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INVERSE THEOREM FOR AN ITERATIVE COMBINATION OF BERNSTEIN-DURRMEYER POLYNOMIALS

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Abstract. The Bernstein-Durrmeyer polynomial

$$[M_n(f;t) = (n+1)\sum_{k=0}^{n} p_{n,k}(t)\int_0^1 p_{n,k}(u)f(u)\,du,$$

where $p_{n,k}(t) = \binom{n}{k} t^k(1-t)^{n-k}$, $t \in [0,1]$ defined on $L_B[0,1]$, the space of bounded and integrable functions on $[0,1]$ were introduced by Durrmeyer [5] and extensively studied by Derriennic [3] and other researchers (see [1]-[3], [5], [6], [8]). It turns out that the order of approximation by these operators is, at best, $O(n^{-1})$ however smooth the function may be. In order to improve the rate of approximation we consider an iterative combination $T_{n,k}(f;t)$ of the operators $M_n(f;t)$. This technique was given by Micchelli [9] who first used it to improve the order of approximation by Bernstein polynomials $B_n(f;t)$. In the paper [1] some direct theorems in ordinary and simultaneous approximation for the operators $T_{n,k}(f;t)$ in the uniform norm, have been established. The paper [10] is a study of some direct results in the L_p- approximation by the operators $T_{n,k}(f;t)$. The object of the present paper is to study the corresponding inverse theorem in L_p- approximation by the operators $T_{n,k}(f;t)$.

1. Introduction

For $f \in L_p[0,1], 1 \leqslant p < \infty$ the operators M_n can be expressed as

$$M_n(f;t) = \int_0^1 W_n(u,t)f(u)\,du,$$

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where $W_n(u,t) = (n+1)\sum_{k=0}^{n} p_{n,k}(t)p_{n,k}(u)$ is the kernel of the operators.

For $m \in N^0$ (the set of non-negative integers), the mth order moment for the operators M_n is defined as

$$\mu_{n,m}(t) = M_n\left((u-t)^m; t\right).$$

The iterative combination $T_{n,k} : L_p[0,1] \to C^\infty[0,1]$ of the operators is defined as

$$T_{n,k}(f;t) = \left(I - (I - M_n)^k\right)(f;t) = \sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r} M_n^r(f;t),\ k \in N,$$

where $M_n^0 \equiv I$ and $M_n^r \equiv M_n(M_n^{r-1})$ for $r \in N$.

Throughout the present paper we assume that $I = [0,1]$ and $I_j = [a_j, b_j]$, $j = 1,2,3,\ 0 < a_1 < a_2 < a_3 < b_3 < b_2 < b_1 < 1$ and by C we mean the positive constant not necessarily the same at each occurrence.

In [10], we obtained following direct theorem:

Theorem 1. *If $p \geqslant 1, f \in L_p[0,1]$. Then for all n sufficiently large there holds*

$$\|T_{n,k}(f;.) - f\|_{L_p(I_2)} \leqslant C_k\left(\omega_{2k}\Big(f, \frac{1}{\sqrt{n}}, p, I_1.\Big) + n^{-k}\|f\|_{L_p[0,1]}\right), \qquad (1.1)$$

where C_k is a constant independent of f and n.

Remark 1. *From above theorem it follows that if $\omega_{2k}(f,\tau,p,I_2) = O(\tau^\alpha)$ as $\tau \to 0$ then $\|T_{n,k}(f,.) - f\|_{L_p(I_2)} = O(n^{-\alpha/2})$ as $n \to \infty$, where $0 < \alpha < 2k$.*

The aim of this paper is to establish a corresponding local inverse theorem for the operators $T_{n,k}(f,t)$ in the L_p-norm i.e. the characterization of the class of functions for which $\|T_{n,k}(f,.) - f\|_{L_p(I_2)} = O(n^{-\alpha/2})$ as $n \to \infty$, where $0 < \alpha < 2k$.

Thus we prove the following theorem (*inverse theorem*):

Theorem 2. *Let $f \in L_p[0,1], 1 \leqslant p < \infty,\ 0 < \alpha < 2k$ and $\|T_{n,k}(f,.) - f\|_{L_p(I_1)} = O(n^{-\alpha/2})$ as $n \to \infty$. Then, $\omega_{2k}(f,\tau,p,I_2) = O(\tau^\alpha)$ as $\tau \to 0$.*

2. **Preliminaries**

In this section we give some results which are useful in establishing our main theorem.

Lemma 1. [1] *For the function* $\mu_{n,m}(t)$, *we have*

$$\mu_{n,0}(t) = 1, \mu_{n,1}(t) = \frac{(1-2t)}{(n+2)}$$

and for $m \geqslant 1$ *there holds the recurrence relation*

$$(n+m+2)\mu_{n,m+1}(t) = t(1-t)\left\{\mu'_{n,m}(t) + 2m\mu_{n,m-1}(t)\right\} + (m+1)(1-2t)\mu_{n,m}(t).$$

Consequently,

(i) $\mu_{n,m}(t)$ *is a polynomial in* t *of degree* m;

(ii) *for every* $t \in [0,1], \mu_{n,m}(t) = O\left(n^{[(m+1)/2]}\right)$, *where* $[\beta]$ *is the integer part of* β.

Lemma 2. [8]*For the function* $p_{n,k}(t)$, *there holds the result*

$$t^r(1-t)^r D^r\left(p_{n,k}(t)\right) = \sum_{\substack{2i+j\leq m \\ i,j\geq 0}} n^i (k-nt)^j q_{i,j,r}(t) p_{n,k}(t),$$

where $D \equiv \frac{d}{dt}$ *and* $q_{i,j,r}(t)$ *are certain polynomials in* t *independent of* n *and* k.

Lemma 3. [1] *For* $k, l \in N$, *there holds* $T_{n,k}\left((u-t)^l; t\right) = O(n^{-k})$.

Lemma 4. *If* $f \in L_p[0,1]$ *then there holds the estimate*

$$\left\| \frac{d^m}{dt^m}\left(T_{n,k}(f;\bullet)\right) \right\|_{L_p[c,d]} \leqslant C\, n^{m/2} \|f\|_{L_p[0,1]},$$

where $[c,d]$ *is any closed interval contained in* $(0,1)$.

Proof. We have

$$\frac{d^m}{dt^m}\left(M_n^k(f;t)\right) = \frac{d^m}{dt^m} \int_0^1 W_n(u,t) M_n^{k-1}(f;u)\, du$$

$$= (n+1)\sum_{\nu=0}^{n} p_{n,\nu}(t) \sum_{\substack{2i+j\leq m \\ i,j\geqslant 0}} n^i \frac{(\nu-nt)^j q_{i,j,m}(t)}{(t(1-t))^m} \times \int_0^1 p_{n,\nu}(u) M_n^{k-1}(f;u)\, du, \qquad (2.1)$$

Using Holder's inequality for summation, we obtain

$$\frac{d^m}{dt^m}\left(M_n^k(f;t)\right)\Big| \leqslant C(n+1)\sum_{\nu=0}^{n} \sum_{\substack{2i+j\leqslant m \\ i,j\geqslant 0}} p_{n,\nu}(t) n^i |\nu - nt|^j \Big(\int_0^1 p_{n,\nu}(u)\, du\Big)^{1/q}$$

$$\times\left(\int_0^1 p_{n,\nu}(u)\left|M_n^{k-1}(f;u)\right|^p du\right)^{1/p}$$

$$\leqslant C(n+1)^{1-1/q}\sum_{\substack{2i+j\leqslant m\\ i,j\geqslant 0}} n^i\Big(\sum_{\nu=0}^n p_{n,\nu}(t)|\nu-nt|^{qj}\Big)^{1/q}$$

$$\times\left(\sum_{\nu=0}^n p_{n,\nu}(t)\int_0^1 p_{n,\nu}(u)\left|M_n^{k-1}(f;u)\right|^p du\right)^{1/p}$$

$$\leqslant C(n+1)^{1/p}\sum_{\substack{2i+j\leqslant m\\ i,j\geqslant 0}} n^i.n^{j/2}$$

$$\times\left(\sum_{\nu=0}^n p_{n,\nu}(t)\int_0^1 p_{n,\nu}(u)\left|M_n^{k-1}(f;u)\right|^p du\right)^{1/p} \tag{2.2}$$

Therefore, applying Fubini's theorem, we get

$$\left\|\frac{d^m}{dt^m}\big(M_n^k(f;t)\big)\right\|_{L_p[c,d]}\leqslant C(n+1)^{1/p}n^{m/2}\times$$

$$\left(\int_c^d\sum_{\nu=0}^n p_{n,\nu}(t)\int_0^1\left|M_n^{k-1}(f;u)\right|^p p_{n,\nu}(u)\,du\,dt\right)^{1/p}$$

$$\leqslant C(n+1)^{1/p}n^{m/2}\left\{\sum_{\nu=0}^n\left(\int_c^d p_{n,\nu}(t)\,dt\right)\times\right.$$

$$\left.\left(\int_0^1 p_{n,\nu}(u)\left|M_n^{k-1}(f;u)\right|^p du\right)\right\}^{1/p}$$

$$\leqslant Cn^{m/2}\left\{\int_0^1\sum_{\nu=0}^n p_{n,\nu}(u)\left|M_n^{k-1}(f;u)\right|^p du\right\}^{1/p}$$

$$\leqslant Cn^{m/2}\left\|M_n^{k-1}(f;u)\right\|_{L_p[0,1]}\leqslant Cn^{m/2}\|f\|_{L_p[0,1]}. \tag{2.3}$$

Since $T_{n,k}$ are linear combinations of the iterates M_n, and the r.h.s. in (2.3) is independent of k, the lemma follows from (2.3). □

Lemma 5. *If $f\in L_p[0,1]$ is such that $f^{(m-1)}\in AC(I)$ and $f^{(m)}\in L_p(I)$, then*

$$\left\|\frac{d^m}{dt^m}\big(T_{n,k}(f;\bullet)\big)\right\|_{L_p[c,d]}\leqslant M\|f^{(m)}\|_{L_p[0,1]},$$

where $[c,d]\subset(0,1)$.

Proof. It is sufficient to find the estimate for $\frac{d^m}{dt^m}\big(M_n^k(f;\bullet)\big)$. Thus, we have

$$\begin{aligned}\frac{d^m}{dt^m}\big(M_n^r(f;\bullet)\big) &= \frac{d^m}{dt^m}\big[M_n\big((M_n^{k-1}(f;u_k);u);t\big)\big]\\ &= \sum_{i=0}^{m-1}\frac{f^{(i)}(t)}{i!}\frac{d^m}{dt^m}\big[M_n\big((M_n^{k-1}(u_k-t)^i;u);t\big)\big)\big]\\ &+ \frac{1}{(m-1)!}\frac{d^m}{dt^m}\left[M_n\Big(M_n^{k-1}\Big(\int_t^{u_k}(u_k-w)^{m-1}f^{(m)}(w)\,dw;u\Big);t\Big)\right]\end{aligned}$$

The term inside the summation is polynomial of degree $(m-1)$ and hence vanish. In order to estimate the second term we break the integral as follows. There exists a non-negative integer $r=r(n)$ such that $r/\sqrt{n}\leqslant \max|u_k-t|\leqslant (r+1)/\sqrt{n}$. Hence, we get

$$\begin{aligned}I &= \int_0^1 W_n(u_k,u_{k-1})|u_k-t|^{m-1}\left|\int_t^{u_k}\big|f^{(m)}(w)\big|\,dw\right|du_k\\ &\leqslant \sum_{l=0}^{r}\left\{\int_{t+\frac{l}{\sqrt{n}}}^{t+\frac{l+1}{\sqrt{n}}} W_n(u_k,u_{k-1})|u_k-t|^{m-1}\int_t^{t+\frac{l+1}{\sqrt{n}}}\big|f^{(m)}(w)\big|\,dw\,du_k\right.\\ &+ \left.\int_{t-\frac{l+1}{\sqrt{n}}}^{t-\frac{l}{\sqrt{n}}} W_n(u_k,u_{k-1})|u_k-t|^{m-1}\int_{t-\frac{l+1}{\sqrt{n}}}^{t}\big|f^{(m)}(w)\big|\,dw\,du_k\right\}\end{aligned}\tag{2.4}$$

Now, $|u_k-t|>l/\sqrt{n}$ and

$$|u_k-t|^{m+3}\leqslant\sum_{s=0}^{m+3}\binom{m+3}{s}|u_k-u_{k-1}|^{m+3-s}|u_{k-1}-t|^s$$

Hence a typical term of (2.4) is estimated as

$$\leqslant\sum_{r=0}^{m+3}\int_{t+\frac{l}{\sqrt{n}}}^{t+\frac{l+1}{\sqrt{n}}} W_n(u_k,u_{k-1})|u_k-u_{k-1}|^{m+3-r}|u_{k-1}-t|^r$$

$$\times\binom{m+3}{r}\frac{n^2}{l^4}\int_t^{t+\frac{l+1}{\sqrt{n}}}\big|f^{(m)}(w)\big|\,dw\,du_k$$

$$\leqslant \sum_{r=0}^{m+3} C\frac{n^2}{l^4}\frac{1}{n^{(m+3-r)/2}} \int\limits_{t}^{t+\frac{l+1}{\sqrt{n}}} \left|f^{(m)}(w)\right| dw$$

Proceeding recursively we reach

$$\frac{d^m}{dt^m}\left[\int\limits_0^1 W_n(u_1,t)|u_1-t|^s\frac{n^2}{l^4}\frac{1}{n^{(m+3-r)/2}}\left(\int\limits_{t}^{t+\frac{l+1}{\sqrt{n}}} f^{(m)}(w)\,dw\right)du_1\right]$$

$$= (n+1)\sum_{\substack{2i+j\leq m\\ i,j\geqslant 0}}\sum_{\nu=0}^{n} n^i q_{i,j,m}(t)(\nu-nt)^j p_{n,\nu}(t)\left(\int\limits_0^1 p_{n,\nu}(u_1)|u_1-t|^s\,du_1\right)$$

$$\times\frac{n^2}{l^4}\frac{1}{n^{(m+3-r)/2}}\left(\int\limits_{t}^{t+\frac{l+1}{\sqrt{n}}}\left|f^{(m)}(w)\right|dw\right)$$

Using Holder's inequality and moment estimates for M_n, we obtain

$$\left|\frac{d^m}{dt^m}\left[\int\limits_0^1 W_n(u_1,t)|u_1-t|^s\frac{n^2}{l^4}\frac{1}{n^{(m+3-r)/2}}\left(\int\limits_{t}^{t+\frac{l+1}{\sqrt{n}}} f^{(m)}(w)\,dw\right)du_1\right]\right|$$

$$\leqslant C\sum_{l=0}^{r}\frac{n^2}{l^4}\frac{n^{m/2-s/2}}{n^{\frac{m+3-s}{2}}}\left(\int\limits_{t}^{t+\frac{l+1}{\sqrt{n}}}\left|f^{(m)}(w)\right|dw\right)$$

This implies

$$\left\|\frac{d^m}{dt^m}M_n^k(f;t)\right\|_{L_p[x_2',y_2']} \leqslant C\sum_{l=0}^{r}\frac{1}{l^4}(l+1)\left\|f^{(m)}\right\|_{L_p[0,1]} \leqslant C\left\|f^{(m)}\right\|_{L_p[0,1]}$$

This completes the proof of the lemma. □

3. Proof of the main theorem

Proof. We prove the theorem by induction on k.

When $k=1$ the operator $T_{n,k}$ becomes the well known Bernstein Durrmeyer operator M_n for which we prove the inverse result. Thus, we prove that

$$\|M_n(f;t)-f(t)\|_{L_p(I_1)} = O\left(n^{-\alpha/2}\right) \Rightarrow \omega_2(f,\tau,I_2) = O(\tau^\alpha);0<\alpha<2.$$

Let $g \in C_0^\infty$ be such that $supp\, g \in (a_2, b_2)$ with $g(t) = 1$ on I_3. Further, let $\bar{f} = fg$. Now,

$$\left\|\Delta_\tau^2 \bar{f}(t)\right\|_{L_p(I_3)} \leqslant \left\|\Delta_\tau^2(\bar{f}(t) - M_n(\bar{f};t))\right\|_{L_p(I_3)} + \left\|\Delta_\tau^2 M_n(\bar{f};t)\right\|_{L_p(I_3)} = I_1 + I_2. \quad (3.1)$$

In I_1

$$(fg)(t) - M_n\left(f(u)\big(g(t) + (u-t)g'(t) + ...\big); t\right)$$
$$= g(t)\left(f(t) - M_n(f;t)\right) - g'(t) M_n\left(f(u)(u-t);t\right) + ... \quad (3.2)$$

By hypothesis,

$$\|M_n(f;t) - f(t)\|_{L_p(I_1)} = O\left(n^{-\alpha/2}\right). \quad (3.3)$$

and by dual moment estimate,

$$\|M_n\left(f(u)(u-t);t\right)\|_{L_p(I_1)} = \|f\|/n^{1/2}. \quad (3.4)$$

Now

$$I_2 = \left\|\Delta_\tau^2 M_n(\bar{f};t)\right\|_{L_p(I_1)} \leq \tau^2 \left\|\frac{d^2}{dt^2}\left(M_n(\bar{f};t)\right)\right\|_{L_p(I_1)}$$
$$\leq \tau^2 \left\|\frac{d^2}{dt^2}\left(M_n(\bar{f} - \bar{f}_\eta;t)\right)\right\|_{L_p(I_1)} + \tau^2 \left\|\frac{d^2}{dt^2}\left(M_n(\bar{f}_\eta;t)\right)\right\|_{L_p(I_1)}$$
$$\leq \tau^2 \left(n\ \omega_2(\eta, \bar{f}) + \frac{1}{\eta^2}\ \omega_2(\eta, \bar{f})\right) \quad (3.5)$$
$$\therefore \omega_2(\tau, \bar{f}) \leq \frac{M}{n^{1/2}} + \tau^2 \left(n + \frac{1}{\eta^2}\right) \omega_2(\eta, \bar{f})$$
$$\Rightarrow \omega_1(\tau, \bar{f}) = O\left(\tau \left|\ln\, \tau\right|\right) \quad (3.6)$$

We use (3.6) in (3.2) and (3.3). Now,

$$\begin{aligned} M_n\left(f(u)(u-t);t\right) &= M_n\left((f(u) - f(t))(u-t);t\right) \\ &+ f(t) M_n((u-t);t) \\ &\leq M_n\left(|u-t|^2 \left|\ln|u-t|\right|; t\right) + O\left(\frac{1}{n}\right) \\ &\leq M_n\left(|u-t|^{2-\epsilon}; t\right) + O\left(\frac{1}{n}\right) = O\left(\frac{1}{n^{1-\epsilon}}\right). \end{aligned}$$

From,(3.1),(3.2) and (3.3)

$$\omega_2(\tau,\bar{f}) \leq O\left(\frac{M}{n^{1-\epsilon}}\right) + \tau^2\left(n+\frac{1}{\eta^2}\right)\omega_2(\eta,\bar{f})$$

$$\Rightarrow \omega_2(\tau,\bar{f}) = O\left(\tau^{2-\epsilon}\right).$$

Hence theorem is proved for $k=1$.

Now, suppose it is true for a certain k i.e.

$$\omega_{2k}(f,\tau,p,I_2) = O(\tau^{\alpha}) \tag{3.7}$$

Let

$$\|T_{n,k+1}(f,.)-f\|_{L_p(I_1)} = O(n^{-(\alpha+2)/2}) \tag{3.8}$$

We will show that

$$\omega_{2k+2}(f,\tau,p,I_2) = O(\tau^{\alpha+2})$$

Let $a_1 < x_1 < x_2 < x_3 < a_2 < b_2 < y_3 < y_2 < y_1 < b_1$ and $g \in C_0^{\infty}$ be such that $supp\, g \in (x_2,y_2)$ with $g(t)=1$ on $[x_3,y_3]$. Further, let $\bar{f}=fg$. Then we have

$$\begin{aligned}\left\|\Delta_{\tau}^{2k+2}T_{n,k+1}(\bar{f};t)\right\|_{L_p[x_2,y_2]} &\leqslant \left\|\Delta_{\tau}^{2k+2}(\bar{f}(t)-T_{n,k+1}(\bar{f};t))\right\|_{L_p[x_2,y_2]}\\ &+ \tau^{2k+2}\left\|\frac{d^{2k+2}}{dt^{2k+2}}\left(T_{n,k+1}(\bar{f}-\bar{f}_{\eta,2k+2};t)\right)\right\|_{L_p[x_2',y_2']}\\ &+ \tau^{2k+2}\left\|\frac{d^{2k+2}}{dt^{2k+2}}\left(T_{n,k+1}(\bar{f}_{\eta,2k+2};t)\right)\right\|_{L_p[x_2',y_2']}\end{aligned} \tag{3.9}$$

where $x_2'=x_2$ and $y_2'=y_2+(2k+2)\tau$.

For the first term, we have the estimate

$$\begin{aligned}\left\|\Delta_{\tau}^{2k+2}(\bar{f}(t)-T_{n,k+1}(\bar{f};t))\right\|_{L_p[x_2,y_2]} &\leqslant C\left\|\bar{f}(t)-T_{n,k+1}(\bar{f};t))\right\|_{L_p[x_2',y_2']}\\ \leqslant C\left\|f(t)g(t)-T_{n,k+1}\left(f(u)\Big[\sum_{i=0}^{\infty}\frac{g^{(i)}(t)}{i!}(u-t)^i\Big];t\right)\right\|_{L_p[x_2',y_2']}\\ \leqslant C\|g\|_{L_p[x_2,y_2]}\,\|f(t)-T_{n,k+1}(f;t)\|_{L_p[x_2',y_2']}\\ +\|g'\|_{L_p[x_2',y_2']}\,\|T_{n,k+1}(f(u)(u-t);t)\|_{L_p[x_2',y_2']}+\dots\end{aligned} \tag{3.10}$$

Using smoothness of f in second term of (3.10), we get

$$
\begin{aligned}
&\big\|T_{n,k+1}(f(u)(u-t);t)\big\|_{L_p[x_2',y_2']}\\
&\leqslant \quad \Bigg\|\sum_{i=0}^{2k-1}\frac{f^{(i)}(t)}{i!}T_{n,k+1}\big((u-t)^i;t\big)+\frac{1}{(2k-2)!}\\
&\times \quad T_{n,k+1}\Bigg((u-t)^{2k-1}\Bigg|\int_t^u\big(f^{(2k-1)}(w)-f^{(2k-1)}(t)\big)\,dw\Bigg|;t\Bigg)\Bigg\|_{L_p[x_2',y_2']}\\
&\leqslant \quad O\Big(\frac{1}{n^{k+1}}\Big)+C\sum_{m=1}^{k}\Bigg\|M_n^m\Big(|u-t|^{2k-1}\times\\
&\qquad\qquad \times\Bigg|\int_t^u\Big|f^{(2k-1)}(w)-f^{(2k-1)}(t)\Big|\,dw\Bigg|;t\Big)\Bigg\|_{L_p[x_2',y_2']}\\
&\leqslant \quad O\Big(\frac{1}{n^{k+1}}\Big)+C\Bigg\|M_n\Big(|u-t|^{2k-1}\times\\
&\qquad\qquad \times\Bigg|\int_t^u\Big|f^{(2k-1)}(w)-f^{(2k-1)}(t)\Big|\,dw\Bigg|;t\Big)\Bigg\|_{L_p[x_2',y_2']}
\end{aligned}
\tag{3.11}
$$

Now, we have

$$I=\Bigg|M_n\Bigg(|u-t|^{2k-1}\Bigg|\int_t^u\Big|f^{(2k-1)}(w)-f^{(2k-1)}(t)\Big|\,dw\Bigg|;t\Bigg)\Bigg|^p$$

$$\leqslant\Big(\int_0^1 W_n(u,t)\,du\Big)^{1/p}\Bigg(\int_0^1 W_n(u,t)\Bigg|\int_t^u\Big|f^{(2k-1)}(w)-f^{(2k-1)}(t)\Big|\,dw\Bigg|^p du\Bigg)$$

$$\leqslant\int_0^1 W_n(u,t)\Bigg|\int_t^u dw\Bigg|^{p/q}\Bigg|\int_t^u\Big|f^{(2k-1)}(w)-f^{(2k-1)}(t)\Big|\,dw\Bigg|^p$$

$$\leqslant\int_0^1 W_n(u,t)|u-t|^{(2k-1)p+p/q}\Bigg|\int_t^u\Big|f^{(2k-1)}(w)-f^{(2k-1)}(t)\Big|\,dw\Bigg|\,du \tag{3.12}$$

Now, in order to estimate the quantity in the right, we divide the integral once again as in Lemma 5 and use the moment estimates given in Lemma 1. Thus, from

(3.12) we get the following

$$\begin{aligned} I &\leqslant \sum_{l=0}^{r} \Bigg\{ \int_{t+\frac{l}{\sqrt{n}}}^{t+\frac{l+1}{\sqrt{n}}} \frac{n^2}{l^4} |u-t|^{4+(2k-1)p+p/q} W_n(u,t) \\ &\times \int_{t}^{t+\frac{l+1}{\sqrt{n}}} \left| f^{(2k-1)}(w) - f^{(2k-1)}(t) \right|^p dw\, du \\ &+ \int_{t-\frac{l+1}{\sqrt{n}}}^{t-\frac{l}{\sqrt{n}}} \frac{n^2}{l^4} |u-t|^{4+(2k-1)p+p/q} W_n(u,t) \\ &\times \int_{t-\frac{l+1}{\sqrt{n}}}^{t} \left| f^{(2k-1)}(w) - f^{(2k-1)}(t) \right|^p dw\, du \Bigg\} \\ &\leqslant \sum_{l=0}^{r} C \frac{n^2}{l^4} \frac{1}{n^{2+(2k-1)p/2+p/2q}} \Bigg(\int_{t}^{t+\frac{l+1}{\sqrt{n}}} \left| f^{(2k-1)}(w) - f^{(2k-1)}(t) \right|^p dw \\ &+ \int_{t-\frac{l+1}{\sqrt{n}}}^{t} \left| f^{(2k-1)}(w) - f^{(2k-1)}(t) \right|^p dw \Bigg) \end{aligned} \tag{3.13}$$

Now,

$$\int_{x_2'}^{y_2'} \int_{t}^{t+\frac{l+1}{\sqrt{n}}} \left| f^{(2k-1)}(w) - f^{(2k-1)}(t) \right|^p dw\, dt = \int_{0}^{\frac{l+1}{\sqrt{n}}} \int_{x_2'}^{y_2'} \left| f^{(2k-1)}(x+t) - f^{(2k-1)}(t) \right|^p dx\, dt$$

$$= \int_{x_2'}^{y_2'} \int_{0}^{1} \left| f^{(2k-1)}(x+t) - f^{(2k-1)}(t) \right|^p \chi(x)\, dx\, dt \leqslant \int_{0}^{1} x^{\theta p} \chi(x)\, dx$$

(where χ is the characteristic function of $[0, (l+1)/\sqrt{n}]$)

$$= \int_{0}^{1} \int_{x_2'}^{y_2'} \left| f^{(2k-1)}(x+t) - f^{(2k-1)}(t) \right|^p \chi(x)\, dt\, dx \leqslant C \frac{(l+1)^{p\theta+1}}{n^{\frac{p\theta+1}{2}}}, \ (\text{where}\, 0 < \theta < 1). \tag{3.14}$$

Combining (3.12),(3.13) and (3.14), we get

$$\left\| M_n\Big(|u-t|^{2k-1}\int\limits_t^u \Big|f^{(2k-1)}(w)-f^{(2k-1)}(t)\Big|\,dw\Big|;t\Big)\right\|_{L_p[x_2',y_2']}$$

$$\begin{aligned} &\leqslant C\left\{\sum_{l=0}^{r}\frac{n^2}{l^4}\frac{1}{n^{2+(2k-1)p/2+p/2q}}\frac{(l+1)^{p\theta+1}}{n^{(p\theta+1)/2}}\right\}^{1/p} \\ &\leqslant C\big(n^{-(k+\theta/2)}\big). \end{aligned} \tag{3.15}$$

Similarly the rest terms in (3.10) give the required order.

By (3.8), (3.11) and (3.15) we obtain the estimate

$$\begin{aligned} \big\|\Delta_\tau^{2k+2}(\bar f(t)-T_{n,k+1}(\bar f;t))\big\|_{L_p[x_2,y_2]} &\leqslant C\left\{\frac{1}{n^{k+1}}+\frac{1}{n^{k+\theta/2}}\right\} \\ &\leqslant C\frac{1}{n^{k+\theta/2}}. \end{aligned} \tag{3.16}$$

Combining (3.9), (3.16), Lemma 4 and Lemma 5 and in view of properties of the Steklov means we get

$$\big\|\Delta_\tau^{2k+2}\bar f(t)\big\|_{L_p[x_2,y_2]}\leqslant C\frac{1}{n^{k+\theta/2}}+\tau^{2k+2}\left(n^{k+1}+\frac{1}{\eta^{2k+2}}\right)\omega_{2k+2}(\bar f,\eta,[x_2,y_2])$$

Taking $\tau\leqslant r$

$$\omega_{2k+2}(\bar f,r,[x_2,y_2])=O\big(r^{2k+\theta}\big) \tag{3.17}$$

This implies that $\bar f^{(2k)}$ exists and belongs to $\operatorname{Lip}\theta$. This is reiterated into second term of (3.10) as

$$f(u)=\sum_{i=0}^{2k}\frac{f^{(i)}(t)}{i!}(u-t)^i+\frac{1}{(2k-1)!}\int\limits_t^u(u-w)^{2k-1}\left(f^{(2k-1)}(w)-f^{(2k-1)}(t)\right)dw$$

Thus we get

$$\big\|T_{n,k+1}(f(u)(u-t);t)\big\|_{L_p[x_2',y_2']}\leqslant\frac{C}{n^{k+1/2+\theta/2}}$$

This implies $\omega_{2k+2}(\bar f,r,p,[x_2,y_2])=O(r^{2k+1+\theta})$ which further implies

$$\omega_{2k+2}(f,\tau,p,I_2)=O(\tau^{2k+1+\theta}).$$

Thus the theorem is completed by induction. □

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