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NEW ESTIMATE FOR THE NUMERICAL RADIUS OF A GIVEN MATRIX AND BOUNDS FOR THE ZEROS OF POLYNOMIALS

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Abstract. In this paper we find new estimate for the numerical radius of a given matrix, and we prove that, this estimate is better than any estimate for the numerical radius. We present also new bounds for the zero of polynomials by using new estimate for the numerical radius of a companion matrix of a given polynomial and matrix inequalities.

1. **Introduction**

Numerical radii estimate of companion matrices have been invoked by Linden (1999) [7] and Kittaneh [6]. Also, Kittaneh (2003) found that

$$w(A) \leq \frac{1}{2}(\|A\| + \|A^2\|^{\frac{1}{2}}).$$

Also, we know that

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|.$$

In this paper, we find that

$$w(A) \leq \|A^2\|^{\frac{1}{2}} \leq \frac{1}{2}\left(\|A\| + \|A^2\|^{\frac{1}{2}}\right) \leq \|A\|,$$

whenever, A^2does not converge to the zero matrix.

Also, if $A^2 = [0]_{n\times n}$, then $w(A) = \frac{1}{2}\|A\|$, and from the new estimate and matrix inequalities we find new bounds for the zeros of polynomials.

In this work, let $M_n(\mathbb{C})$ denote the algebra of all $n \times n$ complex matrices.

Definition 1.1 *If* $A \in M_n(\mathbb{C})$, *then*

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(i) The spectral norm (or the operator norm) is defined by

$$\|A\| = \max\left\{\|Ax\| : \|x\| = 1\right\} = \max\left\{\frac{\|Ax\|}{\|x\|} : \|x\| \neq 0\right\}.$$

(ii) *The numerical radius of A is defined by*

$$w(A) = \max\left\{|(Ax, x)| : x \in \mathbb{C}^n, \|x\| = 1\right\}.$$

(iii) *The spectral radius of A is defined by*

$$r(A) = \max\left\{|\lambda| : \lambda \text{ is an eigenvalue of } A\right\}.$$

Now, we list some known results as a background and reminder for the reader.

Theorem 1.1

(i) If $A \in M_n(\mathbb{C})$, then $\frac{1}{2}|A\| \leq w(A) \leq \|A\|$ (see e.g.[2]).

(ii) If $A \in M_n(\mathbb{C})$, then $w(A) \leq \frac{1}{2}(\|A\| + \|A^2\|^{\frac{1}{2}})$ (see e.g.[6]).

(iii) If $A \in M_n(\mathbb{C})$, then there exists a unitary matrix $U \in M_n(\mathbb{C})$ such that $A = U|A|$, where $|A| = (A^*A)^{\frac{1}{2}}$.

(iv) Let $A = [a_{ij}] \in M_n(\mathbb{C})$ be written in partitioned form as

$$A = \begin{bmatrix} \widetilde{A} & x \\ x^* & a_{nn} \end{bmatrix},$$

where $x \in \mathbb{C}^{n-1}$ and $\widetilde{A} \in M_{n-1}(\mathbb{C})$, then

$$det(A) = a_{nn} det(\widetilde{A}) - x^*(adj\widetilde{A})x,$$

where $adj\widetilde{A}$ is the classical adjoint of $\widetilde{A}$.(see e.g[4]).

(v) Let $A = [a_{ij}] \in M_n(\mathbb{C})$ be partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where A_{ij} is an $n_i \times n_j$ matrix for $i, j = 1, 2$, with $n_1 + n_2 = n$.

If

$$\widetilde{A} = \begin{bmatrix} \|A_{11}\| & \|A_{12}\| \\ \|A_{21}\| & \|A_{22}\| \end{bmatrix},$$

then

$$\|A\| \leq \|\widetilde{A}\|, r(A) \leq r(\widetilde{A}), \textit{and } w(A) \leq w(\widetilde{A}).$$

2. **Main results**

In the following theorem, we find a new estimate for the numerical radius of a given matrix.

Theorem 2.1 *Let* $A \in M_n(\mathbb{C})$*, then*

(i) $w(A) \leq \|A^2\|^{\frac{1}{2}}$,*if* A^2does not converge to the zero matrix.

(ii) $w(A) = \frac{1}{2}\|A\|$,*if* A^2*is the zero matrix*

Proof. (i) Let $A = u\,|A|$, for some unitary matrix $u \in M_n(\mathbb{C})$.

Now,

$$w(A) = \max\left\{|(Ax, x)| : x \in \mathbb{C}^n, \|x\| = 1\right\}.$$

So

$$\begin{aligned}
(Ax, x) &= (u\,|A|\,x, x) = (|A|\,x, u^*x) \\
&\leq (|A|\,x, x)^{\frac{1}{2}} \left(|A|\,u^*x, u^*x\right)^{\frac{1}{2}} \\
&\leq (|A|\,x, x)^{\frac{1}{2}} \left(u\,|A|\,u^*x, x\right)^{\frac{1}{2}} \\
&\leq (|A|\,x, x)^{\frac{1}{2}} \left(|A^*|\,x, x\right)^{\frac{1}{2}} \\
&\leq \left(x^*\,|A|\,xx^*\,|A^*|\,x\right)^{\frac{1}{2}} \\
&\leq \left(x^*\,|A|\,|A^*|\,x\right)^{\frac{1}{2}} \\
&\leq \left(|A|\,|A^*|\,x, x\right)^{\frac{1}{2}} \\
&\leq \||A|\,|A^*|\|^{\frac{1}{2}},
\end{aligned}$$

since

$$S_1^2\left(|A|\,|A^*|\right) = S_1^2\left(A^2\right),$$

where $S_1^2\left(A^2\right)$ denotes the largest singular value of A^2 and

$$S_1\left(A^2\right) = \|A^2\|.$$

So we get

$$\||A|\,|A^*|\|^2 = \|A^2\|^2 \text{ and } \||A|\,|A^*|\| = \|A^2\|,$$

hence

$$w(A) \leq \|A^2\|^{\frac{1}{2}}. \qquad \square$$

(ii) We know that

$$\frac{1}{2}\|A\| \leq w(A), \tag{1}$$

since $A^2 = [0]_{n\times n}$, so

$$w(A) \leq \frac{1}{2}(\|A\| + \|A^2\|^{\frac{1}{2}}) \leq \frac{1}{2}\|A\|. \tag{2}$$

From (1) and (2) we get the result. $\square$

In Theorem 2.1, the estimate of the numerical radius is a uniform estimate, because

$$w(A) \leq \|A^2\|^{\frac{1}{2}} \leq \|A\|,$$

also, since

$$\|A^2\|^{\frac{1}{2}} \leq \|A\|,$$

we have

$$w(A) \leq \|A^2\|^{\frac{1}{2}} \leq \frac{1}{2}\left(\|A\| + \|A^2\|^{\frac{1}{2}}\right).$$

Corollary 2.1 *Since $r(A) \leq w(A)$, we get that,*

$$r(A) \leq \|A^2\|^{\frac{1}{2}}, \text{if } A^2 \textit{does not converge to the zero matrix.}$$

Also,

$$r(A) \leq \frac{1}{2}\|A\|, \text{if } A^2 \text{is the zero matrix.}$$

Corollary 2.2 *If $A \in M_n(\mathbb{C})$ is a normal matrix, then*

$$r(A) = w(A) = \|A^2\|^{\frac{1}{2}} = \|A\|.$$

3. New bounds for the zeros of polynomials

In this section, we find new bounds for the zeros of the monic polynomials

$$p(z) = z^n + a_n z^{n-1} + a_{n-1} z^{n-2} + \ldots + a_z z + a_1 \tag{3}$$

with complex coefficients $a_1, a_2, a_3, ..., a_n$ where, $a_1 \neq 0$ and $n \geq 3$ by using numerical radius, and matrix inequalities of the companion matrix

$$C(p) = \begin{bmatrix} -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{4}$$

In [6] Kittaneh found $\|C^2(p)\|$,where

$$C^2(p) = \begin{bmatrix} a_n^2 - a_{n-1} & a_n a_{n-1} - a_{n-2} & a_n a_{n-2} - a_{n-3} & \cdots & a_n a_1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots 1 & 0 \end{bmatrix}. \tag{5}$$

And hence by using Theorem 2.1, we find new bounds for zeros of polynomials as in the following theorem.

Theorem 3.1 *If z is a zero of $p(z)$ as in (3), then*

$$|z| \leq \sqrt[4]{\frac{(\delta+1)\left[(\delta+1)^2 + 4\delta'\right]^{\frac{1}{2}}}{2}},$$

where

$$\delta = \frac{1}{2}\left[(\alpha+\beta) + \left((\alpha+\beta)^2 + 4|\gamma|^2\right)^{\frac{1}{2}}\right],$$

and

$$\alpha = \sum_{j=1}^{n} |a_j|^2, \beta = \sum_{j=1}^{n} |L_j|^2,$$

$$\gamma = -\sum_{j=1}^{n} L_j \overline{a_j}, \;\; L_j = a_n a_j - a_{j-1}$$

for $j = 1, 2, \ldots, n$ *with* $a_0 = 0$,*and*

$$\delta' = \frac{1}{2}\left[(\alpha' + \beta') + \sqrt{(\alpha' + \beta')^2 + 4|\gamma'|^2}\right],$$

where

$$\alpha' = \sum_{j=3}^{n} |a_j|^2, \beta' = \sum_{j=3}^{n} |L_j|^2 \text{ and } \gamma' = -\sum_{j=3}^{n} L_j a_j.$$

Proof. Let $C(p)$ be the companion matrix of $p(z)$.Since $C^2(p) \neq [0]_{n\times n}$,we have

$$wC(p) \leq \left\|C^2(p)\right\|^{\frac{1}{2}}.$$

Kittaneh found $\left\|C^2(p)\right\|$ in [6], and hence we get the result. □

Now since

$$\left\|C^2(p)\right\|^{\frac{1}{2}} \leq \frac{1}{2}\left(\|C(p)\| + \left\|C^2(p)\right\|^{\frac{1}{2}}\right).$$

Therefore, the bound in Theorem 3.1, is better than Kittaneh bound in [6].

Kittaneh found new bound for the zeros of a polynomial $p(z)$ by using matrix inequality as in the following theorem.

Theorem 3.2 *(see[5]) If z is a zero of $p(z)$ as in (3), then*

$$|z| \leq \frac{1}{2}\left[(1 + |a_n|) + \sqrt{(1 + |a_n|)^2 + 4\left(\sum_{j=1}^{n-1} |a_j|^2\right)^{\frac{1}{2}}}\right].$$

In the following theorem, we find new bounds for the zeros of $p(z)$ by using matrix inequalities.

Theorem 3.3. *If z is any zero of $p(z)$ as in (3), then*

$$|z| \leq \frac{1}{2}\left[\beta + \sqrt{\beta^2 + 4|a_1|}\right],$$

where

$$\beta = \left[\frac{\left(1 + \sum_{j=2}^{n} |a_j|^2\right) + \sqrt{\left(\sum_{j=2}^{n} |a_j|^2 - 1\right)^2 + 4|a_2|^2}}{2}\right]^{\frac{1}{2}}.$$

Proof. Let $C(p)$ be the companion matrix of $p(z)$ as in (4). Then

$$C(p) = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix},$$

where

$$C_{11} = \begin{bmatrix} -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots 1 & 0 \end{bmatrix},$$

$$C_{12} = \begin{bmatrix} -a_1 & 0 & 0 & \cdots & 0 \end{bmatrix}^t,$$

$$C_{21} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

$$C_{22} = \begin{bmatrix} 0 \end{bmatrix}.$$

Known,

$$rC(p) \le r\left(\begin{bmatrix} \beta & |a_1| \\ 1 & 0 \end{bmatrix} \right).$$

By using (iv) in Theorem 1.2, we get

$$\beta = \|C_{11}\| = \left[\frac{\left(1 + \sum_{j=2}^{n} |a_j|^2\right) + \sqrt{\left(\sum_{j=2}^{n} |a_j|^2 - 1\right)^2 + 4\,|a_2|^2}}{2} \right]^{\frac{1}{2}}. \qquad \square$$

That is, the desired result.

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