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FUNCTIONALS ON NORMED SEQUENCE SPACES AND UNIFORM EXPONENTIAL INSTABILITY OF EVOLUTION OPERATORS

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Abstract. In this paper, we present necessary and sufficient conditions for uniform exponential instability of evolution operators in Banach spaces. Variants for uniform exponential instability of some well-known results due to Datko, Neerven and Zabczyk are given. As consequences, some results proved in [6] are obtained.

1. **Introduction**

One of the most remarkable result in stability theory of evolution operators in Banach spaces has been obtained by Datko in [3]. An important generalization of Datko's result was proved by Rolewicz in [12]. A new and interesting idea has been presented by Neerven in [9], where an unified treatment of the preceding results is given and the exponential stability of C_0-semigroups has been characterized in terms of functionals on Banach function spaces. Some generalizations of these results for the case of linear evolution operators have been presented in [1], [5] and [6].
In this paper, we shall present characterizations for exponential instability of linear evolution operators in the spirit of Neerven's approach. Thus we obtain the versions of the theorems due to Datko, Zabczyck and Neerven for the case of exponential instability. As consequences, we obtain some results presented in [6].

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2. **Linear evolution operators**

Let X be a Banach space. The norm on X and on the space $\mathcal{B}(X)$ of all bounded linear operators from X into itself will be denoted by $\|.\|$.
Let T be the set defined by

$$T = \{(t, t_0) \in \mathbb{R}_+^2 : t_0 \le t\}.$$

Definition 2.1. An application $E : T \to \mathcal{B}(X)$ is called *evolution operator* on X, if it satisfies the following conditions :

(i) $E(t, t) = I$ (the identity operator on X);
(ii) $E(t, s)E(s, t_0) = E(t, t_0)$, for all $(t, s) \in T$ and $(s, t_0) \in T$;
(iii) there exist $M, \omega > 0$ such that

$$\|E(t, t_0)\| \le Me^{\omega(t-t_0)}, \quad \text{for all} \quad (t, t_0) \in T.$$

Particular classes of evolution operators are given by:

Definition 2.2. An evolution operator E is said to be

(i) *strongly measurable*, if for every $(t_0, x) \in \mathbb{R}_+ \times X$ the mapping $E(\cdot, t_0)x$ is measurable;
(ii) *injective*, if for every $(t, t_0) \in \mathbb{R}_+^2$ the linear operator $E(t, t_0)$ is injective.
(iii) *uniformly exponentially instable*, if there are $N, \nu > 0$ such that

$$\|E(t, t_0)x\| \ge Ne^{\nu(t-t_0)}\|x\|, \quad \text{for all} \quad (t, t_0, x) \in T \times X.$$

A characterization of the exponential instability property is given by:

Proposition 2.1. *An evolution operator E is uniformly exponentially instable if and only if there exists a nondecreasing sequence* $f : \mathbb{N} \to \mathbb{R}_+^* = (0, \infty)$ *with* $\lim\limits_{n\to\infty} f(n) = \infty$ *and*

$$\|E(n + t_0, t_0)x\| \ge f(n)\|x\|, \quad \textit{for all} \quad m, n \in \mathbb{N} \quad \textit{and} \quad x \in X.$$

Proof. *Necessity* is trivial.
Sufficiency. Let $(t, t_0) \in T$ and $n \in \mathbb{N}$ such that $n \le t - t_0 < n + 1$. Then for every

$x \in X$ with $\|x\| = 1$, we have that

$$\begin{aligned}\|E(t,t_0)x\| &= \|E(t,t_0+n)E(t_0+n,t_0)x\| \geq f(t-t_0-n)\|E(t_0+n,t_0)x\| \\ &\geq f(0)f(1)\|E(t_0+(n-1),t_0)x\| \geq \cdots \geq f(0)f(1)^n\|x\| \\ &= f(0)e^{\nu n}\|x\| \geq Ne^{\nu(n+1)}\|x\| \geq Ne^{\nu(t-t_0)}\|x\|,\end{aligned}$$

where $\nu = lnf(1) > 0$ and $N = f(0)/f(1) > 0$, which shows that E is uniformly exponentially instable. □

3. Main results

Let $\mathcal{S}(\mathbb{R})$ the set of all real sequences. By $\mathcal{S}^+(\mathbb{R})$ we denote the set of all $s \in \mathcal{S}(\mathbb{R})$ with $s(n) \geq 0$, for all $n \in \mathbb{N}$.

Let $\mathcal{F}$ be the set of all functions $F : \mathcal{S}^+(\mathbb{R}) \to [0,\infty]$ with the properties:

(f_1) if s_1, $s_2 \in \mathcal{S}^+(\mathbb{R})$ with $s_1 \leq s_2$ then $F(s_1) \leq F(s_2)$;

(f_2) there exists $\alpha > 0$ such that $F(c\chi_{\{n\}}) \geq \alpha c$, for all $c > 0$ and $n \in \mathbb{N}$;

(f_3) there exists $f \in \mathcal{S}^+(\mathbb{R}_+)$ with $\lim\limits_{n\to\infty} f(n) = \infty$ such that $F(c\chi_{\{0,\ldots,n\}}) \geq f(n)$, for all $c > 0$ and $n \in \mathbb{N}$.

Here χ_A denotes the characteristic function of the set A.

For every injective evolution operator E and every $x \in X$ with $\|x\| = 1$, we associate the following sequences:

$$e_x^{t_0}(n) = \frac{1}{\|E(n+t_0,t_0)x\|}, \quad e_x^{m,t_0}(n) = e_x^{t_0}(m+n), \quad v_x^{m,t_0}(n) = \frac{e_x^{m,t_0}(n)}{e_x^{t_0}(m)},$$

for all $m, n \in \mathbb{N}$.

Remark 3.1. If the evolution operator E is uniformly exponentially instable then there exists $F \in \mathcal{F}$ with the properties:

(i) $\sup\limits_{\substack{\|x\|=1\\ t_0\geq 0}} F(e_x^{t_0}) < \infty$;

(ii) there exists $N > 0$ such that $F(e_x^{m,t_0}) \leq Ne_x^{t_0}(m)$, for all $m \in \mathbb{N}$, $t_0 \geq 0$ and $x \in X$ with $\|x\| = 1$;

(iii) $\sup\limits_{\substack{\|x\|=1\\ (m,t_0)\in\mathbb{N}\times\mathbb{R}_+}} F(e_x^{m,t_0}) < \infty$;

(iv) $\sup\limits_{\substack{\|x\|=1 \\ (m,t_0)\in\mathbb{N}\times\mathbb{R}_+}} F(v_x^{m,t_0}) < \infty.$

Indeed, if we consider the function $F : \mathcal{S}^+(\mathbb{R}) \to [0,\infty]$ defined by

$$F(s) = \sum_{n=0}^{\infty} s(n)$$

then it is easy to verify that the uniformly exponentially instability property of E implies the conditions (i), (ii), (iii) and (iv).

The main result of this paper is :

Proposition 3.1. *An injective evolution operator E is uniformly exponentially instable if and only if there is $F \in \mathcal{F}$ such that*

$$\sup_{\substack{\|x\|=1 \\ t_0\geq 0}} F(e_x^{t_0}) < \infty.$$

Proof. *Necessity.* It results from Remark 3.1.

Sufficiency. We observe that

$$e_x^{t_0} = \sum_{k=0}^{\infty} e_x^{t_0}(k)\chi_{\{k\}} \geq \sum_{k=0}^{n} e_x^{t_0}(k)\chi_{\{k\}} \geq e_x^{t_0}(n)\chi_{\{n\}}.$$

Let $M = \sup\limits_{\substack{\|x\|=1 \\ t_0\geq 0}} F(e_x^{t_0}) < \infty$. Using the hypothesis, we obtain that

$$M \geq F(e_x^{t_0}) \geq F(e_x^{t_0}(n)\chi_{\{n\}}) \geq \alpha \cdot e_x^{t_0}(n),$$

and hence $e_x^{t_0}(n) \leq M/\alpha$, for all $t_0 \geq 0, n \in \mathbb{N}$ and $x \in X$ with $\|x\| = 1$.

The last inequality becomes

$$\|E(n+t_0,t_0)x\| \geq \frac{\alpha}{M},$$

for all $n \in \mathbb{N}, t_0 \geq 0$ and all $\|x\| = 1$.

This implies that

$$\|E(n+t_0,k+t_0)E(k+t_0,t_0)x\| \geq \frac{\alpha}{M}\|x\|, \quad \text{for all} \quad x \in X \quad \text{with} \quad \|x\| = 1,$$

and hence

$$\|E(n+t_0,t_0)x\| \geq \frac{\alpha}{M}\|E(k+t_0,t_0)x\|, \quad \text{for all} \quad k,n \in \mathbb{N}, \quad k \leq n, \quad t_0 \geq 0$$

and $x \in X$ with $\|x\| = 1$.

We have that

$$e_x^{t_0} = \sum_{k=0}^{\infty} e_x^{t_0}(k)\chi_{\{k\}} \geq \sum_{k=0}^{n} e_x^{t_0}(k)\chi_{\{k\}} \geq \frac{\alpha}{M} e_x^{t_0}(n)\chi_{\{0,\ldots,n\}}.$$

It follows that

$$M \geq F(e_x^{t_0}) \geq \frac{\alpha}{M} e_x^{t_0}(n) f(n)$$

and hence

$$\|E(n+t_0,t_0)x\| \geq \frac{\alpha}{M^2} f(n), \quad \text{for all} \quad x \in X \quad \text{with} \quad \|x\| = 1.$$

By Proposition 2.1 it results that E is uniformly exponentially instable. □

Corollary 3.1. *An injective evolution operator E is uniformly exponentially instable if and only if there exist $N > 0$ and $F \in \mathcal{F}$ such that*

$$F(e_x^{m,t_0}) \leq N e_x^{t_0}(m),$$

for all $m \in \mathbb{N}$, $t_0 \geq 0$ and $x \in X$ with $\|x\| = 1$.

Proof. *Necessity.* It results from Remark 3.1.

Sufficiency. We observe that

$$\sup_{\substack{\|x\|=1 \\ t_0 \geq 0}} F(e_x^{t_0}) = \sup_{\substack{\|x\|=1 \\ t_0 \geq 0}} F(e_x^{0,t_0}) \leq N e_x^{t_0}(0) = N < \infty$$

and by Proposition 3.1 it follows that E is uniformly exponentially instable. □

Corollary 3.2. *An injective evolution operator E is uniformly exponentially instable if and only if there is $F \in \mathcal{F}$ such that*

$$\sup_{\substack{\|x\|=1 \\ (m,t_0) \in \mathbb{N} \times \mathbb{R}_+}} F(e_x^{m,t_0}) < \infty.$$

Proof. *Necessity.* It results from Remark 3.1.

Sufficiency. It results from Proposition 3.1 taking into account that

$$\sup_{\substack{\|x\|=1 \\ t_0 \geq 0}} F(e_x^{t_0}) = \sup_{\substack{\|x\|=1 \\ t_0 \geq 0}} F(e_x^{0,t_0}) \leq \sup_{\substack{\|x\|=1 \\ (m,t_0) \in \mathbb{N} \times \mathbb{R}_+}} F(e_x^{m,t_0}) < \infty.$$

□

Similarly, we obtain:

Corollary 3.3. *An injective evolution operator E is uniformly exponentially instable if and only if there is $F \in \mathcal{F}$ such that*

$$\sup_{\substack{\|x\|=1 \\ (m,t_0)\in\mathbb{N}\times\mathbb{R}_+}} F(v_x^{m,t_0}) < \infty.$$

Remark 3.2. The preceding results are discrete versions of a Neerven's theorem ([9]) for the case of instability property.

We shall denote by Φ the set of all nondecreasing functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(0) = 0$ and $\varphi(t) > 0$, for every $t > 0$.

Corollary 3.4. *An injective evolution operator E is uniformly exponentially instable if and only if there exists $\varphi \in \Phi$ such that*

$$\sup_{\substack{\|x\|=1 \\ t_0\geq 0}} \sum_{n=0}^{\infty} \varphi(e_x^{t_0}(n)) < \infty.$$

Proof. *Necessity.* It is trivial for $\varphi(t) = t$.

Sufficiency. It results from Proposition 3.1 for

$$F(s) = \sum_{n=0}^{\infty} \varphi(s(n)).$$

□

Remark 3.3. The preceding corollary extends a Zabczyk's theorem ([13]) for the case of exponential instability.

For the particular case $\varphi(t) = t^p$, we obtain:

Corollary 3.5. *An injective evolution operator E is uniformly exponentially instable if and only if there exists $p \in [1, \infty)$ such that*

$$\sup_{\substack{\|x\|=1 \\ t_0\geq 0}} \sum_{n=0}^{\infty} [e_x^{t_0}(n)]^p < \infty.$$

Remark 3.4. Corollary 3.5 is a discrete version of Datko's theorem ([3]) for the case of exponential instability. It can be also considered as a variant for the exponential instability of a theorem proved by Przyluski and Rolewicz in [11] for the case of exponential stability.

Let $\mathcal{B}(\mathbb{N})$ be the set of all normed sequence spaces B ([8]) with the properties:

(i) $\chi_{\{0,\dots,n\}} \in B$, for all $n \in \mathbb{N}$;

(ii) $\lim\limits_{n\to\infty} |\chi_{\{0,\dots,n\}}|_B = \infty$;

(iii) there exists $\alpha > 0$ such that $|\chi_{\{n\}}|_B \geq \alpha$, for all $n \in \mathbb{N}$.

Corollary 3.6. *An injective evolution operator E is uniformly exponentially instable if and only if there exists a normed sequence space $B \in \mathcal{B}(\mathbb{N})$ such that for every $x \in X$ with $\|x\| = 1$, we have that $e_x^{t_0} \in B$ and*

$$\sup_{\substack{\|x\|=1\\ t_0\geq 0}} |e_x^{t_0}|_B < \infty.$$

Proof. *Necessity.* It is immediate for $B = l^1$.

Sufficiency. Let $F : \mathcal{S}^+(\mathbb{R}) \to [0, \infty]$ be the function defined by

$$F(s) = \sup_{n\in\mathbb{N}} |s \cdot \chi_{\{0,\dots,n\}}|_B.$$

It is easy to see that $F \in \mathcal{F}$ and

$$e_x^{t_0}\chi_{\{0,\dots,n\}} \leq e_x^{t_0}, \quad \text{for all} \quad n \in \mathbb{N}, \quad t_0 \geq 0 \quad \text{and} \quad x \in X \quad \text{with} \quad \|x\| = 1.$$

Then

$$\sup_{\substack{\|x\|=1\\ t_0\geq 0}} F(e_x^{t_0}) \leq \sup_{\substack{\|x\|=1\\ t_0\geq 0}} |e_x^{t_0}|_B < \infty.$$

By Proposition 3.1 it results that E is uniformly exponentially instable. □

Remark 3.5. The Corollary 3.6 is a discrete variant for exponential instability of Theorem 3.1.5 from [8].

As a particular case, for the Banach sequence space

$$B = \{s \in \mathcal{S}^+(\mathbb{R}) : \beta s \in l^p\}$$

we obtain:

Corollary 3.7. *An injective evolution operator E is uniformly exponentially instable if and only if there are $p \in [1, \infty)$ and $\beta \in \mathcal{S}^+(\mathbb{R})$ with $\beta > 0$ and $\sum_{n=0}^{\infty} \beta(n) = \infty$ such that*

$$\sup_{\substack{\|x\|=1 \\ t_0 \geq 0}} \sum_{n=0}^{\infty} \beta^p(n)[e_x^{t_0}(n)]^p < \infty.$$

Remark 3.6. The preceding corollary is an extension of Corollary 3.1.6. from [8].

References

[1] Buşe, C., Dragomir, S.S., *A theorem of Rolewicz type in solid function spaces*, Glasgow Math. J., **44**(2002), 125-135.

[2] Chicone, C, Latushkin, Y., *Evolution Semigroups in Dynamical Systems and Differential Equations*, Math. Surveys Monogr., **70**(1999).

[3] Datko, R., *Extending a theorem of Liapunov to Hilbert spaces*, J. Math. Anal. Appl., **32**(1970), 610-616.

[4] Datko, R., *Uniform asimptotic stability of evolutionary processes in Banach spaces*, SIAM J. Math. Anal., **3**(1972), 428-445.

[5] Megan, M., Sasu, B., Sasu, A.L., *On uniform exponential stability of evolution families,* Rev. Mat. Univ. Parma, **4**(2001), 27-43.

[6] Megan, M., Sasu, B., Sasu, A.L., *Banach functions spaces and exponential instability of evolution families*, Archivum Mathematicum (Brno), **39**(2003), 277-286.

[7] Van Neerven, J.M.A.M., *Exponential stability of operators and operator semigroups*, Journal Funct. Anal., **130**(1995), 293-309.

[8] Van Neerven, J.M.A.M., *The Asymptotic Behaviour of Semigroups of Linear Operators*, Theory, Advances and Applications, **88**(1996), Birkhäuser.

[9] Van Neerven, J.M.A.M., *Lower semicontinuity and the theorem of Datko and Pazy*, Integral Equation and Operatory Theory, **42**(2002), 482-492.

[10] Pazy, A., *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag 1983.

[11] Przyluski, K.M., Rolewicz, S., *On stability of linear time-varying infinite dimensional discrete systems*, Systems and Control Lett., **4**(1984), 307-315.

[12] Rolewicz, S., *On uniform N-equstability*, J. Math. Anal. Appl., **115**(1986), 434-441.

[13] Zabczyk, J., *Remarks on the control of discrete time distributed parameter systems*, SIAM J. Control, **12**(1974), 721-735.

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