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ULAM STABILITY OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. In this paper we present four types of Ulam stability for ordinary differential equations: Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability. Some examples and counterexamples are given.

1. **Introduction**

The basic statements of data dependence in the theory of ordinary differential equations are the following (see for example [2], [5], [6], [8], [17], [20], [23], [24]): monotony w.r.t. data, continuity w.r.t. data, differentiability w.r.t. parameters, Liapunov stability, asymptotic behavior, structural stability, analiticity of solutions, regularity of solutions, G-convergences. On the other hand, in the theory of functional equations, there are some special kind of data dependence (see [9], [10], [4], [7], [3], [18], [19]). There are some results of this type for some differential equations ([8], [11], [12], [14]-[16]) and some integral equations ([13], [21] and [22]).

With these results in mind we shall present, in this paper, four types of Ulam stability for ordinary differential equations: Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability. Some examples and some counterexamples are given.

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2. **General definitions and remarks**

Let $(\mathbb{B}, |\cdot|)$ be a (real or complex) Banach space, $a \in \mathbb{R}$, $b \in \overline{\mathbb{R}}$, $a < b \leq +\infty$, ε a positive real number, $f : [a, b) \times \mathbb{B} \to \mathbb{B}$ be a continuous operator and $\varphi : [a, b) \to \mathbb{R}_+$ be a continuous function. We consider the following differential equation

$$x'(t) = f(t, x(t)),\ \forall\ t \in [a, b) \tag{2.1}$$

and the following differential inequations

$$|y'(t) - f(t, y(t))| \leq \varepsilon,\ \forall\ t \in [a, b) \tag{2.2}$$

$$|y'(t) - f(t, y(t))| \leq \varphi(t),\ \forall\ t \in [a, b) \tag{2.3}$$

and

$$|y'(t) - f(t, y(t))| \leq \varepsilon\varphi(t), \quad t \in [a, b). \tag{2.4}$$

Definition 2.1. The equation (2.1) is Ulam-Hyers stable if there exists a real number $c_f > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in C^1([a, b), \mathbb{B})$ of (2.2) there exists a solution $x \in C^1([a, b), \mathbb{B})$ of (2.1) with

$$|y(t) - x(t)| \leq c_f\varepsilon,\ \forall\ t \in [a, b).$$

Definition 2.2. The equation (2.1) is generalized Ulam-Hyers stable if there exists $\theta_f \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\theta_f(0) = 0$, such that for each solution $y \in C^1([a, b), \mathbb{B})$ of the inequation (2.2) there exists a solution $x \in C^1([a, b), \mathbb{B})$ of the equation (2.1) with

$$|y(t) - x(t)| \leq \theta_f(\varepsilon),\ \forall\ t \in [a, b).$$

Definition 2.3. The equation (2.1) is Ulam-Hyers-Rassias stable with respect to φ if there exists $c_{f,\varphi} > 0$ such that for each $\varepsilon > 0$ and for each solution $y \in C^1([a, b), \mathbb{B})$ of (2.4) there exists a solution $x \in C^1([a, b), \mathbb{B})$ of (2.1) with

$$|y(t) - x(t)| \leq c_{f,\varphi}\varepsilon\varphi(t),\ \forall\ t \in [a, b).$$

Definition 2.4. The equation (2.1) is generalized Ulam-Hyers-Rassias stable with respect to φ if there exists $c_{f,\varphi} > 0$ such that for each solution $y \in C^1([a, b), \mathbb{B})$ of

(2.3) there exists a solution $x \in C^1([a,b),\mathbb{B})$ of (2.1) with

$$|y(t) - x(t)| \leq c_{f,\varphi}\varphi(t),\ \forall\ t \in [a,b).$$

Remark 2.1. A function $y \in C^1([a,b),\mathbb{B})$ is a solution of (2.2) if and only if there exists a function $g \in C([a,b),\mathbb{B})$ (which depend on y) such that

(i) $|g(t)| \leq \varepsilon,\ \forall\ t \in [a,b)$

(ii) $y'(t) = f(t,y(t)) + g(t),\ \forall\ t \in [a,b)$.

We have similar remarks for the inequations (2.3) and (2.4).

So, the Ulam stabilities of the differential equations are some special types of data dependence of the solutions of differential equations.

Remark 2.2. If $y \in C^1([a,b),\mathbb{B})$ is a solution of the inequation (2.2), then y is a solution of the following integral inequation

$$\left|y(t) - y(a) - \int_a^t f(s,y(s))ds\right| \leq (t-a)\varepsilon,\ \forall\ t \in [a,b).$$

Indeed, by Remark 2.1 we have that

$$y'(t) = f(t,y(t)) + g(t), \quad t \in [a,b).$$

This implies that

$$y(t) = y(a) + \int_a^t f(s,y(s))ds + \int_a^t g(s)ds, \quad t \in [a,b).$$

From this it follows that

$$\begin{aligned}\left|y(t) - y(a) - \int_a^t f(s,y(s))ds\right| &\leq \left|\int_a^t g(s)ds\right| \\ &\leq \int_a^t |g(s)|ds \leq \varepsilon(t-a).\end{aligned}$$

We have similar remarks for the solutions of the inequations (2.3) and (2.4).

Remark 2.3. A solution of the inequation (2.2) is called an ε-solution of the equation (2.1) (see for example [2], p. 94-95; [8], p. 14-18; [24], p. 233).

Remark 2.4. The case $b < +\infty$ and the case $b = +\infty$ are two distinct cases as the following example shows.

Example 2.1. We consider in the case $\mathbb{B} := \mathbb{R}$ the equation

$$x'(t) = 0, \quad t \in [a, b) \tag{2.5}$$

and the inequation

$$|y'(t)| \leq \varepsilon, \quad t \in [a, b). \tag{2.6}$$

Let $y \in C^1[a, b)$ be a solution of (2.6). Then there exists $g \in C[a, b]$ such that:

(i) $|g(t)| \leq \varepsilon, \ t \in [a, b)$

(ii) $y'(t) = g(t), \ t \in [a, b)$.

We have, for all $c \in \mathbb{R}$,

$$\begin{aligned} |y(t) - c| &\leq |y(0) - c| + \int_a^t |g(s)| ds \\ &\leq |y(0) - c| + \varepsilon(t - a), \ t \in [a, b). \end{aligned}$$

If we take $c := y(0)$, then

$$|y(t) - y(0)| \leq \varepsilon(t - a), \quad t \in [a, b).$$

If $b < +\infty$, then

$$|y(t) - y(0)| \leq (b - a)\varepsilon.$$

So, the equation (2.5) is Ulam-Hyers stable.

Let $b = +\infty$. The function $y(t) = \varepsilon t$ is a solution of the inequation (2.6) and

$$|y(t) - c| = |\varepsilon t - c| \to +\infty \text{ as } t \to +\infty.$$

So, the equation (2.5) is not Ulam-Hyers stable on the interval $[a, +\infty)$.

Let us consider the inequation

$$|y'(t)| \leq \varphi(t), \quad t \in [a, +\infty). \tag{2.7}$$

Let y be a solution of (2.7) and $x(t) = y(0)$, $t \in [a, +\infty)$ a solution of (2.5). We have that

$$|y(t) - x(t)| = |y(t) - y(0)| \leq \int_a^t \varphi(s) ds, \quad t \in [a, +\infty).$$

If there exists $c_\varphi \in \mathbb{R}_+$ such that

$$\int_a^t \varphi(s)ds \le c_\varphi \varphi(t), \quad t \in [a, +\infty)$$

then the equation (2.5) is generalized Ulam-Hyers-Rassias stable on $[a, +\infty)$ with respect to φ.

Remark 2.5. For the Ulam-Hyers-Rassias stability of the differential equation

$$y' - \lambda y = 0$$

in a Banach space see [16]. For other results see [1], [11], [12], [14] and [15].

3. **Generalized Ulam-Hyers-Rassias stability**

Let us consider the equation (2.1) and the inequation (2.3) in the case $b = \infty$. We suppose that:

(i) $f \in C([a, +\infty) \times \mathbb{B}, \mathbb{B})$ and $\varphi \in C([a, +\infty), \mathbb{R}_+)$ be an increasing function;

(ii) there exists $l_f \in L^1[a, +\infty)$ such that

$$|f(t, u) - f(t, v)| \le l_f(t)|u - v|, \ \forall\ u, v \in \mathbb{B}, \ \forall\ t \in [a, +\infty);$$

(iii) there exists $\lambda_\varphi > 0$ such that

$$\int_a^t \varphi(s)ds \le \lambda_\varphi \varphi(t), \ \forall\ t \in [0, a + \infty).$$

We have

Theorem 3.1. *In the conditions (i), (ii), (iii) the equation (2.1)* ($b = +\infty$) *is generalized Ulam-Hyers-Rassias stable.*

Proof. Let $y \in C^1([a, +\infty), \mathbb{B})$ be a solution of the inequation (2.3) ($b = +\infty$). Denote by x the unique solution of the Cauchy problem

$$x'(t) = f(t, x(t)), \ t \in [a, +\infty)$$
$$x(a) = y(a).$$

We have that

$$x(t) = y(a) + \int_a^t f(s, x(s))ds, \quad t \in [a, +\infty)$$

and

$$\left|y(t) - y(a) - \int_a^t f(s, y(s))ds\right| \leq \int_a^t \varphi(s)ds \leq \lambda_\varphi \varphi(t), \quad t \in [a, +\infty).$$

From these relation it follows

$$\begin{aligned} |y(t) - x(t)| &\leq \left|y(t) - y(a) - \int_a^t f(s, y(s))ds\right| \\ &+ \int_a^t |f(s, y(s)) - f(s, x(s))|ds \\ &\leq \lambda_\varphi \varphi(t) + \int_a^t l_f(s)|y(s) - x(s)|ds. \end{aligned}$$

By a Gronwall lemma (see [22], [23], [5]) we have that

$$\begin{aligned} |y(t) - x(t)| &\leq \lambda_\varphi \varphi(t) e^{\int_a^t l_f(s)ds} \\ &\leq [\lambda_\varphi l^{\int_a^{+\infty} l_f(s)ds}]\varphi(t) = c_{f,\varphi}\varphi(t),\ t \in [a, +\infty), \end{aligned}$$

i.e. the equation (2.1) ($b = +\infty$) is generalized Ulam-Hyers-Rassias stable.

Remark 3.1. For the case $\mathbb{B} := \mathbb{C}$ see [13], [15].

Remark 3.2. If we take $\mathbb{B}$ a Banach space of sequences in $\mathbb{K} = \mathbb{R} \vee \mathbb{C}$ $(C(\mathbb{K}), C_0(\mathbb{K}), l^p(\mathbb{K}), \dots)$ then we have some results for an infinite system of differential equations.

Remark 3.3. For the Ulam stability of some integral equations see [13] and [21].

Remark 3.4. If we have a differential equation of n-order in a Banach space $\mathbb{B}$ then we reduce it to a differential equation of first order in the Banach space $\mathbb{B}^n$. If the order n is even we can use the Green function technique as the following example shows.

For simplicity we shall consider the following second order differential equation

$$-x''(t) = f(t, x(t)), \quad t \in [a, b] \tag{3.1}$$

where $a < b < +\infty$ and $f \in C([a, b] \times \mathbb{R})$.

Let us denote by G the Green function of the following boundary value problem (see [6], [17], [20], [23])

$$-y'' = h(t)$$

$$y(a) = 0,\ y(b) = 0$$

The function $G : [a, b] \times [a, b] \to \mathbb{R}$ is defined by

$$G(t,s) := \begin{cases} \dfrac{(s-a)(b-t)}{b-a} & \text{if} \quad s \leq t, \\ \dfrac{(t-a)(b-s)}{b-a} & \text{if} \quad s \geq t. \end{cases}$$

We have

Theorem 3.2. *We suppose that:*

(i) $f \in C([a, b] \times \mathbb{R})$*;*

(ii) there exists $L_f > 0$ *such that*

$$|f(t, u) - f(t, v)| \leq L_f |u - v|,\ \forall\ t \in [a, b],\ \forall\ u, v \in \mathbb{R};$$

(iii) $L_f \dfrac{(b-a)^2}{4} < 1.$

Then the equation (3.1) is Ulam-Hyers stable.

Proof. Let $y \in C^2[a, b]$ be a solution of the inequation

$$| - y'' - f(t, y(t))| \leq \varepsilon,\ \forall\ t \in [a, b].$$

First of all we remark that y is a solution of the following inequation

$$\left| y(t) - \frac{t-a}{b-a} y(b) - \frac{b-t}{b-a} y(a) - \int_a^b G(t,s) f(s, y(s)) ds \right|$$

$$\leq \varepsilon \left[\frac{t^2}{2} - \frac{a+b}{2} t + \frac{ab}{2} \right], \quad t \in [a, b].$$

Now we take x the solution of the following boundary value problem ([8], p. 186; [20], p. 99)

$$-x''(t) = f(t, x(t)), \quad t \in [a, b],$$

$$x(a) = y(a), \quad x(b) = y(b).$$

It is clear that

$$x(t) = \frac{t-a}{b-a}y(b) + \frac{b-t}{b-a}y(a) + \int_a^b G(t,s)f(s,x(s))ds, \quad t \in [a,b]$$

and we estimate $|y(t) - x(t)|$ in a similar way as in the proof of Theorem 3.1.

References

[1] Alsina, C., and Ger, R., *On some inequalities and stability results related to the exponential function*, J. Inequal. Appl., **2**(1998), 373-380.

[2] Amann, H., *Ordinary Differential Equations*, Walter de Gruyter, Berlin, 1990.

[3] Breckner, W.W., and Trif, T., *Convex Functions and Related Functional Equations*, Cluj Univ. Press, Cluj-Napoca, 2008.

[4] Cădariu, L., *Stabilitatea Ulam-Hyers-Bourgin pentru ecuaţii funcţionale*, Editura Universităţii de Vest, Timişoara, 2007.

[5] Chicone, C., *Ordinary Differential Equations with Applications*, Springer New York, 2006.

[6] Corduneanu, C., *Principles of Differential and Integral Equations*, Chelsea Publ. Company, New York, 1971.

[7] Czerwik, S., *Functional Equations and Inequalities in Several Variables*, World Scientific, 2002.

[8] Hsu, S.-B., *Ordinary Differential Equations with Applications*, World Scientific, New Jersey, 2006.

[9] Hyers, D.H., Isac, G., and Rassias, Th.M., *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.

[10] Jung, S.-M., *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, 2001.

[11] Jung, S.-M., *Hyers-Ulam stability of linear differential equations of first order, III*, J. Math. Anal. Appl., **311**(2005), 139-146.

[12] Jung, S.-M., *Hyers-Ulam stability of first order linear differential equations with constant coefficients*, J. Math. Anal. Appl., **320**(2006), 549-561.

[13] Jung, S.-M., *A fixed point approach to the stability of a Volterra integral equation*, Fixed Point Theory and Applications, Vol. 2007, 9 pages.

[14] Jung, S.-M. and Lee, K.-S., *Hyers-Ulam-Rassias stability of linear differential equations of second order*, J. Comput. Math. Optim., **3**(2007), no. 3, 193-200.

[15] Jung, S.-M. and Rassias, Th.M., *Generalized Hyers-Ulam stability of Riccati differential equation*, Math. Ineq. Appl., **11**(2008), No. 4, 777-782.

[16] Miura, T., Jung, S.-M. and Takahasi, S.-E., *Hyers-Ulam-Rassias stability of the Banach space valued linear differential equation* $y' = \lambda y$, J. Korean Math. Soc., **41**(2004), 995-1005.

[17] Piccinini, L.C., Stampacchia, G. and Vidossich, G., *Ordinary Differential Equations in* $\mathbb{R}^n$, Springer, Berlin, 1984.

[18] Popa, D., *Hyers-Ulam-Rassias stability of a linear recurrence*, J. Math. Anal. Appl., **309**(2005), 591-597.

[19] Radu, V., *The fixed point alternative and the stability of functional equations*, Fixed Point Theory, **4**(2003), No. 1, 91-96.

[20] Rus, I.A., *Ecuaţii diferenţiale, ecuaţii integrale şi sisteme dinamice*, Transilvania Press, Cluj-Napoca, 1996.

[21] Rus, I.A., *Gronwall lemma approach to the Ulam-Hyers-Rassias stability of an integral equation* (to appear).

[22] Rus, I.A., *Gronwall lemmas: ten open problems*, Scientiae Mathematicae Japonicae (to appear).

[23] Ver Eecke, P., *Applications du calcul différentiel*, Presses Univ. de France, Paris, 1985.

[24] Vrabie, I.I., *Co-Semigroups and Applications*, Elsevier, Amsterdam, 2003.

[25] Vrabie, I.I., *Differential Equations*, World Scientific, New Jersey, 2004.

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