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STATISTICAL APPROXIMATION BY DOUBLE PICARD SINGULAR INTEGRAL OPERATORS

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Abstract. We first construct a sequence of double smooth Picard singular integral operators which do not have to be positive in general. After giving some useful estimates, we mainly show that it is possible to approximate a function by these operators in statistical sense even though they do not obey the positivity condition of the statistical Korovkin theory.

1. **Introduction**

In the classical Korovkin theory, the positivity condition of linear operators and the validity of their (ordinary) limits are crucial points in approximating a function by these operators (see [1, 22]). However, there are many approximation operators that do not have to be positive, such as Picard, Poisson-Cauchy and Gauss-Weierstrass singular integral operators (see, e.g., [2, 3, 4, 8, 9, 10, 19]). Furthermore, using the concept of statistical convergence from the summability theory which is a weaker method than the usual convergence, it is possible to approximate (in statistical sense) a function by means of a sequence of positive linear operators although the limit of the sequence fails (see, e.g., [5, 6, 11, 12, 13, 14, 15, 16, 23, 24]).

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The aim of the present paper is to construct a sequence of linear operators that are not necessarily positive and to investigate its statistical approximation properties. Hence, we demonstrate that it is possible to find some statistical approximation operators that are not in general positive.

This paper is organized as follows. In the first section we recall some definitions and set the main notation used in the paper, while, in the second section, we construct the double smooth Picard singular integral operators which do not have to be positive. In the third section, we give some useful estimates on these operators. In the forth section, we obtain some statistical approximation theorems for our operators. The last section of the paper is devoted to the concluding remarks and discussion.

Let $A := [a_{jn}]$, $j, n = 1, 2, ...$, be an infinite summability matrix and assume that, for a given sequence $x = (x_n)_{n\in\mathbb{N}}$, the series $\sum_{n=1}^{\infty} a_{jn}x_n$ converges for every $j \in \mathbb{N}$. Then, by the A-transform of x, we mean the sequence $Ax = ((Ax)_j)_{j\in\mathbb{N}}$ such that, for every $j \in \mathbb{N}$,

$$(Ax)_j := \sum_{n=1}^{\infty} a_{jn}x_n.$$

A summability matrix A is said to be regular (see [20]) if for every $x = (x_n)_{n\in\mathbb{N}}$ for which $\lim_{n\to\infty} x_n = L$ we get $\lim_{j\to\infty} (Ax)_j = L$. Now, fix a non-negative regular summability matrix A. In [18] Freedman and Sember introduced a convergence method, the so-called A-statistical convergence, as in the following way. A given sequence $x = (x_n)_{n\in\mathbb{N}}$ is said to be A-statistically convergent to L if, for every $\varepsilon > 0$,

$$\lim_{j\to\infty} \sum_{n\,:\,|x_n-L|\geq\varepsilon} a_{nj} = 0.$$

This limit is denoted by $st_A - \lim_n x_n = L$.

Observe that if $A = C_1 = [c_{jn}]$, the Cesáro matrix of order one defined to be $c_{jn} = 1/j$ if $1 \leq n \leq j$, and $c_{jn} = 0$ otherwise, then C_1-statistical convergence coincides with the concept of statistical convergence, which was first introduced by Fast [17]. In this case, we use the notation $st - \lim$ instead of $st_{C_1} - \lim$ (see Section 5 for this situation). Notice that every convergent sequence is A-statistically convergent to the same value for any non-negative regular matrix A, however, its converse is not always true. Actually, Kolk [21] proved that A-statistical convergence is stronger than (usual) convergence if $A = [a_{jn}]$ is any nonnegative regular summability matrix satisfying the condition $\lim_j \max_n \{a_{jn}\} = 0$. Not all properties of convergent sequences hold true for A-statistical convergence (or statistical convergence). For instance, although it is well-known that a subsequence of a convergent sequence is convergent, this is not always true for A-statistical convergence. Another example is that every convergent sequence must be bounded, however it does not need to be bounded of an A-statistically convergent sequence. Of course, with these properties, the usage of A-statistical convergence in the approximation theory provides us many advantages.

2. Construction of the operators

Throughout the paper, for $r \in \mathbb{N}$ and $m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we use

$$\alpha_{j,r}^{[m]} := \begin{cases} (-1)^{r-j} \binom{r}{j} j^{-m} & \text{if } j = 1, 2, ..., r, \\ 1 - \sum\limits_{j=1}^{r} (-1)^{r-j} \binom{r}{j} j^{-m} & \text{if } j = 0. \end{cases} \tag{2.1}$$

and

$$\delta_{k,r}^{[m]} := \sum_{j=1}^{r} \alpha_{j,r}^{[m]} j^k, \quad k = 1, 2, ..., m \in \mathbb{N}. \tag{2.2}$$

Then observe that

$$\sum_{j=0}^{r} \alpha_{j,r}^{[m]} = 1 \tag{2.3}$$

and

$$-\sum_{j=1}^{r} (-1)^{r-j} \binom{r}{j} = (-1)^r \binom{r}{0}. \tag{2.4}$$

We now define the double smooth Picard singular integral operators as follows:

$$P_{r,n}^{[m]}(f; x, y) = \frac{1}{2\pi\xi_n^2} \sum_{j=0}^{r} \alpha_{j,r}^{[m]} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x + sj, y + tj) \, e^{-(\sqrt{s^2+t^2})/\xi_n} ds dt \right), \tag{2.5}$$

where $(x, y) \in \mathbb{R}^2$, $n, r \in \mathbb{N}$, $m \in \mathbb{N}_0$, $f : \mathbb{R}^2 \to \mathbb{R}$ is a Lebesgue measurable function, and also $(\xi_n)_{n \in \mathbb{N}}$ is a bounded sequence of positive real numbers.

Remark 2.1. The operators $P_{r,n}^{[m]}$ are not in general positive. For example, consider the function $\varphi(u,v) = u^2 + v^2$ and also take $r = 2$, $m = 3$, $x = y = 0$. Observe that $\varphi \geq 0$, however

$$\begin{aligned} P_{2,n}^{[3]}(\varphi;0,0) &= \frac{1}{2\pi\xi_n^2}\left(\sum_{j=1}^{2} j^2\alpha_{j,2}^{[3]}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(s^2+t^2\right)e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt \\ &= \frac{2}{\pi\xi_n^2}\left(\alpha_{1,2}^{[3]} + 4\alpha_{2,2}^{[3]}\right)\int_{0}^{\infty}\int_{0}^{\infty}\left(s^2+t^2\right)e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt \\ &= \frac{2}{\pi\xi_n^2}\left(-2+\frac{1}{2}\right)\int_{0}^{\pi/2}\int_{0}^{\infty} e^{-\rho/\xi_n}\rho^3 d\rho d\theta \\ &= -9\xi_n^2 < 0. \end{aligned}$$

Lemma 2.1. *The operators $P_{r,n}^{[m]}$ given by* (2.5) *preserve the constant functions in two variables.*

Proof. Let $f(x,y) = C$, where C is any real constant. By (2.1) and (2.3), we get, for every $r, n \in \mathbb{N}$ and $m \in \mathbb{N}_0$, that

$$\begin{aligned} P_{r,n}^{[m]}(C;x,y) &= \frac{C}{2\pi\xi_n^2}\sum_{j=0}^{r}\alpha_{j,r}^{[m]}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt\right) \\ &= \frac{C}{2\pi\xi_n^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt \\ &= \frac{2C}{\pi\xi_n^2}\int_{0}^{\infty}\int_{0}^{\infty} e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt \\ &= \frac{2C}{\pi\xi_n^2}\int_{0}^{\pi/2}\int_{0}^{\infty} e^{-\rho/\xi_n}\rho d\rho d\theta \\ &= \frac{C}{\xi_n^2}\int_{0}^{\infty} e^{-\rho/\xi_n}\rho d\rho \\ &= C, \end{aligned}$$

which completes the proof. □

Lemma 2.2. *Let $k \in \mathbb{N}_0$. Then, it holds, for each $\ell = 0, 1, ..., k$ and for every $n \in \mathbb{N}$, that*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s^{k-\ell}t^{\ell}e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt = \begin{cases} 0 & \text{if } k \text{ is odd} \\ 2B\left(\frac{k-\ell+1}{2}, \frac{\ell+1}{2}\right)\xi_n^{k+2}(k+1)! & \text{if } k \text{ is even} \end{cases}$$

where $B(a, b)$ denotes the Beta function.

Proof. It is clear that if k is odd, then the integrand is a odd function with respect to s and t; and hence the above integral is zero. Also, if k is even, then the integrand is an even function with respect to s and t. So, we may write that

$$\begin{aligned} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} s^{k-\ell}t^{\ell}e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt &= 4\int_{0}^{\infty}\int_{0}^{\infty} s^{k-\ell}t^{\ell}e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt \\ &= 4\left(\int_0^{\pi/2}(\cos\theta)^{k-\ell}(\sin\theta)^{\ell}d\theta\right)\left(\int_0^{\infty}\rho^{k+1}e^{-\rho/\xi_n}d\rho\right) \\ &= 2B\left(\frac{k-\ell+1}{2}, \frac{\ell+1}{2}\right)\xi_n^{k+2}(k+1)! \end{aligned}$$

whence the result. □

3. **Estimates for the operators (2.5)**

Let $f \in C_B(\mathbb{R}^2)$, the space of all bounded and continuous functions on $\mathbb{R}^2$. Then, the rth (double) modulus of smoothness of f is given by (see, e.g., [7])

$$\omega_r(f; h) := \sup_{\sqrt{u^2+v^2}\leq h} \left\|\Delta_{u,v}^r(f)\right\| < \infty, \quad h > 0, \tag{3.1}$$

where $\|\cdot\|$ is the sup-norm and

$$\Delta_{u,v}^r\left(f(x,y)\right) = \sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}f(x+ju, y+jv). \tag{3.2}$$

Let $m \in \mathbb{N}$. By $C^{(m)}\left(\mathbb{R}^2\right)$ we mean the space of functions having m times continuous partial derivatives with respect to the variables x and y. Assume now that a function $f \in C^{(m)}\left(\mathbb{R}^2\right)$ satisfies the condition

$$\left\|\frac{\partial^m f(\cdot,\cdot)}{\partial^{m-\ell}x\partial^{\ell}y}\right\| := \sup_{(x,y)\in\mathbb{R}^2}\left|\frac{\partial^m f(x,y)}{\partial^{m-\ell}x\partial^{\ell}y}\right| < \infty \tag{3.3}$$

for every $\ell = 0, 1, ..., m$. Then, we consider the function

$$G_{x,y}^{[m]}(s,t) := \frac{1}{(m-1)!} \sum_{j=0}^{r} \binom{r}{j} \int_0^1 (1-w)^{m-1}$$

$$\times \left\{ \sum_{\ell=0}^{m} \binom{m}{m-\ell} \left| \frac{\partial^m f(x+jsw, y+jtw)}{\partial^{m-\ell} x \partial^\ell y} \right| \right\} dw \tag{3.4}$$

for $m \in \mathbb{N}$ and $(x,y), (s,t) \in \mathbb{R}^2$. Notice that the condition (3.3) implies that $G_{x,y}^{[m]}(s,t)$ is well-defined for each fixed $m \in \mathbb{N}$.

We first estimate the case of $m \in \mathbb{N}$ in (2.5).

Theorem 3.1. *Let $m \in \mathbb{N}$ and $f \in C^{(m)}\left(\mathbb{R}^2\right)$ for which (3.3) holds. Then, for the operators $P_{r,n}^{[m]}$, we have*

$$\left| P_{r,n}^{[m]}(f;x,y) - f(x,y) - \frac{1}{\pi} \sum_{i=1}^{[m/2]} (2i+1) \delta_{2i,r}^{[m]} \xi_n^{2i} \right.$$

$$\left. \times \left\{ \sum_{\ell=0}^{2i} \binom{2i}{2i-\ell} \frac{\partial^{2i} f(x,y)}{\partial^{2i-\ell} x \partial^\ell y} B\left(\frac{2i-\ell+1}{2}, \frac{\ell+1}{2} \right) \right\} \right|$$

$$\leq \frac{1}{2\pi \xi_n^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_{x,y}^{[m]}(s,t) (|s|^m + |t|^m) e^{-(\sqrt{s^2+t^2})/\xi_n} ds dt. \tag{3.5}$$

The sums in the left hand side of (3.5) collapse when $m = 1$.

Proof. Let $(x,y) \in \mathbb{R}^2$ be fixed. By Taylor's formula, we may write that

$$\begin{aligned} f(x+js, y+jt) &= \sum_{k=0}^{m-1} \frac{j^k}{k!} \sum_{\ell=0}^{k} \binom{k}{k-\ell} s^{k-\ell} t^\ell \frac{\partial^k f(x,y)}{\partial^{k-\ell} x \partial^\ell y} \\ &+ \frac{j^m}{(m-1)!} \int_0^1 (1-w)^{m-1} \left\{ \sum_{\ell=0}^{m} \binom{m}{m-\ell} s^{m-\ell} t^\ell \right. \\ &\times \left. \frac{\partial^m f(x+jsw, y+jtw)}{\partial^{m-\ell} x \partial^\ell y} \right\} dw, \end{aligned}$$

which implies that

$$\begin{aligned} f(x+js,y+jt)-f(x,y) &= \sum_{k=1}^{m}\frac{j^k}{k!}\sum_{\ell=0}^{k}\binom{k}{k-\ell}s^{k-\ell}t^{\ell}\frac{\partial^k f(x,y)}{\partial^{k-\ell}x\partial^{\ell}y} \\ &-\frac{j^m}{(m-1)!}\int_0^1(1-w)^{m-1} \\ &\times\left\{\sum_{\ell=0}^{m}\binom{m}{m-\ell}s^{m-\ell}t^{\ell}\frac{\partial^m f(x,y)}{\partial^{m-\ell}x\partial^{\ell}y}\right\}dw \\ &+\frac{j^m}{(m-1)!}\int_0^1(1-w)^{m-1}\left\{\sum_{\ell=0}^{m}\binom{m}{m-\ell}s^{m-\ell}t^{\ell}\right. \\ &\times\left.\frac{\partial^m f(x+jsw,y+jtw)}{\partial^{m-\ell}x\partial^{\ell}y}\right\}dw. \end{aligned}$$

Now multiplying both sides of the above equality by $\alpha_{j,r}^{[m]}$ and summing up from 0 to r we obtain

$$\begin{aligned} \sum_{j=0}^{r}\alpha_{j,r}^{[m]}\left(f(x+js,y+jt)-f(x,y)\right) &= \sum_{k=1}^{m}\frac{\delta_{k,r}^{[m]}}{k!}\sum_{\ell=0}^{k}\binom{k}{k-\ell}s^{k-\ell}t^{\ell}\frac{\partial^k f(x,y)}{\partial^{k-\ell}x\partial^{\ell}y} \\ &+\frac{1}{(m-1)!}\int_0^1(1-w)^{m-1}\varphi_{s,t}^{[m]}(w)dw, \end{aligned}$$

where

$$\begin{aligned} \varphi_{x,y}^{[m]}(w;s,t) &= \sum_{j=0}^{r}\alpha_{j,r}^{[m]}j^m\left\{\sum_{\ell=0}^{m}\binom{m}{m-\ell}s^{m-\ell}t^{\ell}\frac{\partial^m f(x+jsw,y+jtw)}{\partial^{m-\ell}x\partial^{\ell}y}\right\} \\ &-\delta_{m,r}^{[m]}\sum_{\ell=0}^{m}\binom{m}{m-\ell}s^{m-\ell}t^{\ell}\frac{\partial^m f(x,y)}{\partial^{m-\ell}x\partial^{\ell}y}. \end{aligned}$$

We first estimate $\varphi_{x,y}^{[m]}(w;s,t)$. Using (2.1), (2.2) and (2.4), we have

$$\varphi_{x,y}^{[m]}(w;s,t)=\sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}\left\{\sum_{\ell=0}^{m}\binom{m}{m-\ell}s^{m-\ell}t^{\ell}\frac{\partial^m f(x+jsw,y+jtw)}{\partial^{m-\ell}x\partial^{\ell}y}\right\}$$

$$-\sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}\left\{\sum_{\ell=0}^{m}\binom{m}{m-\ell}s^{m-\ell}t^{\ell}\frac{\partial^m f(x,y)}{\partial^{m-\ell}x\partial^{\ell}y}\right\}$$

$$= \sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}\left\{\sum_{\ell=0}^{m}\binom{m}{m-\ell}s^{m-\ell}t^{\ell}\frac{\partial^m f(x+jsw,y+jtw)}{\partial^{m-\ell}x\partial^{\ell}y}\right\}$$

$$+(-1)^{r}\binom{r}{0}\left\{\sum_{\ell=0}^{m}\binom{m}{m-\ell}s^{m-\ell}t^{\ell}\frac{\partial^m f(x,y)}{\partial^{m-\ell}x\partial^{\ell}y}\right\}$$

$$= \sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j}\left\{\sum_{\ell=0}^{m}\binom{m}{m-\ell}s^{m-\ell}t^{\ell}\frac{\partial^m f(x+jsw,y+jtw)}{\partial^{m-\ell}x\partial^{\ell}y}\right\}.$$

In this case, we see that

$$\left|\varphi_{x,y}^{[m]}(w;s,t)\right| \leq \left(|s|^m+|t|^m\right)\sum_{j=0}^{r}\binom{r}{j}\left\{\sum_{\ell=0}^{m}\binom{m}{m-\ell}\left|\frac{\partial^m f(x+jsw,y+jtw)}{\partial^{m-\ell}x\partial^{\ell}y}\right|\right\}. \tag{3.6}$$

After integration and some simple calculations, and also using Lemma 2.1, we obtain, for every $n\in\mathbb{N}$, that

$$\begin{aligned}
P_{r,n}^{[m]}(f;x,y)-f(x,y) &= \frac{1}{2\pi\xi_n^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\sum_{j=0}^{r}\alpha_{j,r}^{[m]}\left(f\left(x+sj,y+tj\right)-f(x,y)\right)\right\} \\
&\quad\times e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt \\
&= \frac{1}{2\pi\xi_n^2}\sum_{k=1}^{m}\frac{\delta_{k,r}^{[m]}}{k!}\sum_{\ell=0}^{k}\binom{k}{k-\ell}\frac{\partial^k f(x,y)}{\partial^{k-\ell}x\partial^{\ell}y} \\
&\quad\times\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}s^{k-\ell}t^{\ell}e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt\right\} \\
&\quad+R_n^{[m]}(x,y)
\end{aligned}$$

where

$$R_n^{[m]}(x,y) := \frac{1}{2\pi\xi_n^2(m-1)!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_0^1(1-w)^{m-1}\varphi_{x,y}^{[m]}(w;s,t)dw\right)e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt.$$

By (3.4) and (3.6), it is clear that

$$\left|R_n^{[m]}(x,y)\right| \leq \frac{1}{2\pi\xi_n^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G_{x,y}^{[m]}(s,t)\left(|s|^m+|t|^m\right)e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt.$$

Then, combining these results with Lemma 2.2, we immediately get (3.5). The proof is completed. □

The next estimate answers the case of $m=0$ in (2.5).

Theorem 3.2. *Let $f\in C_B\left(\mathbb{R}^2\right)$. Then, we have*

$$\left|P_{r,n}^{[0]}(f;x,y)-f(x,y)\right|\leq\frac{2}{\pi\xi_n^2}\int\limits_0^\infty\int\limits_0^\infty\omega_r\left(f;\sqrt{s^2+t^2}\right)e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt. \tag{3.7}$$

Proof. Taking $m=0$ in (2.1) we observe that

$$\begin{aligned}P_{r,n}^{[0]}(f;x,y)-f(x,y)&=\frac{1}{2\pi\xi_n^2}\int\limits_{-\infty}^\infty\int\limits_{-\infty}^\infty\left\{\sum_{j=1}^r\alpha_{j,r}^{[0]}\left(f\left(x+sj,y+tj\right)-f(x,y)\right)\right\}\\&\times e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt\\&=\frac{1}{2\pi\xi_n^2}\int\limits_{-\infty}^\infty\int\limits_{-\infty}^\infty\left\{\sum_{j=1}^r(-1)^{r-j}\binom{r}{j}\left(f\left(x+sj,y+tj\right)-f(x,y)\right)\right\}\\&\times e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt\\&=\frac{1}{2\pi\xi_n^2}\int\limits_{-\infty}^\infty\int\limits_{-\infty}^\infty\left\{\sum_{j=1}^r(-1)^{r-j}\binom{r}{j}f\left(x+sj,y+tj\right)\right.\\&\left.+\left(-\sum_{j=1}^r(-1)^{r-j}\binom{r}{j}\right)f(x,y)\right\}e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt.\end{aligned}$$

Now using (2.4) we have

$$\begin{aligned}P_{r,n}^{[0]}(f;x,y)-f(x,y)&=\frac{1}{2\pi\xi_n^2}\int\limits_{-\infty}^\infty\int\limits_{-\infty}^\infty\left\{\sum_{j=1}^r(-1)^{r-j}\binom{r}{j}f\left(x+sj,y+tj\right)\right.\\&\left.+(-1)^r\binom{r}{0}f(x,y)\right\}e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt\\&=\frac{1}{2\pi\xi_n^2}\int\limits_{-\infty}^\infty\int\limits_{-\infty}^\infty\left\{\sum_{j=0}^r(-1)^{r-j}\binom{r}{j}f\left(x+sj,y+tj\right)\right\}\\&\times e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt,\end{aligned}$$

and hence, by (3.2),

$$P_{r,n}^{[0]}(f;x,y)-f(x,y)=\frac{1}{2\pi\xi_n^2}\int\limits_{-\infty}^\infty\int\limits_{-\infty}^\infty\Delta_{s,t}^r\left(f(x,y)\right)e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt.$$

Therefore, we obtain from (3.1) that

$$\begin{aligned}\left|P_{r,n}^{[0]}(f;x,y)-f(x,y)\right| &\leq \frac{1}{2\pi\xi_n^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\Delta_{s,t}^{r}\left(f(x,y)\right)\right|e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt \\ &\leq \frac{1}{2\pi\xi_n^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\omega_r\left(f;\sqrt{s^2+t^2}\right)e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt \\ &= \frac{2}{\pi\xi_n^2}\int_{0}^{\infty}\int_{0}^{\infty}\omega_r\left(f;\sqrt{s^2+t^2}\right)e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt\end{aligned}$$

which completes the proof. □

4. Statistical approximation by the operators (2.5)

We first get the following statistical approximation theorem for the operators (2.5) in case of $m\in\mathbb{N}$.

Theorem 4.1. *Let $A=[a_{jn}]$ be a non-negative regular summability matrix, and let $(\xi_n)_{n\in\mathbb{N}}$ be a bounded sequence of positive real numbers for which*

$$st_A-\lim_n \xi_n=0 \tag{4.1}$$

holds. Then, for each fixed $m\in\mathbb{N}$ and for all $f\in C^{(m)}\left(\mathbb{R}^2\right)$ satisfying (3.3), *we have*

$$st_A-\lim_n\left\|P_{r,n}^{[m]}(f)-f\right\|=0. \tag{4.2}$$

Proof. Let $m\in\mathbb{N}$ be fixed. Then, we obtain from the hypothesis and (3.5) that

$$\begin{aligned}\left\|P_{r,n}^{[m]}(f)-f\right\| \leq &\sum_{i=1}^{[m/2]}(2i+1)K_i\delta_{2i,r}^{[m]}\xi_n^{2i} \\ &+\frac{1}{2\pi\xi_n^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\|G_{x,y}^{[m]}(s,t)\right\|(|s|^m+|t|^m)e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt,\end{aligned}$$

where

$$K_i:=\frac{1}{\pi}\sum_{\ell=0}^{2i}\binom{2i}{2i-\ell}\left\|\frac{\partial^{2i}f(\cdot,\cdot)}{\partial^{2i-\ell}x\partial^{\ell}y}\right\|B\left(\frac{2i-\ell+1}{2},\frac{\ell+1}{2}\right)$$

for $i = 1, ..., \left[\frac{m}{2}\right]$. By (3.4) we get that

$$\begin{aligned}\left\|G_{x,y}^{[m]}(s,t)\right\| &\leq \frac{2^r}{(m-1)!}\left(\sum_{\ell=0}^{m}\binom{m}{m-\ell}\left\|\frac{\partial^m f(\cdot,\cdot)}{\partial^{m-\ell}x\partial^\ell y}\right\|\right)\int_0^1 (1-w)^{m-1}dw \\ &= \frac{2^r}{m!}\sum_{\ell=0}^{m}\binom{m}{m-\ell}\left\|\frac{\partial^m f(\cdot,\cdot)}{\partial^{m-\ell}x\partial^\ell y}\right\|,\end{aligned}$$

thus we obtain

$$\begin{aligned}\left\|P_{r,n}^{[m]}(f)-f\right\| &\leq \sum_{i=1}^{[m/2]}(2i+1)K_i\delta_{2i,r}^{[m]}\xi_n^{2i} \\ &\quad +\frac{2^{r+1}}{\pi m!\xi_n^2}\left(\sum_{\ell=0}^{m}\binom{m}{m-\ell}\left\|\frac{\partial^m f(\cdot,\cdot)}{\partial^{m-\ell}x\partial^\ell y}\right\|\right) \\ &\quad \times\int_0^\infty\int_0^\infty (s^m+t^m)e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt.\end{aligned}$$

Then, we have

$$\begin{aligned}\left\|P_{r,n}^{[m]}(f)-f\right\| &\leq \sum_{i=1}^{[m/2]}(2i+1)K_i\delta_{2i,r}^{[m]}\xi_n^{2i} \\ &\quad +L_m\int_0^\infty\int_0^\infty (s^m+t^m)e^{-(\sqrt{s^2+t^2})/\xi_n}dsdt \\ &= \sum_{i=1}^{[m/2]}(2i+1)K_i\delta_{2i,r}^{[m]}\xi_n^{2i} \\ &\quad +L_m\int_0^{\pi/2}\int_0^\infty (\cos^m\theta+\sin^m\theta)\rho^{m+1}e^{-\rho/\xi_n}d\rho d\theta,\end{aligned}$$

where

$$L_m := \frac{2^{r+1}}{\pi m!\xi_n^2}\left(\sum_{\ell=0}^{m}\binom{m}{m-\ell}\left\|\frac{\partial^m f(\cdot,\cdot)}{\partial^{m-\ell}x\partial^\ell y}\right\|\right).$$

After some simple calculations, we see that

$$\left\|P_{r,n}^{[m]}(f)-f\right\| \leq \sum_{i=1}^{[m/2]}(2i+1)K_i\delta_{2i,r}^{[m]}\xi_n^{2i}+L_m\xi_n^{m+2}(m+1)!U_m,$$

where

$$U_m := \int_0^{\pi/2} (\cos^m \theta + \sin^m \theta) d\theta = B\left(\frac{m+1}{2}, \frac{1}{2}\right),$$

which yields

$$\left\| P_{r,n}^{[m]}(f) - f \right\| \leq S_m \left\{ \xi_n^{m+2} + \sum_{i=1}^{[m/2]} \xi_n^{2i} \right\}, \tag{4.3}$$

where

$$S_m := (m+1)! U_m L_m + \max_{i=1,2,\ldots,[m/2]} \left\{ (2i+1) K_i \delta_{2i,r}^{[m]} \right\}.$$

Now for a given $\varepsilon > 0$, define the following sets:

$$\begin{aligned} D &: = \left\{ n \in \mathbb{N} : \left\| P_{r,n}^{[m]}(f) - f \right\| \geq \varepsilon \right\}, \\ D_i &: = \left\{ n \in \mathbb{N} : \xi_n^{2i} \geq \frac{\varepsilon}{(1+[m/2]) S_m} \right\}, \ i = 1, ..., \left[\frac{m}{2}\right], \\ D_{1+[m/2]} &: = \left\{ n \in \mathbb{N} : \xi_n^{m+2} \geq \frac{\varepsilon}{(1+[m/2]) S_m} \right\}. \end{aligned}$$

Then, the inequality (4.3) gives that

$$D \subseteq \bigcup_{i=1}^{1+[m/2]} D_i,$$

and hence, for every $j \in \mathbb{N}$,

$$\sum_{n \in D} a_{jn} \leq \sum_{i=1}^{1+[m/2]} \sum_{n \in D_i} a_{jn}.$$

Now taking limit as $j \to \infty$ in the both sides of the above inequality and using the hypothesis (4.1), we obtain that

$$\lim_j \sum_{n \in D} a_{jn} = 0,$$

which implies (4.2). So, the proof is completed. □

Finally, we investigate the statistical approximation properties of the operators (2.5) when $m = 0$. We need the following result.

Lemma 4.1. *Let $A = [a_{jn}]$ be a non-negative regular summability matrix, and let $(\xi_n)_{n\in\mathbb{N}}$ be a bounded sequence of positive real numbers for which* (4.1) *holds. Then, for every $f \in C_B\left(\mathbb{R}^2\right)$, we have*

$$st_A - \lim_n \omega_r\left(f;\xi_n\right) = 0. \tag{4.4}$$

Proof. By the right-continuity of $\omega_r\left(f;\cdot\right)$ at zero, we may write that, for a given $\varepsilon > 0$, there exists a $\delta > 0$ such that $\omega_r\left(f;h\right) < \varepsilon$ whenever $0 < h < \delta$. Hence, $\omega_r\left(f;h\right) \geq \varepsilon$ implies that $h \geq \delta$. Now replacing h by ξ_n, for every $\varepsilon > 0$, we see that

$$\{n : \omega_r\left(f;\xi_n\right) \geq \varepsilon\} \subseteq \{n : \xi_n \geq \delta\},$$

which guarantees that, for each $j \in \mathbb{N}$,

$$\sum_{n:\omega_r(f;\xi_n)\geq\varepsilon} a_{jn} \leq \sum_{n:\xi_n\geq\delta} a_{jn}.$$

Also, by (4.1), we get

$$\lim_j \sum_{n:\xi_n\geq\delta} a_{jn} = 0,$$

which implies

$$\lim_j \sum_{n:\omega_r(f;\xi_n)\geq\varepsilon} a_{jn} = 0.$$

So, the proof is completed. □

Theorem 4.2. *Let $A = [a_{jn}]$ be a non-negative regular summability matrix, and let $(\xi_n)_{n\in\mathbb{N}}$ be a bounded sequence of positive real numbers for which* (4.1) *holds. Then, for all $f \in C_B\left(\mathbb{R}^2\right)$, we have*

$$st_A - \lim_n \left\| P_{r,n}^{[0]}(f) - f \right\| = 0. \tag{4.5}$$

Proof. By (3.7), we can write

$$\left\| P_{r,n}^{[0]}(f) - f \right\| \leq \frac{2}{\pi\xi_n^2} \int_0^\infty \int_0^\infty \omega_r\left(f;\sqrt{s^2+t^2}\right) e^{-(\sqrt{s^2+t^2})/\xi_n} dsdt.$$

Now using the fact that $\omega_r(f;\lambda u) \leq (1+\lambda)^r \omega_r(f;u)$, $\lambda, u > 0$, we get

$$\begin{aligned}
\left\| P_{r,n}^{[0]}(f) - f \right\| &\leq \frac{2}{\pi\xi_n^2} \int_0^\infty \int_0^\infty \omega_r\left(f;\xi_n \frac{\sqrt{s^2+t^2}}{\xi_n}\right) e^{-(\sqrt{s^2+t^2})/\xi_n} ds dt \\
&\leq \frac{2\omega_r(f;\xi_n)}{\pi\xi_n^2} \int_0^\infty \int_0^\infty \left(1+\frac{\sqrt{s^2+t^2}}{\xi_n}\right)^r e^{-(\sqrt{s^2+t^2})/\xi_n} ds dt \\
&= \frac{2\omega_r(f;\xi_n)}{\pi\xi_n^2} \int_0^{\pi/2} \int_0^\infty \left(1+\frac{\rho}{\xi_n}\right)^r \rho e^{-\rho/\xi_n} d\rho d\theta \\
&= \omega_r(f;\xi_n) \int_0^\infty (1+u)^r u e^{-u} du \\
&\leq \omega_r(f;\xi_n) \int_0^\infty (1+u)^{r+1} e^{-u} du \\
&= \left(\sum_{k=0}^{r+1} \binom{r+1}{k} k! \right) \omega_r(f;\xi_n),
\end{aligned}$$

and hence

$$\left\| P_{r,n}^{[0]}(f) - f \right\| \leq K_r \omega_r(f;\xi_n), \tag{4.6}$$

where

$$K_r := \sum_{k=0}^{r+1} \binom{r+1}{k} k!.$$

Then, from (4.6), for a given $\varepsilon > 0$, we observe that

$$\left\{ n \in \mathbb{N} : \left\| P_{r,n}^{[0]}(f) - f \right\| \geq \varepsilon \right\} \subseteq \left\{ n \in \mathbb{N} : \omega_r(f;\xi_n) \geq \frac{\varepsilon}{K_r} \right\},$$

which implies that

$$\sum_{n:\left\| P_{r,n}^{[0]}(f)-f \right\| \geq \varepsilon} a_{jn} \leq \sum_{n:\omega_r(f;\xi_n) \geq \varepsilon/K_r} a_{jn} \tag{4.7}$$

holds for every $j \in \mathbb{N}$. Now, taking limit as $j \to \infty$ in the both sides of inequality (4.7) and also using (4.4), we obtain that

$$\lim_j \sum_{n:\left\| P_{r,n}^{[0]}(f)-f \right\| \geq \varepsilon} a_{jn} = 0,$$

which means (4.5). Hence, the proof is completed. □

5. Concluding remarks

In this section, we give some special cases of our results obtained in the previous sections.

Taking $A = C_1$, the Cesáro matrix of order one, and also combining Theorems 4.1 and 4.2, we immediately get the following result.

Corollary 5.1. *Let $(\xi_n)_{n\in\mathbb{N}}$ be a bounded sequence of positive real numbers for which*

$$st - \lim_n \xi_n = 0$$

holds. Then, for each fixed $m \in \mathbb{N}_0$ and for all $f \in C^{(m)}\left(\mathbb{R}^2\right)$ satisfying (3.3), *we have*

$$st - \lim_n \left\| P_{r,n}^{[m]}(f) - f \right\| = 0.$$

Furthermore, choosing $A = I$, the identity matrix, in Theorems 4.1 and 4.2, we have the next approximation theorems with the usual convergence.

Corollary 5.2. *Let $(\xi_n)_{n\in\mathbb{N}}$ be a bounded sequence of positive real numbers for which*

$$\lim_n \xi_n = 0$$

holds. Then, for each fixed $m \in \mathbb{N}_0$ and for all $f \in C^{(m)}\left(\mathbb{R}^2\right)$ satisfying (3.3), *the sequence $\left\{P_{r,n}^{[m]}(f)\right\}$ is uniformly convergent to f on $\mathbb{R}^2$.*

Now we define a special sequence $(\xi_n)_{n\in\mathbb{N}}$ as follows:

$$\xi_n := \begin{cases} 1, & \text{if } n = k^2,\ k = 1, 2, ... \\ \frac{1}{n}, & \text{otherwise.} \end{cases} \tag{5.1}$$

Then, observe that $st - \lim_n \xi_n = 0$. In this case, taking $A = C_1$, we obtain from Corollary 5.1 (or, Theorems 4.1 and 4.2) that

$$st - \lim_n \left\| P_{r,n}^{[m]}(f) - f \right\| = 0$$

holds for each $m \in \mathbb{N}_0$ and for all $f \in C^{(m)}\left(\mathbb{R}^2\right)$ satisfying (3.3). However, since the sequence $(\xi_n)_{n\in\mathbb{N}}$ given by (5.1) is non-convergent, the classical approximation to a function f by the operators $P_{r,n}^{[m]}(f)$ is impossible.

Notice that Theorems 4.1, 4.2 and Corollary 5.1 are also valid when $\lim \xi_n = 0$ because every convergent sequence is A-statistically convergent, and so statistically convergent. But, as in the above example, our theorems still work although $(\xi_n)_{n\in\mathbb{N}}$ is non-convergent. Therefore, this non-trivial example clearly demonstrates the power of our statistical approximation method in Theorems 4.1 and 4.2 with respect to Corollary 5.2.

In the end, we should remark that, so far, almost all statistical approximation results have dealt with positive linear operators. Of course, in this case, one has the following natural problem:

- Can we use the concept of A-statistical convergence in the approximation by non-positive approximation operators?

The same question was also asked as an open problem by Duman et. al. in [13]. With this paper we find affirmative answers to this problem by using the double smooth Picard singular integral operators given by (2.5). However, some similar arguments may be valid for other non-positive operators. Thus, in the future studies, it would be very interesting to improve such structures.

References

[1] Altomare, F. and Campiti, M., *Korovkin Type Approximation Theory and Its Application*, Walter de Gruyter Publ., Berlin, 1994.

[2] Anastassiou, G. A., *L_p convergence with rates of smooth Picard singular operators*, Differential & Difference Equations and Applications, 31-45, Hindawi Publ. Corp., New York, 2006.

[3] Anastassiou, G. A., *Basic convergence with rates of smooth Picard singular integral operators*, J. Comput. Anal. Appl., **8** (2006), 313-334.

[4] Anastassiou, G. A., *Global smoothness and uniform convergence of smooth Picard singular operators*, Comput. Math. Appl., **50** (2005), 1755-1766.

[5] Anastassiou, G. A. and Duman, O., *A Baskakov type generalization of statistical Korovkin theory*, J. Math. Anal. Appl., **340** (2008), 476-486.

[6] Anastassiou, G. A. and Duman, O., *Statistical fuzzy approximation by fuzzy positive linear operators*, Comput. Math. Appl., **55** (2008), 573-580.

[7] Anastassiou, G. A. and Gal, S. G., *Approximation Theory: Moduli of Continuity and Global Smoothness Preservation*, Birkhäuser Boston, Inc., Boston, MA, 2000.

[8] Anastassiou, G. A. and Gal, S. G., *Convergence of generalized singular integrals to the unit, multivariate case*, *Applied Mathematics Reviews*, **1** (2000), no. 1-8, World Sci. Publ., River Edge, NJ.

[9] Anastassiou, G. A. and Gal, S. G., *Convergence of generalized singular integrals to the unit, univariate case*, Math. Inequal. Appl., **4** (2000), 511-518.

[10] Anastassiou, G. A. and Gal, S. G., *General theory of global smoothness preservation by singular integrals, univariate case*, J. Comput. Anal. Appl., **1** (1999), 289-317.

[11] Duman, O. and Erkuş, E., *Approximation of continuous periodic functions via statistical convergence*, Comput. Math. Appl., **52** (2006), 967-974.

[12] Duman, O., Erkuş, E. and Gupta, V., *Statistical rates on the multivariate approximation theory*, Math. Comput. Modelling, **44** (2006), 763-770.

[13] Duman, O., Khan, M. K. and Orhan, C., *A-Statistical convergence of approximating operators*, Math. Inequal. Appl., **6** (2003), 689-699.

[14] Duman, O., Özarslan, M. A. and Doğru, O., *On integral type generalizations of positive linear operators*, Studia Math., **174** (2006), 1-12.

[15] Erkuş, E. and Duman, O., *A Korovkin type approximation theorem in statistical sense*, Studia Sci. Math. Hungar., **43** (2006), 285-294.

[16] Erkuş, E. and Duman, O., *A-statistical extension of the Korovkin type approximation theorem*, Proc. Indian Acad. Sci., Math. Sci., **115** (2005), 499-508.

[17] Fast, H., *Sur la convergence statistique*, Colloq. Math., **2** (1951), 241-244.

[18] Freedman, A. R. and Sember, J. J., *Densities and summability*, Pacific J. Math., **95** (1981), 293-305.

[19] Gal, S. G., *Degree of approximation of continuous functions by some singular integrals*, Rev. Anal. Numér. Théor. Approx., Cluj, **27** (1998), 251-261.

[20] Hardy, G. H., *Divergent Series*, Oxford Univ. Press, London, 1949.

[21] Kolk, E., *Matrix summability of statistically convergent sequences*, Analysis, **13** (1993), 77-83.

[22] Korovkin, P. P., *Linear Operators and Approximation Theory*, Hindustan Publ. Corp., Delhi, 1960.

[23] Özarslan, M. A., Duman, O. and Doğru, O., *Rates of A-statistical convergence of approximating operators*, Calcolo, **42** (2005), 93-104.

[24] Özarslan, M. A., Duman, O. and Srivastava, H. M., *Statistical approximation results for Kantorovich-type operators involving some special functions*, Math. Comput. Modelling, **48** (2008), 388-401.

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