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ERROR BOUND FOR THE SOLUTION OF A POLYLOCAL PROBLEM WITH A COMBINED METHOD

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Abstract. Consider the problem:

$$-y''(t) + q(t)y(t) = r(t), \qquad t \in [a, b]$$
$$y(c) = \alpha$$
$$y(d) = \beta, \qquad c, d \in (a, b).$$

The aim of this paper is to give an error bound for the solution of this problem using a collocation with B-spline method combined with a Runge-Kutta method. A numerical example is also given.

1. **Introduction**

Consider the problem:

$$-y''(t) + q(t)y(t) = r(t), \qquad t \in [a, b] \tag{1}$$
$$y(d) = \alpha \tag{2}$$
$$y(e) = \beta, \qquad d, e \in (a, b), d < e. \tag{3}$$

where $q, r \in C[a, b]$, $\alpha, \beta \in \mathbb{R}$. *This is not a two-point boundary value problem* (BVP), since $d, e \in (a, b)$.

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If the solution of the two-point boundary value problem

$$\begin{aligned} -y''(t) + q(t)y(t) &= r(t), \qquad t \in (d, e) \\ y(d) &= \alpha \\ y(e) &= \beta, \end{aligned} \tag{4}$$

exists and it is unique, then the requirement $y \in C^2[a, b]$ assures the existence and the uniqueness of (1)+(2)+(3).

We have two initial value problems on $[a, d]$ and $[e, b]$, respectively, and the existence and the uniqueness for (4) assure existence and uniqueness of these problems. It is possible to solve this problem by dividing it into the three above-mentioned problems and to solve each of these problem separately.

This decomposition strategy allows us to solve the problem using a new combined method (collocation + Runge-Kutta) and to give an error estimation.

2. **Principles of the method**

We decompose our problem into a two-point BVP:

$$-y''(t) + q(t)y(t) = r(t), \qquad t \in (d, e) \tag{5}$$

$$y(d) = \alpha \tag{6}$$

$$y(e) = \beta, \tag{7}$$

and two initial value problems (IVP)

$$-y''(t) + q(t)y(t) = r(t), \qquad t \in [a, d] \tag{8}$$

$$y(d) = \alpha \tag{9}$$

$$y'(d) = \alpha' \tag{10}$$

and

$$-y''(t) + q(t)y(t) = r(t), \qquad t \in [e, b] \tag{11}$$

$$y(e) = \beta \tag{12}$$

$$y'(e) = \beta'. \tag{13}$$

The values of the differential y' at d and e required for the solution of problems (8)+(9)+(10) and (11)+(12)+(13) are approximated during the solution of the problem (5)+(6)+(7).

For the first problem we use a collocation method with nonuniform B-splines of order $k+2$, $k \in \mathbb{N}^*$ [1, 10, 3]. For properties of B-spline and basic algorithms see [5].

Consider the mesh (see [2, 3])

$$\Delta : d = x_1 < x_2 < \cdots < x_N < x_{N+1} = e, \tag{14}$$

and the step sizes

$$h_i := x_{i+1} - x_i, \qquad i = 1, \ldots, N.$$

The multiplicity of e and d is $k+2$ and the inner points have the multiplicity k. Within each subinterval we consider k points

$$\xi_{i,j} := x_i + h_i \rho_j, \qquad j = 1, \ldots, k, \qquad i = 1, \ldots, N,$$

where

$$0 \leq \rho_1 < \rho_2 < \cdots < \rho_k \leq 1,$$

are the roots of the kth Legendre's orthogonal polynomial on $[0, 1]$ [7, 8]. We add the points d and e to the set of collocation points.

We shall choose the basis such that the following conditions hold:

(C1) the solution verifies the differential equation (1) at $\xi_{i,j}$;

(C2) the solution verifies the conditions (2), (3).

We need a basis having $n = Nk + 2$ cubic B-spline functions.

One renumbers the collocation points (ξ_k), such that the first point is d and the last is e.

The form of solution is

$$y_\Delta(t) = \sum_{i=1}^{n} c_i B_i(t), \tag{15}$$

where $B_i(t)$ is the $k+2$ order B-spline with knots $x_i, \ldots, x_{i+k}$.

The conditions (C1) +(C2) yield a linear system $Ac = \gamma$, with n equations and n unknowns (the coefficients c_i, $i = 1, \ldots, n$).

Its matrix is

$$A = [a_{ij}]_{i,j=1,\ldots,n},$$

where

$$a_{ij} = \begin{cases} -B_j''(\xi_i) + q(\xi_i)B_j(\xi_i), & \text{for } i = 2, ..., n-1 \\ B_j(d), & \text{for } i = 1 \\ B_j(e), & \text{for } i = n. \end{cases} \tag{16}$$

The system matrix is banded with at most $k+2$ nonzero elements on each line ($k+2$ nonzero splines at each inner collocation point and only one four at d and e), since a $k+2$ order B-splines is nonzero only on $k+2$ consecutive subintervals. The right-hand side of the system is

$$\gamma = [\alpha, r(\xi_2), \ldots, r(\xi_{n-1}), \beta]^T.$$

The paper [9] gives a Maple implementation based on a different B-spline basis.

For the solution of problems (8)+(9)+(10) and (11)+(12)+(13) we consider a Runge-Kutta method with sufficiently high order. For the left IVP we consider negative steps. The values α' and β' are obtained by differentiating the B-spline solution of the BVP at points d and e, respectively.

3. Main result

Our estimation is inspired from [3, Chapter 5]. If the mesh is sufficiently fine, the condition number of matrix A given by (16) is not too high and the order of

Runge-Kutta method is sufficiently high we can obtain an acceptable upper bound of error.

Theorem 1. *Suppose there exists a $p \geq k \geq 2$ such that*

(a) *The linear problem (5) with boundary conditions (6)+(7) is well-posed, that is, the equivalent problem*

$$\begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ q(x) & 0 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} + \begin{bmatrix} 0 \\ -r(x) \end{bmatrix}$$

has a condition number $\kappa_k = cond(A)$ of moderate size, $q,\ r \in C^p[a,b]$;

(b) *The linear problem (5) with boundary conditions (6)+(7) has a unique solution;*

(c) *The collocation points $\rho_1, \ldots, \rho_k$ satisfy the orthogonality condition*

$$\int_0^1 \Phi(t) \prod_{\ell=1}^{k} (t-\rho_\ell) dt = 0, \qquad \Phi \in \mathbb{P}_{p-k},\ (p \leq 2k),$$

where $\mathbb{P}_{p-k}$ is a set of polynomials at most degree $p-k$.

Then, for $h = \max_{i=1,\ldots,N} h_i$ sufficiently small, our method (collocation+two Runge-Kutta) is stable with constant $\kappa_k N$ and leads to a unique solution $y_\Delta(x)$. Furthermore, at mesh points x_i it holds

$$\left| y^{(j)}(x_i) - y_\Delta^{(j)}(x_i) \right| = O(h^p), \qquad j = 0,1;\ i = 1, ..., N+1, \tag{17}$$

while, on the other hands, for $i = 1, \ldots, N$, $x \in [x_i, x_{i+1}]$

$$\left| y^{(j)}(x) - y_\Delta^{(j)}(x) \right| = O(h_i^{k+2-j}) + O(h^p), \qquad j = 0, \ldots, k+1. \tag{18}$$

Remark 2. *The condition* (c) *means that (ρ_ℓ) are the roots of kth Legendre polynomial.*

Proof. Using a result from [3, Theorem 5.140, page 253] we obtain the estimations (17)+(18) for the BVP (5)+(6)+(7). The error obtained by approximating α' and β' with $y_\Delta'(d)$ and $y_\Delta'(e)$ is $O(h^p)$, then, we use [7, Theorem 5.4.1, page 293]. If we choose an embedded pair of Runge-Kutta method of order at least $(p, p+1)$, the conditions in the hypothesis of theorem are fulfilled and the final error is $O(h^p)$. So, if the mesh is sufficiently fine, the embedded pair of Runge-Kutta methods does not increase the order of error. □

Remark 3. *The condition number may grow rapidly when h is small. The paper* [2, page 129] *gives the following estimation*

$$\kappa_\Delta \approx K \sum_{i=1}^{N} h_i^{-2} \max_{j=1,\ldots,N+1} \int_{x_i}^{x_{i+1}} |G(x_j, t)|\, dt,$$

where K is a generic constant and G is the Green's function for the BVP problem.

4. **Numerical examples**

Our implementation is based on ideas from [5, 4]. We implement the method in MATLAB[1], using the Spline Toolbox™ 3 [6]. If $d = a$ and $e = b$, our problem becomes a classical BVP. If $d = a$ or $e = b$, our problem is decomposed into a BVP and one IVP. As a numerical example, we chose a problem with oscillatory solution:

$$-y''(x) - 243y(x) = x, \qquad x \in [0, 1] \tag{19}$$

$$y\left(\frac{1}{4}\right) = \frac{1}{243}\frac{\sin\left(\frac{9\sqrt{3}}{4}\right)}{\sin 9\sqrt{3}} - \frac{1}{972}$$

$$y\left(\frac{3}{4}\right) = \frac{1}{243}\frac{\sin\left(\frac{27\sqrt{3}}{4}\right)}{\sin 9\sqrt{3}} - \frac{1}{324}.$$

If we chose $k = 3$, the order of spline will be 5, and $p = 4$. For the initial value problems we choose the solver `ode45` (order 4). The exact solution is

$$y(x) = \frac{1}{243}\frac{\sin 9\sqrt{3}x}{\sin 9\sqrt{3}} - \frac{1}{243}x.$$

We plot the exact solution and approximate solution and the error in a semi-logarithmic scale for $n = 2$ and $k = 3$ in Figures 1 and 2, respectively.

[1]MATLAB is a trademark of the MathWorks, Inc.

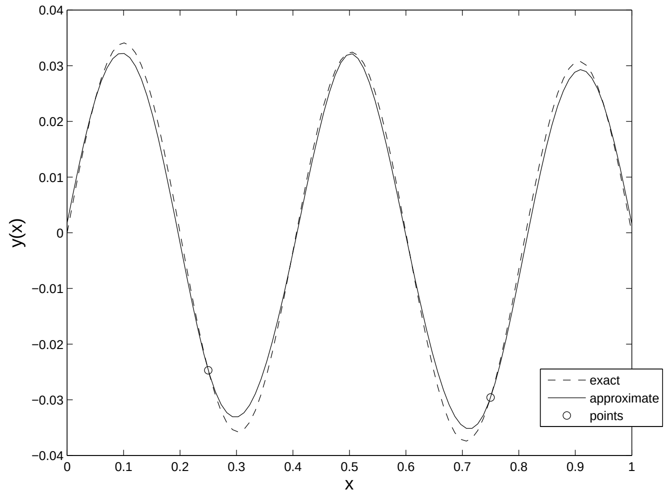

Figure 1. Exact and approximate solution of (19)

The collocation matrix for the BVP is

$$\begin{pmatrix} 1.000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -301.782 & 187.397 & -91.380 & -35.996 & -1.239 & 0 & 0 & 0 \\ -63.187 & -60.750 & 4.875 & -92.343 & -31.593 & 0 & 0 & 0 \\ -2.477 & -34.757 & -91.380 & 36.506 & -150.891 & 0 & 0 & 0 \\ 0 & 0 & 0 & -150.891 & 36.506 & -91.380 & -34.757 & -2.4779 \\ 0 & 0 & 0 & -31.593 & -92.343 & 4.875 & -60.750 & -63.1875 \\ 0 & 0 & 0 & -1.239 & -35.996 & -91.380 & 187.397 & -301.782 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.000 \end{pmatrix}$$

and its condition number is κ_Δ =2.5422e+003.

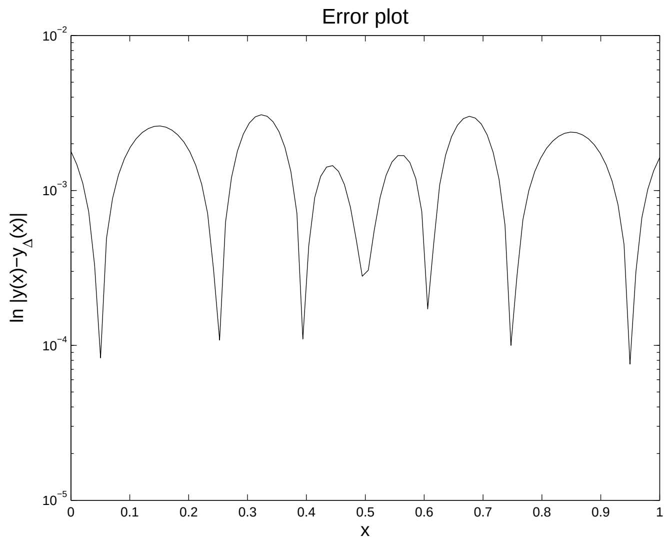

FIGURE 2. Error plot for Example (19)

5. Conclusions

The error estimation does not depend on the number of collocation points. Nevertheless, the Runge-Kutta method requires an order greater or equal to the order of error for the derivatives at d and e. We can conclude collocation combined with Runge-Kutta is an effective method for polylocal problem.

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References

[1] Ascher, U., Christiansen, J., Russell, R.D., *A Collocation Solver for Mixed Order Systems of Boundary Value Problems*, Mathematics of Computation **33**(1979), no. 146, 659-679.

[2] Ascher, U., Pruess, S., Russell, R.D., *On spline basis selections for solving differential equations*, SIAM J. Numer. Anal., **20**(1983), no. 1, 121-142.

[3] Ascher, U.M., Mattheij, R.M.M., Russel, R.D., *Numerical Solution of Boundary Value Problems for Ordinary Differential Equations*, SIAM, 1997.

[4] de Boor, C., *Package for calculating with B-splines*, SIAM J. Numer. Anal., **14**(1977), no. 3, 441-472.

[5] de Boor, C., *A Practical Guide to Splines*, Springer Verlag, Berlin, Heidelberg, New York, 1978.

[6] de Boor, C., *Spline Toolbox 3*, MathWorks Inc., Nattick, MA, 2008.

[7] Gautschi, W., *Numerical Analysis, An Introduction*, Birkhäuser, Basel, 1997.

[8] Lupaş, Al., *Numerical Methods*, Constant, Sibiu, 2000, (Romanian).

[9] Pop, D.N., Trîmbiţas, R., *New Trends in Approximation, Optimization and Classification*, ch. Solution of a polylocal problem - a Computer Algebra based approach, Lucian Blaga University Press, Sibiu, 2008, Proceedings of International Workshop New Trends in Approximation, Optimization and Classification, (D. Simian ed.), pp. 82-93.

[10] Schultz, M.H., *Spline Analysis*, Prentice Hall, Englewood Cliffs, N.J., 1972.

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