STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LV**, Number 1, March 2010

A SPECIAL DIFFERENTIAL SUPERORDINATION IN THE COMPLEX PLANE

CAMELIA MĂDĂLINA BĂLĂEŢI

Abstract. In this paper some differential superordinations, by applying the integral operator, are introduced.

1. Introduction and preliminaries

Denote by U the unit disc of the complex plane:

$$U = \{z \in \mathbb{C} :| z |< 1\},$$

for $0 < r < 1$, we let

$$U_r = \{z \in \mathbb{C} : |z| < r\},$$

and

$$\dot{U} = U \setminus \{0\}.$$

Let $\mathcal{H}(U)$ be the space of holomorphic functions in U.

For $a \in \mathbb{C}$ and $n \in \mathbb{N}^*$ we let:

$$\mathcal{H}[a, n] = \{f \in \mathcal{H}(U),\ f(z) = a + a_n z^n + \dots,\ z \in U\}$$

and

$$A_n = \left\{f \in \mathcal{H}(U),\ f(z) = z + a_{n+1} z^{n+1} + \dots,\ z \in U\right\}$$

with $A = A_1$.

Let f and F be members of $\mathcal{H}(U)$. The function f is said to be subordinate to F, or F is said to be superordinate to f, if there exists a function w analytic in

Received by the editors: 30.10.2008.

2000 *Mathematics Subject Classification.* 30C45.

Key words and phrases. differential subordination, differential superordination, integral operator.

U, with $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = F(w(z))$; in such a case we write $f(z) \prec F(z)$. If F is univalent, then $f(z) \prec F(z)$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$.

Let Ω be any set in the complex plane $\mathbb{C}$, let p be analytic in the unit disk U and let $\psi(\gamma, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}$.

In a series of articles the S.S. Miller and P.T. Mocanu, D.J. Hallenbeck and S. Ruscheweyh have determined properties of functions p that satisfy the differential subordination

$$\{\psi(p(z), zp'(z), z^2p''(z); z) |\ z \in U\} \subset \Omega.$$

In this article we consider the dual problem of determining properties of functions p that satisfy the differential superordination

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z); z) |\ z \in U\}.$$

The following definitions, comments and lemmas have been presented in [5].

Definition 1.1. *Let $\varphi : \mathbb{C}^2 \times U \to \mathbb{C}$ and let h be analytic in U. If p and $\varphi(p(z), zp'(z); z)$ are univalent in U and satisfy the (first-order) differential superordination*

$$h(z) \prec \varphi(p(z), zp'(z); z) \tag{1.1}$$

then p is called a solution of the differential superordination. An analytic function q is called a subordinant of the solutions of the differential superordination, or more simply a subordinant if $q \prec p$ for all p satisfying (1.1). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1.1) is said to be the best subordinant.

Note that the best subordinant is unique up to a rotation of U.

For Ω a set in $\mathbb{C}$, with φ and p as given in Definition 1.1, suppose (1.1) is replaced by

$$\Omega \subset \{\varphi(p(z), zp'(z); z) |\ z \in U\}. \tag{1.2}$$

Although this more general situation is a "differential containment", the condition in (1.2) will also be referred to as a differential superordination, and the definitions of solution, subordinant and best dominant as given above can be extend to this generalization.

Before obtaining some of the main results we need to introduce a class of univalent functions defined on the unit disc that have some nice boundary properties.

Definition 1.2. *We denote by Q the set of functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where*

$$E(f) = \left\{ \zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty \right\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

The subclass of Q for which $f(0) = a$ is denoted by $Q(a)$.

Definition 1.3. *Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}[a,n]$ with $q'(z) \neq 0$. The class of admissible functions $\Phi_n[\Omega, q]$, consist of those functions $\varphi : \mathbb{C}^2 \times \overline{U} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\varphi\left(q(z), \frac{zq'(z)}{m}; \zeta\right) \in \Omega \tag{1.3}$$

where $z \in U$, $\zeta \in \partial U$ and $m \geq n \geq 1$.

In order to prove the new results we shall use the following lemmas:

Lemma 1.4. [5] *Let h be convex in U, with $h(0) = a$, $\gamma \neq 0$ and $\mathrm{Re}\gamma \geq 0$. If $p \in \mathcal{H}[a,n] \cap Q$ and $p(z) + \frac{zp'(z)}{\gamma}$ is univalent in U with*

$$h(z) \prec p(z) + \frac{zp'(z)}{\gamma}$$

then

$$q(z) \prec p(z)$$

where

$$q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t) t^{\frac{\gamma}{n}-1} dt$$

The function q is convex and is the best subordinant.

Lemma 1.5. [5] *Let q be convex in U and let h be defined by*

$$h(z) = q(z) + \frac{zq'(z)}{\gamma}, \quad z \in U,$$

with $\gamma \neq 0$, $\operatorname{Re} \gamma \geq 0$. *If* $p \in \mathcal{H}[a,n] \cap Q$, $p(z) + \dfrac{zp'(z)}{\gamma}$ *is univalent in* U, *and*

$$q(z) + \frac{zq'(z)}{\gamma} \prec p(z) + \frac{zp'(z)}{\gamma}, \quad z \in U$$

then

$$q(z) \prec p(z),$$

where

$$q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t) t^{\frac{\gamma}{n}-1} dt$$

The function q is the best subordinant.

Definition 1.6. [6] *For $f \in A_n$ and $m \geq 0$, $m \in \mathbb{N}$, the operator $I^m f$ is defined by*

$$I^0 f(z) = f(z)$$

$$I^1 f(z) = If(z) = \int_0^z f(t)\, t^{-1} dt$$

$$I^m f(z) = I\left[I^{m-1} f(z)\right], z \in U.$$

Remark 1.7. *If we denote $l(z) = -\log(1-z)$, then*

$$I^m f(z) = [\underbrace{(l * l * \cdots * l)}_{n-times} * f](z), f \in \mathcal{H}(U), f(0) = 0$$

By " $*$ " we denote the Hadamard product or convolution (i.e. if $f(z) = \sum_{j=0}^{\infty} a_j z^j$, $g(z) = \sum_{j=0}^{\infty} b_j z^j$ *then* $(f * g)(z) = \sum_{j=0}^{\infty} a_j b_j z^j$*).*

Remark 1.8. $I^m f(z) = \int_0^z \int_0^{t_m} \ldots \int_0^{t_2} \frac{f(t_1)}{t_1 t_2 \ldots t_m} dt_1 dt_2 \ldots dt_m$

Remark 1.9. $D^m I^m f(z) = I^m D^m f(z) = f(z)$, $f \in \mathcal{H}(U)$, $f(0) = 0$, *where $D^m f$ is the Sălăgean differential operator.*

2. Main results

Theorem 2.1. *Let $h \in \mathcal{H}(U)$ be a convex function in U, with $h(0) = 1$ and $f \in A_n$, $n \in \mathbb{N}^*$.*

Assume that $[I^m f(z)]'$ is univalent with $\left[I^{m+1} f(z)\right]' \in \mathcal{H}[1, n] \cap Q$.

If

$$h(z) \prec [I^m f(z)]', z \in U \tag{2.1}$$

then

$$q(z) \prec [I^{m+1} f(z)]', z \in U \tag{2.2}$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t)\, t^{\frac{1}{n}-1} dt.$$

The function q is convex and is the best subordinant.

Proof. Let $f \in A_n$. By using the properties of the integral operator we have

$$I^m f(z) = z\left[I^{m+1} f(z)\right]', \ z \in U. \tag{2.3}$$

Differentiating (2.3), we obtain

$$[I^m f(z)]' = [I^{m+1} f(z)]' + z[I^{m+1} f(z)]'', \quad z \in U. \tag{2.4}$$

If we denote $p(z) = [I^{m+1} f(z)]'$, $z \in U$, then (2.4) becomes

$$[I^m f(z)]' = p(z) + zp'(z), \quad z \in U.$$

By using Lemma 1.4 for $\gamma = 1$ we deduce that

$$q(z) \prec p(z) = [I^{m+1} f(z)]'$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt.$$

Moreover the function q is the best subordinant. □

As a corollary of Theorem 2.1, we have the next corollary.

Corollary 2.2. *Let $h \in \mathcal{H}(U)$ be a convex function in U, with $h(0) = 1$ and $f \in A$.*

Assume that $[I^m f(z)]'$ is univalent with $[I^{m+1} f(z)]' \in \mathcal{H}[1, 1] \cap Q$.

$$h(z) \prec [I^m f(z)]', \ z \in U \tag{2.5}$$

then

$$q(z) \prec [I^{m+1} f(z)]', z \in U \tag{2.6}$$

where

$$q(z) = \frac{1}{z} \int_0^z h(t)dt.$$

The function q is convex and is the best subordinant.

Theorem 2.3. *Let $h \in \mathcal{H}(U)$ a convex function in U, with $h(0) = 1$ and $f \in A_n$.*

Assume that $[I^m f(z)]'$ is univalent with $\frac{I^m f(z)}{z} \in \mathcal{H}[1, n] \cap Q$.

If

$$h(z) \prec [I^m f(z)]', \ z \in U \tag{2.7}$$

then

$$q(z) \prec \frac{I^m f(z)}{z}, \ z \in U \tag{2.8}$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h\left(t\right) t^{\frac{1}{n}-1} dt.$$

The function q is convex and is the best subordinant.

Proof. If we denote

$$p(z) = \frac{I^m f(z)}{z} \tag{2.9}$$

then

$$I^m f(z) = zp(z). \tag{2.10}$$

Differentiating (2.10) we obtain

$$[I^m f(z)]' = p(z) + zp'(z).$$

By using Lemma 1.4 we have

$$q(z) \prec p(z)$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t)\, t^{\frac{1}{n}-1} dt.$$

Also, the function q is the best subordinant. □

For $n = 1$ we have the following corollary.

Corollary 2.4. *Let $h \in \mathcal{H}(U)$ a convex function in U, with $h(0) = 1$ and $f \in A$. Assume that $[I^m f(z)]'$ is univalent with $\frac{I^m f(z)}{z} \in \mathcal{H}[1,1] \cap Q$.*

If

$$h(z) \prec [I^m f(z)]', \;\; z \in U \tag{2.11}$$

then

$$q(z) = \prec \frac{I^m f(z)}{z}, \;\; z \in U \tag{2.12}$$

where

$$q(z) = \frac{1}{z} \int_0^z h(t) dt.$$

The function q is convex and is the best subordinant.

Theorem 2.5. *Let q be a convex function in U and h defined by*

$$h(z) = q(z) + zq'(z), \;\; z \in U.$$

If $f \in A_n$, $[I^{m+1}]'$ is univalent in U, $[I^{m+1} f(z)]' \in \mathcal{H}[1,n] \cap Q$ and

$$h(z) \prec [I^{m+1} f(z)]' \tag{2.13}$$

then

$$q(z) \prec [I^{m+1} f(z)]' \tag{2.14}$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt.$$

The function q is the best subordinant.

Proof. Let $f \in A_n$. By using the properties of integral operator we have

$$[I^m f(z)]' = [I^{m+1} f(z)]' + z[I^{m+1} f(z)]'', z \in U. \tag{2.15}$$

If we denote

$$p(z) = [I^{m+1} f(z)]'$$

in (2.15) we obtain

$$[I^m f(z)]' = p(z) + zp'(z), z \in U. \tag{2.16}$$

By using Lemma 1.5 we obtain:

$$q(z) \prec [I^{m+1} f(z)]'$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt.$$

Moreover this is the best subordinant. □

For $n = 1$, we have the following corollary.

Corollary 2.6. *Let q convex in U and h defined by*

$$h(z) = q(z) + zq'(z), \;\; z \in U.$$

If $f \in A$, $[I^{m+1} f(z)]'$ is univalent in U, $[I^{m+1} f(z)]' \in \mathcal{H}[1, 1] \cap Q$ and

$$h(z) = q(z) + zq'(z) \prec [I^m f(z)]' \tag{2.17}$$

then

$$q(z) \prec [I^{m+1} f(z)]' \tag{2.18}$$

where

$$q(z) = \frac{1}{z} \int_0^z h(t) dt.$$

Theorem 2.7. *Let q a convex function in U and h defined by*

$$h(z) = q(z) + zq'(z).$$

If $f \in A_n$, $[I^m f(z)]'$ is univalent in U, $\frac{I^m f(z)}{z} \in \mathcal{H}[1, n] \cap Q$ and

$$h(z) \prec [I^m f(z)]' \tag{2.19}$$

then

$$q(z) \prec \frac{I^m f(z)}{z} \tag{2.20}$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt.$$

The function q is convex and is the best subordinant.

Proof. If

$$p(z) = \frac{I^m f(z)}{z}$$

then

$$[I^m f(z)]' = p(z) + zp'(z)$$

From relation (2.19), we have

$$q(z) + zq'(z) \prec p(z) + zp'(z).$$

By using Lemma 1.5 we obtain

$$q(z) \prec \frac{I^m f(z)}{z}$$

where

$$q(z) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t) t^{\frac{1}{n}-1} dt$$

and it is the best subordinant. □

As a corollary of this theorem, we have:

Corollary 2.8. *Let q a convex function in U and h defined by*

$$h(z) = q(z) + zq'(z).$$

If $f \in A$, with $[I^m f(z)]'$ univalent in U, $\frac{I^m f(z)}{z} \in \mathcal{H}[1,1] \cap Q$ and

$$h(z) \prec [I^m f(z)]' \tag{2.21}$$

then

$$q(z) \prec \frac{I^m f(z)}{z} \tag{2.22}$$

where

$$q(z) = \frac{1}{z} \int_0^z h(t) dt.$$

The function q is the best subordinant.

Remark 2.9. *For the special case of the function*

$$h(z) = \frac{1 + (2\alpha - 1)z}{1 + z}$$

this results were obtained in [1].

Acknowledgement. This work is supported by UEFISCSU - CNCSIS, Grant PN-II-ID 524/2007.

References

[1] Bălăeţi, C. M., *Differential superordinations defined by an integral operator* (submitted).

[2] Cătaş, A., *A note on a differential superordination defined by a generalized Sălăgean operator*, (to appear).

[3] Hallenbeck, D. J., Ruscheweyh, S., *Subordination by convex functions*, Proc. Amer. Math. Soc., **52** (1975), 191-195.

[4] Miller, S. S. and Mocanu, P. T., *Differential Subordinations. Theory and Applications*, Marcel Dekker Inc., New York, Basel, 2000.

[5] Miller, S. S. and Mocanu, P. T., *Subordinants of Differential Superordinations*, Complex Variables, 48, **10** (2003), 815-826.

[6] Sălăgean, G. S., *Subclasses of univalent functions*, Lecture Notes in Math., Springer Verlag, **1013** (1983), 362-372.

Departament of Mathematics and Computer Science
University of Petroşani
Str. Universităţii, no. 20
332006 Petroşani, Romania
E-mail address: madalina@upet.ro