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THERMAL STRESSES IN A THIN POROUS PLATE

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Abstract. The thermal stresses that appear in a thin porous plate are analized and numerical results are obtained with FreeFem++.

1. Introduction

The porous plates have been recently studied, in particular, the silicon thin porous plates which are component parts of electronic engines (integrate circuits, transistors) and are often used in nanotechnology.

An existence and uniqueness result for the problem with initial data and boundary conditions was established by Bîrsan [1], using the logarithmic convexity method. Kumar and Rani [6] determined an analytical solution for the equilibrium equations for the generalized thermoelastic half-space with voids using the Laplace and Fourier transforms.

In what follows, based on the representation theory, we shall establish an existence and uniqueness theorem, using the theory of semigroups [8]. In order to obtain numerical results modeled with FreeFem++, it is necessary to give the variational formulation of the limit problem (1.1).

We are interested to study the thermal effect on a thin porous plate (deformation and thermal stresses), not taking into account the chemical and physical phenomena that appear under the action of the thermal field. In order to do that, we need a representation theorem of the solution of the limit problem (1.1).

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Consider a porous media having the form of a rectangular plate that fulfills a domain $B \subset \mathbb{R}^3$. The geometry of the plate is described with respect to an orthonormal positively oriented frame $Ox_1x_2x_3$, having the axis Ox_1 and Ox_2 in the median plane Σ of the plate.

We shall reduce the study of the system to the 2-dimensional case (in the median plane), using the micropolar theory of thermoelastic media introduced by Eringen [3].

Following Lord and Shulman [7], Green and Lindsay [4] and Ieşan [5], the field equations and constitutive relations in a generalized thermoelastic solid with voids, without body forces, heat sources and extrinsic equilibrated body force are:

$$\begin{cases} (\lambda+\mu)\dfrac{\partial}{\partial x_i}(\operatorname{div}\bar{u}) + \mu\Delta u_i + b\dfrac{\partial\phi}{\partial x_i} - \beta\dfrac{\partial\theta}{\partial x_i} + \rho_0 f_i^* = \rho_0\ddot{u}_i, \quad i=\overline{1,3} \\ \alpha\Delta\phi - b(\operatorname{div}\bar{u}) - \xi\phi + m\theta + \rho_0 l^* = \rho_0\chi\ddot{\phi} \\ T_0[\beta(\operatorname{div}\dot{\bar{u}}) + m\dot{\phi} + a\dot{\theta}] = k\Delta\theta + \rho_0 S^* \end{cases}, \tag{1.1}$$

on $B\times(0,t_0)$, where by $\bar{u}$ we denoted the displacement field, θ stands for the variation of the absolute temperature, Φ is the change in volume fraction field, ρ_0 is the density of the medium, λ, μ are the Lame's constants, k is the thermal conduction coefficient and a, b, m, α, β, ξ are the constitutive coefficients.

Denote by $f_i = \rho_0 f_i^*$ the density of the body forces. Assume that $\bar{f} \in C^0(\bar{B}\times(0,t_0))$ and $\bar{f}\in C^{2,1}(B\times(0,t_0))$. Then

$$\bar{f} = \operatorname{grad} Q + \operatorname{rot}\gamma,$$

where $Q,\gamma \in C^{2,1}(B\times(0,t_0))$ and $\operatorname{div}\gamma = 0$. Assume that $\beta \neq 0$. Put

$$\bar{u} = \operatorname{grad}\Phi + \operatorname{rot}\psi. \tag{1.2}$$

The first equation of the system (1.1) becomes

$$\mu\Delta\bar{u} + (\lambda+\mu)\operatorname{grad}(\operatorname{div}\bar{u}) + b\operatorname{grad}\phi - \beta\operatorname{grad}\theta - \rho_0\frac{\partial^2\bar{u}}{\partial t^2} = -\bar{f}$$

$$\Longleftrightarrow \mu\Delta(\operatorname{grad}\Phi + \operatorname{rot}\psi) + (\lambda+\mu)\operatorname{grad}(\operatorname{div}(\operatorname{grad}\Phi + \operatorname{rot}\psi)) + b\operatorname{grad}\phi -$$
$$-\beta\operatorname{grad}\theta - \rho_0\frac{\partial^2}{\partial t^2}(\operatorname{grad}\Phi + \operatorname{rot}\psi) = -(\operatorname{grad}Q + \operatorname{rot}\gamma)$$

$$\Longleftrightarrow \operatorname{grad}[(\lambda+2\mu)\Delta\Phi-\rho_0\ddot{\Phi}+b\phi-\beta\theta+Q]+\operatorname{rot}[\mu\Delta\phi-\rho_0\ddot{\psi}+\gamma]=0.$$

The first equation of the system (1.1) is satisfied if we take

$$\Delta\Phi-\frac{\rho_0}{\lambda+2\mu}\frac{\partial^2\Phi}{\partial t^2}=\frac{1}{\lambda+2\mu}(\beta\theta-b\phi-Q)$$

$$\Longleftrightarrow (\Delta-\frac{1}{c_1^2}\frac{\partial^2}{\partial t^2})\Phi=\frac{1}{\lambda+2\mu}(\beta\theta-b\phi-Q), \tag{1.3}$$

where $c_1=\sqrt{\dfrac{\lambda+2\mu}{\rho_0}}$, and respectively,

$$\Delta\psi-\frac{\rho_0}{\mu}\frac{\partial^2\psi}{\partial t^2}=-\frac{1}{\mu}\gamma$$

$$\Longleftrightarrow (\Delta-\frac{1}{c_2^2}\frac{\partial^2}{\partial t^2})\psi=-\frac{1}{\mu}\gamma,$$

where $c_2=\sqrt{\dfrac{\mu}{\rho_0}}$.

We obtain

$$\theta=\frac{1}{\beta}[(\lambda+2\mu)(\Delta-\frac{1}{c_1^2}\frac{\partial^2}{\partial t^2})\Phi+b\psi+Q],$$

and respectively,

$$\psi=\frac{1}{b}[-(\lambda+2\mu)(\Delta-\frac{1}{c_1^2}\frac{\partial^2}{\partial t^2})\Phi+b\theta-Q].$$

Replacing $\bar{u}$ and θ in the last equation of the system (1.1) we get

$$T_0[\beta\frac{\partial}{\partial t}\operatorname{div}(\operatorname{grad}\Phi+\operatorname{rot}\psi)+m\dot{\psi}+\frac{\alpha}{\beta}\frac{\partial}{\partial t}((\lambda+2\mu)(\delta-\frac{1}{c_1^2}\frac{\partial^2}{\partial t^2})\Phi+b\phi+Q)]=$$

$$=\frac{k}{\beta}\Delta((\lambda+2\mu)(\delta-\frac{1}{c_1^2}\frac{\partial^2}{\partial t^2})\Phi+b\phi+Q)+\rho_0S^*.$$

Multiplying this relation by $\dfrac{\beta}{a(\lambda+2\mu)}$, it becomes

$$[(\frac{k}{a}\Delta-\frac{\partial}{\partial t})(\Delta-\frac{1}{c_1^2}\frac{\partial^2}{\partial t^2})-\frac{\beta^2T_0}{a\rho_0c_1^2}\frac{\partial}{\partial t}\Delta]\Phi=-(\frac{k}{a}\Delta-\frac{\partial}{\partial t})Q-\frac{b}{\lambda+2\mu}[\frac{k}{a}\Delta-$$

$$-(1+\frac{mT_0\beta}{ab})\frac{\partial}{\partial t}]\phi-\frac{\beta\rho_0}{a(\lambda+2\mu)}S^*. \tag{1.4}$$

Replacing $\bar{u}$ and ϕ in the second equation of the system (1.1) we get

$$\frac{\alpha}{b}\Delta[-(\lambda+2\mu)(\Delta-\frac{1}{c_1^2}\frac{\partial^2}{\partial t^2})\Phi+\beta\theta-Q]-b\Delta\Phi-\frac{\xi}{b}[-(\lambda+2\mu)(\Delta-\frac{1}{c_1^2}\frac{\partial^2}{\partial t^2})\Phi+\beta\theta-Q]+$$

$$+m\Theta-\frac{\rho_0\chi}{b}\frac{\partial^2}{\partial t^2}[-(\lambda+2\mu)(\Delta-\frac{1}{c_1^2}\frac{\partial^2}{\partial t^2})\Phi+\beta\theta-Q]=-\rho_0 l^*.$$

Multiplying this relation by $\frac{b}{\rho_0\chi(\lambda+2\mu)}$, it becomes

$$[(\frac{\partial^2}{\partial t^2}-\frac{\alpha}{\rho_0\chi}\Delta)(\Delta-\frac{1}{c_1^2}\frac{\partial^2}{\partial t^2})-(\frac{b^2-\xi(\lambda+2\mu)}{\rho_0\chi}\Delta+\frac{\xi}{\chi}\frac{\partial^2}{\partial t^2})]\Phi=\frac{1}{\lambda+2\mu}(\beta\frac{\partial^2}{\partial t^2}-$$

$$-\frac{\alpha\beta}{\rho_0\chi}\Delta+(\xi\beta-bm))\theta+\frac{1}{\lambda+2\mu}[-\frac{\partial^2}{\partial t^2}+\frac{\alpha}{\rho_0\chi}\Delta-\frac{\xi}{\rho_0\chi}]Q-\frac{b}{\chi(\lambda+2\mu)}l^*. \quad (1.5)$$

Therefore, it holds a Deresiewicz [2] - Zorski [9] theorem:

Theorem 1.1. *Let* $\bar{u}=\operatorname{grad}\phi+\operatorname{rot}\psi$ *and* $\theta=\frac{1}{\beta}[(\lambda+2\mu)(\Delta-\frac{1}{c_1^2}\frac{\partial^2}{\partial t^2})\Phi+b\psi+Q]$*, where* $\Phi\in C^{4,4}(\bar{B}\times(0,t_0))$ *and* $\psi\in C^{3,2}(\bar{B}\times(0,t_0))$ *satisfy the relations (1.3), (1.4), (1.5). Then* $\bar{u},\theta$ *and* ψ *satisfy the system (1.1).*

2. **Existence and uniqueness**

According to the micropolar theory of thermoelasticity for elastic media with voids introduced by Eringen [3], we shall assume

$$\bar{u}^{(1)}=(x_3v_1,x_3v_2,w)$$

$$\bar{u}^{(2)}=\operatorname{grad}(x_3\Phi)$$

$$\phi=x_3\psi,\ \theta=x_3T$$

where the functions v_1,v_2,w,Φ,ψ,T depend on x_1,x_2,t $[(x_1,x_2)\in\Sigma,\ t\in\mathcal{T}]$.

Using the representation theorem 1.1, the equilibrium equations can be reduced to the following systems:

$$\begin{cases}\frac{\partial^2 v_1}{\partial t^2}-\frac{\mu}{\rho_0}\Delta v_1=0\\ \frac{\partial^2 v_2}{\partial t^2}-\frac{\mu}{\rho_0}\Delta v_2=0\\ \frac{\partial^2 w}{\partial t^2}-\frac{\mu}{\rho_0}\Delta w=0\end{cases} \quad (2.1)$$

and

$$\begin{cases} \ddot{\Phi} = \dfrac{\lambda+2\mu}{\rho_0}\Delta\Phi + \dfrac{b}{\rho_0}\psi - \dfrac{\beta}{\rho_0}T \\ \ddot{\psi} = \dfrac{\alpha}{\rho_0\chi}\Delta\psi - \dfrac{b}{\rho_0}\Delta\Phi - \dfrac{\xi}{\rho_0}\psi + \dfrac{m}{\rho_0}T \\ \dot{T} = \dfrac{k}{\rho_0 c_l}\Delta T - \dfrac{\beta T_0}{\rho_0 c_l}\Delta\dot{\Phi} - \dfrac{mT_0}{\rho_0 c_l}\dot{\psi} \end{cases} . \tag{2.2}$$

The last one can be decomposed into

$$\begin{cases} \dot{\Phi} = \zeta \\ \dot{\zeta} = \dfrac{\lambda+2\mu}{\rho_0}\Delta\Phi + \dfrac{b}{\rho_0}\psi - \dfrac{\beta}{\rho_0}T \\ \dot{\psi} = \tau \\ \dot{\tau} = \dfrac{\alpha}{\rho_0\chi}\Delta\psi - \dfrac{b}{\rho_0}\Delta\Phi - \dfrac{\xi}{\rho_0}\psi + \dfrac{m}{\rho_0}T \\ \dot{T} = \dfrac{k}{\rho_0 c_l}\Delta T - \dfrac{\beta T_0}{\rho_0 c_l}\Delta\dot{\Phi} - \dfrac{mT_0}{\rho_0 c_l}\dot{\psi} \end{cases} \tag{2.3}$$

which is equivalent to

$$\begin{cases} \dot{\Phi} = \zeta \\ \dot{\zeta} - \dfrac{\lambda+2\mu}{\rho_0}\Delta\Phi = \dfrac{b}{\rho_0}\psi - \dfrac{\beta}{\rho_0}T \\ \dot{\psi} = \tau \\ \dot{\tau} - \dfrac{\alpha}{\rho_0\chi}\Delta\psi + \dfrac{b}{\rho_0}\Delta\Phi = -\dfrac{\xi}{\rho_0}\psi + \dfrac{m}{\rho_0}T \\ \dot{T} - \dfrac{k}{\rho_0 c_l}\Delta T + \dfrac{\beta T_0}{\rho_0 c_l}\Delta\dot{\Phi} = -\dfrac{mT_0}{\rho_0 c_l}\dot{\psi} \end{cases} . \tag{2.4}$$

Write the system (2.2) as an evolution system of order 1 associated to a strongly elliptic operator A on a Hilbert space.

Define $D(A) := (H^2(\Sigma) \times H^1(\Sigma)) \times (H^2(\Sigma) \times H^1(\Sigma)) \times H^2(\Sigma) =: V(\Sigma)$ and for $W = (\Phi, \zeta, \psi, \tau, T)^t \in D(A)$, let

$$AW := M\Delta W,$$

where $M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ \dfrac{\lambda+2\mu}{\rho_0} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -\dfrac{b}{\rho_0} & 0 & \dfrac{\alpha}{\rho_0\chi} & 0 & 0 \\ 0 & -\dfrac{\beta T_0}{\rho_0 c_l} & 0 & 0 & \dfrac{k}{\rho_0 c_l} \end{pmatrix}.$

Denote by $|||\cdot|||$ the norm $||\cdot||_{V(\Sigma)}$ in the product space $V(\Sigma)$.

The system (2.4) is an evolution system associated to the operator $-\Delta$ [8] and can be written in the operatorial form:

$$\frac{\partial W}{\partial t} - AW = \bar{F}(t, x_1, x_2, W), \tag{2.5}$$

for $\bar{F}(t, x_1, x_2, W) := NW$, where $N = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{b}{\rho_0} & 0 & -\frac{\beta}{\rho_0} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{\xi}{\rho_0} & 0 & \frac{m}{\rho_0} \\ 0 & 0 & 0 & -\frac{mT_0}{\rho_0 c_l} & 0 \end{pmatrix}$.

Consider the initial data $W(0, x_1, x_2) = W_0(x_1, x_2)$ on Σ and the boundary condition $W(t, x_1, x_2) = 0$ for $(x_1, x_2) \in \partial\Sigma$. Following Pazy [8] (chapters 7, 8), we can state:

Proposition 2.1. *Let Σ be a domain in $\mathbb{R}^2$ with smooth boundary and $\bar{F} = (F_1, F_2, F_3, F_4, F_5, F_6)$ with every component continuous locally Lipschitz function of all its arguments. Assume that there is some continuous functions $\eta_i : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}_+$, $1 \leq i \leq 5$, such that*

$$|F_i(t, x_1, x_2, W)| \leq \eta_i(t, |||W|||), \quad 1 \leq i \leq 5$$

and

$$|F_i(t, x_1, x_2, W_1) - F_i(t, x_1, x_2, W_2)| \leq \eta_i(t, |||W_1||| + |||W_2|||), \quad 1 \leq i \leq 5.$$

For every $W_0 \in (H^2(\Sigma) \times H_0^1(\Sigma)) \times ... \times (H^2(\Sigma) \times H_0^1(\Sigma))$, the initial value problem

$$\begin{cases} \frac{\partial W}{\partial t} - AW = \bar{F}(t, x_1, x_2, W) \\ W(0, x_1, x_2) = W_0(x_1, x_2) \quad \text{on } \Sigma \end{cases}$$

- **i):** *has a unique solution $W = (\Phi, \zeta, \psi, \tau, T)^t \in (H^2(\Sigma) \times L^2(\Sigma)) \times (H^2(\Sigma) \times L^2(\Sigma)) \times H^2(\Sigma)$, if $F_i \in C_0^\infty(\Sigma)$ for every $1 \leq i \leq 5$;*
- **ii):** *has a unique solution $W = (\Phi, \zeta, \psi, \tau, T)^t \in V(\Sigma)$, if $\bar{F} \in (H^1(\Sigma) \times L^2(\Sigma)) \times (H^1(\Sigma) \times L^2(\Sigma)) \times H^1(\Sigma)$.*

3. **Numerical results**

We shall give a numerical modeling and simulation for the thermal stress of an elastic, thin, porous plate made up of magnesium, using FreeFem++.

Assume that the heat transport is realized by conduction as long as there exists an internal thermic source whose temperature is constant.

Consider the following initial conditions:

$S_0 = 998$ K - thermal source

$T_0 = 298$ K - initial temperature of plate

$\Phi(x_1, x_2, 0) = 0$

$\psi(x_1, x_2, 0) = 1,0011507$ - porosity

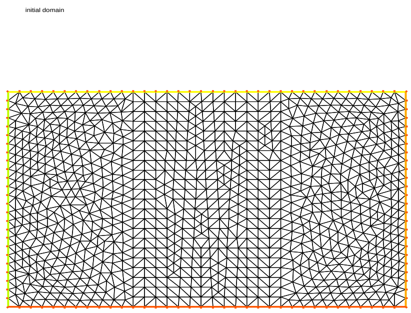
The physical constants of the material and the parameters of voids can be found in [6].

We shall model the case when the body forces are uniformly distributed orthogonal to the median plane of the plate.

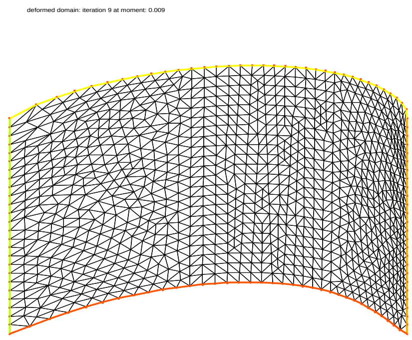
The dependence on temperature of the deformations, of the stresses and of the change in volume fraction field are further showed.

One can notice that after a number of iterations, the plate reached a thermal equilibrium state.

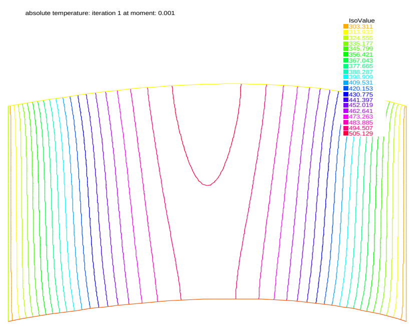
The numerical results are presented in the nearby graphics.



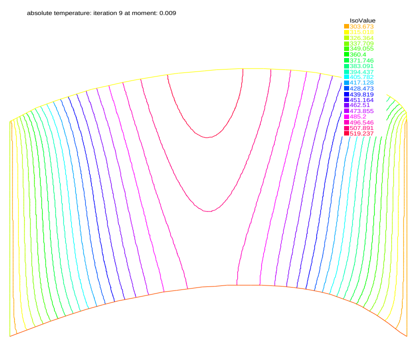
initial domain



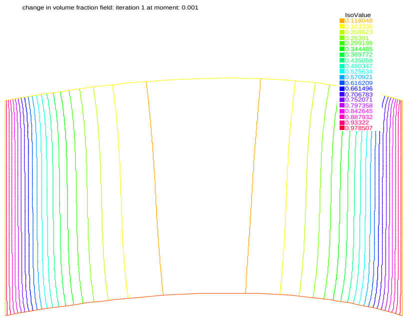
deformed domain: iteration 9



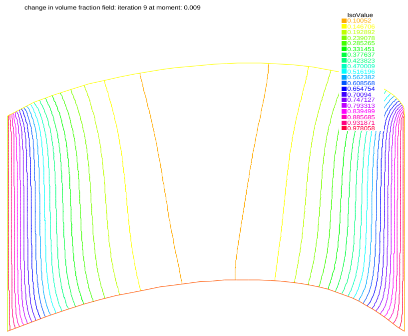
absolute temperature: iteration 1



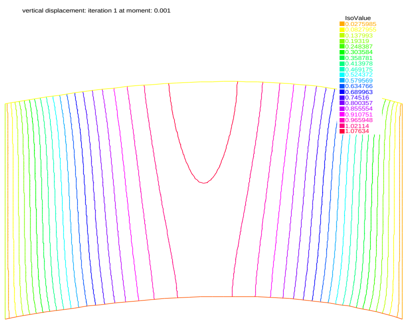
absolute temperature: iteration 9



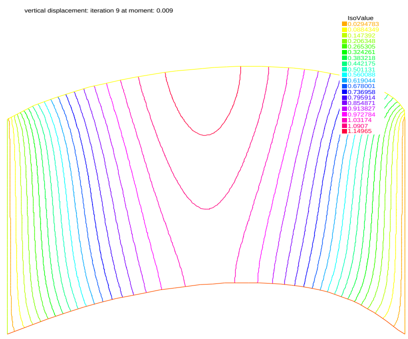
porosity: iteration 1



porosity: iteration 9



vertical displacement: iteration 1



vertical displacement: iteration 9

Final remarks

As the absolute temperature inside the plate is growing, the plate is deforming more and more until it reaches the thermal equilibrium state.

Acknowledgments

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