsilon(t)}{t}|dt \leqslant c_{11}\delta(\varepsilon). \tag{2.31}$$

Now, if we choose

$$\delta(\varepsilon) = \frac{1}{c_2}\varepsilon^{1/4}\ln\frac{1}{\varepsilon^{1/8}} \quad (\delta(\varepsilon)\to 0 \text{ when } \varepsilon\to 0 \tag{2.32}$$

with ε satisfying (2.5) , then we have

$$\int_{-\delta(\varepsilon)}^{\delta(\varepsilon)} |\frac{\psi(t)-\psi_\varepsilon(t)}{t}|dt \leqslant c_{11}\delta(\varepsilon) \leqslant c_{13}\varepsilon^{1/4}\frac{1}{\varepsilon^{1/8}} \leqslant c_{14}\varepsilon^{1/8}. \tag{2.33}$$

On the other hand, from (2.19) and (2.26)

$$\int_{\delta(\varepsilon)\leqslant|t|\leqslant T(\varepsilon)} |\frac{\psi(t)-\psi_\varepsilon(t)}{t}|dt \leqslant c_8\varepsilon^{3/8}\int_{\delta(\varepsilon)}^{T(\varepsilon)} \frac{1}{t}dt = c_8\varepsilon^{3/8}\ln\frac{T(\varepsilon)}{\delta(\varepsilon)}, \tag{2.34}$$

and notice that

$$c_8\varepsilon^{3/8}\ln\frac{T(\varepsilon)}{\delta(\varepsilon)} \leq c_8\varepsilon^{3/8}\ln\frac{T^\alpha(\varepsilon)}{\delta(\varepsilon)} \leq c_8\varepsilon^{3/8}\frac{1}{\varepsilon^{1/4}} = c_8\varepsilon^{1/8}. \tag{2.35}$$

With $T(\varepsilon)$ and $\delta(\varepsilon)$ chosen from conditions (2.19) and (2.32), we shall have:

$$\int_{\delta(\varepsilon)\leqslant|t|\leqslant T(\varepsilon)} |\frac{\psi(t)-\psi_\varepsilon(t)}{t}|dt \leqslant c_8\varepsilon^{3/8}\ln\frac{T(\varepsilon)}{\delta(\varepsilon)} \leqslant c_8\varepsilon^{1/8}. \tag{2.36}$$

By using Esseen's inequality (see [4]) and combine (2.33) with (2.36) we can conclude that

$$\rho(\Psi;\Psi_\varepsilon) \leqslant c_{14}\varepsilon^{1/8} + c_8\varepsilon^{1/8} \leqslant K_2\varepsilon^{1/8},$$

where K_2 is a constant independent of ε. This completes the proof of Theorem 2.3.

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