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OPTIMIZATION PROBLEMS AND η-APPROXIMATED OPTIMIZATION PROBLEMS

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Abstract. In this paper, a so-called η-approximated optimization problem (Ref. [1] and [3]) associated to an optimization problem is considered. The equivalence between the saddle points of the lagrangian of the η-approxiated optimization problem and optimal solutions of the original optimization problem is established.

1. **Introduction**

We consider the optimization problem

$$\begin{aligned} &\min\ f(x) \\ &\text{s.t. } \ x \in X \\ &\qquad g_i(x) \leqq 0, \quad i \in \{1, ..., m\}, \end{aligned} \tag{P}$$

where X is a subset of $\mathbb{R}^n$ and $f, g_1, ..., g_m : X \to \mathbb{R}$ are functions.

Let

$$\mathfrak{F}(P) := \{x \in X : \ g_i(x) \leqq 0, \ i \in \{1, ..., m\}\}$$

denote the set of all feasible solutions of Problem (P).

For solving optimization problem (P), there are various manners to approach. One of these manners is that for Problem (P) one attaches another optimization problem, problem whose solutions give us the (information about) optimal solutions of the initial problem (P).

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Assuming that X is open, and that f and g are differentiable on X, Mangasarian (Ref. [11]) attached to Problem (P) and the point $x^0 \in X$, the problem

$$\begin{aligned} \min\ & f\left(x^0\right) + \left\langle u, \nabla f\left(x^0\right)\right\rangle \\ \text{s.t.}\ & u \in \mathbb{R}^n \\ & g\left(x^0\right) + \left[\nabla g\left(x^0\right)\right](u) \leqq 0. \end{aligned}$$

He took the dual of this linear optimization problem and then considered x^0 to be a variable in X. This last problem is precisely the classical dual of the nonlinear optimization problem, introduced in a different way by Wolfe (Ref. [13]) and investigated extensively (see, for example Ref. [10]). Connections between optimal solutions of the dual and the primal are known (see, for example Ref. [10]).

The above process is repeated but taking nonlinear instead of linear approximation of f and g around some fixed $x^0 \in X$ and taking the dual of the resulting optimization problem. One takes the dual of this nonlinear optimization problem and then one considers x^0 be a variable in X. One obtains the so called higher-order dual problem of Problem (P). In Ref. [11], there are given connections between the optimal solutions of higher-order dual and initial problem (P). D.I. Duca (Ref. [7]) used this idea for optimization problems in complex space.

Another idea came from Antczak (Ref. [3], [2], [1]), who attached to Problem (P) and the point $x^0 \in X$, the following problem

$$\begin{aligned} \min\ & f\left(x^0\right) + \left\langle \nabla f\left(x^0\right), \eta(x)\right\rangle \\ \text{s.t.}\ & x \in X \\ & g\left(x^0\right) + \left[\nabla g\left(x^0\right)\right](\eta(x)) \leqq 0, \end{aligned} \qquad \left(P_\eta\left(x^0\right)\right)$$

where $\eta = \eta_{x^0} : X \to X$ is a function. He studied the connections between the saddle points of Problem $\left(P_\eta\left(x^0\right)\right)$ and optimal solutions of Problem (P).

We attach to Problem (P), the Lagrange function (or the lagrangian) $L : X \times \mathbb{R}^m_+ \to \mathbb{R}$ defined by

$$L(x, v) := f(x) + \langle v, g(x)\rangle, \text{ for all } (x, v) \in X \times \mathbb{R}^m_+,$$

where $g = (g_1, ..., g_m)$.

Definition 1. *We say that* $\left(x^0, v^0\right) \in X \times \mathbb{R}_+^m$ *is a saddle point of the lagrangian* L (*or of Problem* (P)) *if*

$$L\left(x^0, v\right) \leqq L\left(x^0, v^0\right) \leqq L\left(x, v^0\right), \text{ for all } (x, v) \in X \times \mathbb{R}_+^m.$$

The saddle points of the lagrangian L of Problem (P) have been studied by many authors (see for example Ref. [10], [4] and others). A fundamental result of optimization theory is that, in certain conditions, the point x^0 is an optimal solution of Problem (P) if and only if there exists a point $v^0 \in \mathbb{R}_+^m$ such that $\left(x^0, v^0\right)$ is a saddle point of its lagrangian.

More precisely, we have the following results, results which play an important role in optimization theory and economics.

Theorem 2. *If* $\left(x^0, v^0\right) \in X \times \mathbb{R}_+^m$ *is a saddle point of the lagrangian* L *of Problem* (P) *then* x^0 *is an optimal solution of Problem* (P).

Proof. See, for example, Ref. [10]. □

Theorem 3. *Let* x^0 *be an optimal solution of Problem* (P). *Assume that* $f, g_1, ..., g_m$ *are convex at* x^0 *and a suitable constraint qualification (CQ, Ref.* [10]*) is satisfied at* x^0. *Then there exists a point* $v^0 \in \mathbb{R}_+^m$ *such that* $\left(x^0, v^0\right)$ *is a saddle point of the lagrangian of Problem* (P).

Proof. See, for example, Ref. [10]. □

In the last few years, attempts have been made to weaken the convexity hypotheses and thus to explore the existence of optimality conditions applicability. Various classes of generalized convex functions have been suggested for the purpose of weakening the convexity limitation in this result. Among these, the concept of an invex function proposed by Hanson (Ref. [9]) has received more attention. The name of invex (invariant convex) function was given by Craven (Ref. [6])

Definition 4. *Let* X *be a subset of* $\mathbb{R}^n$, x^0 *be an interior point of* X, $f : X \to \mathbb{R}$ *be a differentiable function at* x^0 *and* $\eta = \eta_{x^0} : X \to \mathbb{R}^n$ *be a function. We say that* f *is invex at* x^0 *with respect to* η *if*

$$f(x) - f\left(x^0\right) \geqq \left\langle \nabla f\left(x^0\right), \eta(x)\right\rangle, \text{ for all } x \in X. \tag{1}$$

Hanson defined invex functions which allow the use of the Kuhn-Tucker conditions as sufficient conditions for optimality in constrained optimization problems. Later, Martin (Ref. [12]) proved that invexity hypotheses are not only sufficient but also necessary when using the Kuhn-Tucker optimality conditions for unconstrained optimization problems.

After the works of Hanson and Craven, other types of differentiable functions have appeared with the intent of generalizing invex function from different points of view.

Ben-Israel and Mond (Ref. [5]) defined the so-called pseudoinvex functions, generalizing pseudoconvex functions in the same way that invex functions generalize convex functions.

Definition 5. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta = \eta_{x^0} : X \to \mathbb{R}^n$, and $f : X \to \mathbb{R}$ be a differentiable function at x^0. We say that f is pseudoinvex at x^0 with respect to η if, for each $x \in X$ with the property that*

$$\left\langle \nabla f\left(x^0\right), \eta\left(x\right)\right\rangle \geqq 0,$$

we have

$$f\left(x\right) \geqq f\left(x^0\right).$$

Definition 6. *Let X be a subset of $\mathbb{R}^n$, x^0 be an interior point of X, $\eta = \eta_{x^0} : X \to \mathbb{R}^n$, and $f : X \to \mathbb{R}$ be a differentiable function at x^0. We say that f is quasiinvex at x^0 with respect to η if, for each $x \in X$ with the property that*

$$f\left(x\right) \leqq f\left(x^0\right),$$

we have

$$\left\langle \nabla f\left(x^0\right), \eta\left(x\right)\right\rangle \leqq 0.$$

Remark 7. *Note that, in general, there exists no unique function η such that the function f is invex, respectively pseudoinvex and quasiinvex at the point $x^0 \in X$.*

Indeed, the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f\left(x\right) = \exp x, \ \textit{for all } x \in \mathbb{R},$$

is invex at $x^0 = 0$ *with respect to the function* $\eta : \mathbb{R} \to \mathbb{R}$ *defined by*

$$\eta(x) = x - x_0 = x, \text{ for all } x \in \mathbb{R}.$$

Also, the function f *is invex at* $x^0 = 0$ *with respect to the function* $\eta : \mathbb{R} \to \mathbb{R}$ *defined by*

$$\eta(x) = x + \frac{x^2}{2} + \frac{x^3}{6}, \text{ for all } x \in \mathbb{R}.$$

And also, the function f *is invex at* x^0 *with respect to the function* $\eta : \mathbb{R} \to \mathbb{R}$ *defined by*

$$\eta(x) = x - 2, \text{ for all } x \in \mathbb{R}.$$

In this paper, in more general hypotheses that in Ref. [3], the equivalence between the saddle points of the lagrangian of the η-approximated optimization problem and optimal solutions of the original optimization problem is established.

2. $\eta-$**approximated optimization problem**

In what follows x^0 is an interior point of X, and f and g are differentiable at x^0.

For the function $\eta = \eta_{x^0} : X \to \mathbb{R}^n$, we attach to Problem (P) the optimization problem $\left(P_\eta\left(x^0\right)\right)$, called η-approximated at x^0 of Problem (P).

Remark 8. *If* $X = \mathbb{R}^n$ *and* $\eta(x) = x - x_0$, *for all* $x \in X$, *then Problem* $\left(P_\eta\left(x^0\right)\right)$ *is linear.*

Let

$$\mathfrak{F}\left(P_\eta\left(x^0\right)\right) := \{x \in X : \; g_i\left(x^0\right) + \left\langle \nabla g_i\left(x^0\right), \eta(x)\right\rangle \leqq 0, \; i \in \{1, ..., m\}\},$$

denote the set of all feasible solutions of Problem $\left(P_\eta\left(x^0\right)\right)$.

The lagrangian of Problem $\left(P_\eta\left(x^0\right)\right)$ will be denoted by L_η, i.e. $L_\eta : X \times \mathbb{R}^m_+ \to \mathbb{R}$ is defined by

$$L_\eta(x, v) := f\left(x^0\right) + \left\langle \nabla f\left(x^0\right), \eta(x)\right\rangle + \left\langle v, g\left(x^0\right)\right\rangle + \left\langle v, \left[\nabla g\left(x^0\right)\right](\eta(x))\right\rangle,$$

for all $(x, v) \in X \times \mathbb{R}^m_+$.

Example 9. *Let us consider the optimization problem*

$$\begin{aligned} &\min f(x) = \exp x \\ &s.t.\quad x \in X = \mathbb{R} \\ &\qquad g_1(x) = x^2 - x \leqq 0. \end{aligned} \qquad (\overline{P})$$

We have that $\mathfrak{F}\left(\overline{P}\right) = [0,1]$ *and* $x^0 = 0$ *is the unique optimal solution of Problem* $\left(\overline{P}\right)$.

The functions f *and* g_1 *are invex at* $x^0 = 0$ *with respect to the function* $\eta = \eta_{x^0} : \mathbb{R} \to \mathbb{R}$ *defined by*

$$\eta(x) = x, \text{ for all } x \in \mathbb{R}.$$

Then the η*-approximated optimization problem is*

$$\begin{aligned} &\min(1+x) \\ &s.t.\quad x \in X = \mathbb{R} \\ &\qquad -x \leqq 0, \end{aligned} \qquad (\overline{P}_\eta(x^0))$$

which has the optimal solution $x^0 = 0$.

On the other hand, the lagrangian $\overline{L}_\eta$ *of Problem* $\left(\overline{P}_\eta\left(x^0\right)\right)$ *is defined by*

$$\overline{L}_\eta(x,v) = 1 + x - vx, \text{ for all } (x,v) \in \mathbb{R} \times \mathbb{R}_+.$$

Obviously, $\left(x^0, v^0\right) = (0,1)$ *is a saddle point of the lagrangian* $\overline{L}_\eta$.

In this section we show the equivalence between saddle points of the lagrangian L_η, of Problem $\left(P_\eta\left(x^0\right)\right)$, and optimal solutions of Problem $\left(P_\eta\left(x^0\right)\right)$.

By Theorem 2, the following saddle point theorem follows:

Theorem 10. *If* $\left(x^0, v^0\right) \in X \times \mathbb{R}_+^m$ *is a saddle point of the lagrangian* L_η *of Problem* $\left(P_\eta\left(x^0\right)\right)$, *then* x^0 *is an optimal solution of Problem* $\left(P_\eta\left(x^0\right)\right)$.

Remark 11. *We established Theorem 10, without any assumption about the functions involved in Problem* $\left(P_\eta\left(x^0\right)\right)$.

In order to prove that if $x^0 \in X$ is an optimal solution of Problem $\left(P_\eta\left(x^0\right)\right)$, then there exists a point $v^0 \in \mathbb{R}_+^m$ such that $\left(x^0, v^0\right)$ is a saddle point of the lagrangian of Problem $\left(P_\eta\left(x^0\right)\right)$, let us denote by $F, G_1, ..., G_m : X \to \mathbb{R}$ the functions defined by

$$F(x) := f\left(x^0\right) + \left\langle \nabla f\left(x^0\right), \eta(x)\right\rangle,$$

$$G_i(x) := g_i\left(x^0\right) + \left\langle \nabla g_i\left(x^0\right), \eta(x)\right\rangle,\ i \in \{1, ..., m\},$$

for all $x \in X$.

Obviously, Problem $\left(P_\eta\left(x^0\right)\right)$ can be written as

$$\begin{aligned} &\min F(x) \\ &\text{s.t.} \quad x \in X \\ &\qquad G_i(x) \leqq 0, \quad i \in \{1, ..., m\}. \end{aligned}$$

Now, we can state the converse theorem of Theorem 10.

Theorem 12. *Let $x^0 \in X$ be an optimal solution of Problem $\left(P_\eta\left(x^0\right)\right)$, $\mu = \mu_{x^0} : X \to \mathbb{R}^n$ be a function. Assume that $\eta : X \to \mathbb{R}^n$ is differentiable at x^0, the functions $F, G_1, ..., G_m : X \to \mathbb{R}$ are invex at x^0 with respect to μ and a suitable constraint qualification (CQ, Ref* [10]*) is satisfied at x^0. Then there exists a point $v^0 \in \mathbb{R}^m_+$ such that $\left(x^0, v^0\right)$ is a saddle point of Problem $\left(P_\eta\left(x^0\right)\right)$.*

Proof. Let $G = (G_1, ..., G_m)$. In view of Karush-Kuhn-Tucker theorem, there exists a point $v^0 \in \mathbb{R}^m_+$ such that

$$\nabla F\left(x^0\right) + \left[\nabla G\left(x^0\right)\right]^{\mathrm{T}}\left(v^0\right) = 0, \tag{2}$$

$$\left\langle v^0, G\left(x^0\right)\right\rangle = 0, \tag{3}$$

i.e.

$$\nabla f\left(x^0\right) + \left\langle v^0, \left[\nabla g\left(x^0\right)\right]\left(\nabla \eta\left(x^0\right)\right)\right\rangle = 0,$$

$$\left\langle v^0, g\left(x^0\right) + \left[\nabla g\left(x^0\right)\right]\left(\eta\left(x^0\right)\right)\right\rangle = 0.$$

The functions F, $G_1, ..., G_m$ are invex at x^0 with respect to μ, then, for each $x \in X$, we have

$$F(x) - F\left(x^0\right) \geqq \left\langle \nabla F\left(x^0\right), \mu(x)\right\rangle, \tag{4}$$

$$G_i(x) - G_i\left(x^0\right) \geqq \left\langle \nabla G_i\left(x^0\right), \mu(x)\right\rangle,\ i \in \{1, ..., m\}. \tag{5}$$

Since $v^0 \in \mathbb{R}_+^m$, by (5), we obtain

$$\left\langle v^0, G(x)\right\rangle - \left\langle v^0, G\left(x^0\right)\right\rangle \geqq \left\langle v^0, \left[\nabla G\left(x^0\right)\right](\mu(x))\right\rangle, \text{ for all } x \in X. \tag{6}$$

Then, for each $x \in X$,

$$\begin{aligned} & L_\eta\left(x, v^0\right) - L_\eta\left(x^0, v^0\right) = \\ &= F(x) + \left\langle v^0, G(x)\right\rangle - F\left(x^0\right) - \left\langle v^0, G\left(x^0\right)\right\rangle \geqq \text{ (by (4), and (6))} \\ &\geqq \left\langle \nabla F\left(x^0\right), \mu(x)\right\rangle + \left\langle v^0, \left[\nabla G\left(x^0\right)\right](\mu(x))\right\rangle = \\ &= \left\langle \nabla F\left(x^0\right) + \left[\nabla G\left(x^0\right)\right]^{\mathrm{T}}\left(v^0\right), \mu(x)\right\rangle = \text{ (by (2))} \\ &= 0. \end{aligned}$$

Consequently, the second inequality in the definition of saddle point is satisfied.

In order to prove the first inequality of the definition of saddle point, let $v \in \mathbb{R}_+^m$. Then

$$\begin{aligned} & L_\eta\left(x^0, v^0\right) - L_\eta\left(x^0, v\right) = \\ &= \left\langle v^0, G\left(x^0\right)\right\rangle - \left\langle v, G\left(x^0\right)\right\rangle = \text{ (by (3))} \\ &= -\left\langle v, G\left(x^0\right)\right\rangle \geqq \\ &\geqq 0, \end{aligned}$$

because $G\left(x^0\right) \leqq 0$ □

3. Equivalence between saddle points of η-approximated problem and of the original problem

In this section we will prove the equivalence between the original optimization problem (P) and its associated η-approximated optimization problem $\left(P_\eta\left(x^0\right)\right)$. We establish the results where one assumes that the function $\eta = \eta_{x^0}$ satisfies only the condition $\eta\left(x^0\right) = 0$.

In Ref. [1] one proves the following statement:

Theorem 13. *Let x^0 be a feasible solution of Problem (P). We assume that f and g are invex at x^0 on $\mathfrak{F}(P)$ with respect to $\eta = \eta_{x^0} : X \to \mathbb{R}^n$ satisfying the condition $\eta\left(x^0\right) = 0$. If $\left(x^0, v^0\right) \in \mathfrak{F}(P) \times \mathbb{R}^m_+$ is a saddle point of the η-approximated optimization problem $\left(P_\eta\left(x^0\right)\right)$, then x^0 is an optimal solution of the original optimization problem (P).*

This theorem is true in more general hypotheses.

If x^0 is a feasible solution of Problem (P), then

$$I\left(x^0\right) = \{i \in \{1, ..., m\} : \ g_i\left(x^0\right) = 0\}$$

denote the indices of the active restrictions at x^0.

The following statement is true

Theorem 14. *Let $x^0 \in X$, $\eta = \eta_{x^0} : X \to \mathbb{R}^n$ such that $\eta\left(x^0\right) = 0$, $f : X \to \mathbb{R}$ be pseudoinvex at x^0 with respect to η and $g_1, ..., g_m : X \to \mathbb{R}$ such that g_i, $i \in I\left(x^0\right)$ are quasiinvex at x^0 with respect to η.*

If $\left(x^0, v^0\right) \in X \times \mathbb{R}^m_+$ is a saddle point of the lagrangian L_η of Problem $\left(P_\eta\left(x^0\right)\right)$, then x^0 is an optimal solution of the original problem (P).

Proof. The point $\left(x^0, v^0\right) \in X \times \mathbb{R}^m_+$ is a saddle point of the lagrangian L_η of Problem $\left(P_\eta\left(x^0\right)\right)$; then

$$L_\eta\left(x^0, v\right) \leqq L_\eta\left(x^0, v^0\right), \text{ for all } v \in \mathbb{R}^m_+,$$

i.e.

$$\left(v - v^0\right) g\left(x^0\right) \leqq 0, \text{ for all } v \in \mathbb{R}^m_+, \tag{7}$$

because $\eta\left(x^0\right) = 0$.

Let $i \in \{1, ..., m\}$, and $e^i = (0, ..., 1, ..., 0) \in \mathbb{R}^m$ be the i-th unit vector of $\mathbb{R}^m$. Then, for $v = e^i + v^0 \in \mathbb{R}^m_+$, relation (7) becomes $g_i\left(x^0\right) \leqq 0$. Hence

$$g_i\left(x^0\right) \leqq 0, \text{ for all } i \in \{1, ..., m\}.$$

Consequently,

$$x^0 \in \mathfrak{F}(P).$$

If follows that

$$\left\langle v^0, g\left(x^0\right)\right\rangle \leqq 0, \tag{8}$$

because $v^0 \in \mathbb{R}^m_+$. But, from (7) we deduce

$$\left\langle v^0, g\left(x^0\right)\right\rangle \geqq 0, \tag{9}$$

because $v = 0 \in \mathbb{R}^m_+$.

Thus, by (8) and (9)

$$\left\langle v^0, g\left(x^0\right)\right\rangle = 0. \tag{10}$$

From (10) it follows that

$$v_i^0 = 0, \text{ for all } i \in \{1, ..., m\} \backslash I\left(x^0\right). \tag{11}$$

On the other hand, from

$$L_\eta\left(x^0, v^0\right) \leqq L_\eta\left(x, v^0\right), \text{ for all } x \in X,$$

we deduce that

$$\left\langle \nabla f\left(x^0\right), \eta(x)\right\rangle + \left\langle v^0, \left[\nabla g\left(x^0\right)\right](\eta(x))\right\rangle \geqq 0, \text{ for all } x \in X. \tag{12}$$

In order to prove that x^0 is an optimal solution of Problem (P), let $x \in \mathfrak{F}(P)$. Then

$$g_i(x) \leqq 0, \text{ for all } i \in \{1, ..., m\}.$$

Let $i \in I\left(x^0\right)$. Since

$$g_i(x) - g_i\left(x^0\right) = g_i(x) \leqq 0,$$

and g_i is quasiinvex at x^0 with respect to η, we have

$$\left\langle \nabla g_i\left(x^0\right), \eta(x)\right\rangle \leqq 0,$$

hence

$$v_i^0 \left\langle \nabla g_i\left(x^0\right), \eta(x)\right\rangle \leqq 0,$$

because $v_i^0 \geqq 0$. Then

$$\left\langle v^0, \left[\nabla g\left(x^0\right)\right](\eta(x))\right\rangle \leqq 0, \tag{13}$$

because $v_i^0 = 0$, for all $i \in \{1, ..., m\} \backslash I\left(x^0\right)$.

From (12) and (13) it follows that

$$\left\langle \nabla f\left(x^0\right), \eta\left(x\right)\right\rangle \geqq -\left\langle v^0, \left[\nabla g\left(x^0\right)\right]\left(\eta\left(x\right)\right)\right\rangle \geqq 0. \tag{14}$$

But, the function f is pseudoinvex at x^0 with respect to η, and then, by (14), we deduce that

$$f\left(x\right) \geqq f\left(x^0\right).$$

Consequently, x^0 is an optimal solution of the original problem (P). The theorem is proved. □

Remark 15. *If the functions $f, g_1, ..., g_m$ are invex at x^0 with respect to η, then the hypotheses that f is pseudoinvex at x^0 with respect to η and g_i, $i \in I\left(x^0\right)$ are quasiinvex at x^0 with respect to η are satisfied.*

Remark 16. *The assumption that the function η satisfies the condition $\eta\left(x^0\right) = 0$ is essential in order to have the equivalence between the saddle points of the lagrangian L_η of Problem $\left(P_\eta\left(x^0\right)\right)$, and the optimal solutions of the original problem (P). (see Example 3.4 from Ref.* [1]*)*

Now, we show that, if x^0 is an optimal solution of the original problem (P), then under certain conditions, there exists a point $v^0 \in \mathbb{R}_+^m$ such that $\left(x^0, v^0\right)$ is a saddle point of the η-approximated problem $\left(P_\eta\left(x^0\right)\right)$.

More exactly, the following statement is true:

Theorem 17. *Let $x^0 \in X$ be an optimal solution of the original problem (P) and assume that a suitable constraint qualification is satisfied at x^0 (CQ in Ref.* [10]*). If the function $\eta = \eta_{x^0} : X \to \mathbb{R}^n$ satisfies:*

(i) $\left\langle \nabla f\left(x^0\right), \eta\left(x^0\right)\right\rangle \leqq 0$;

(ii) $g\left(x^0\right) + \left[\nabla g\left(x^0\right)\right]\left(\eta\left(x^0\right)\right) \leqq 0$ *(i.e.* $x^0 \in \mathfrak{F}\left(P_\eta\left(x^0\right)\right)$*)*,

then there exists a point $v^0 \in \mathbb{R}_+^m$ such that $\left(x^0, v^0\right)$ is a saddle point of the lagrangian L_η of the η-approximated problem $\left(P_\eta\left(x^0\right)\right)$.

Proof. Since x^0 is an optimal solution of Problem (P), and some suitable constraint qualification at x^0 is satisfied, by Karush-Kuhn-Tucker' Theorem, there exists a point $v^0 \in \mathbb{R}^m_+$ such that

$$\nabla f\left(x^0\right) + \left[\nabla g\left(x^0\right)\right]^{\mathrm{T}}\left(v^0\right) = 0, \tag{15}$$

$$\left\langle v^0, g\left(x^0\right)\right\rangle = 0. \tag{16}$$

Let $x \in X$. Then, from (15), we have

$$L_\eta\left(x, v^0\right) - L_\eta\left(x^0, v^0\right) = \left\langle \nabla f\left(x^0\right) + \left[\nabla g\left(x^0\right)\right]^{\mathrm{T}}\left(v^0\right), \eta\left(x^0\right)\right\rangle = 0.$$

Consequently, the second inequality from the saddle point definition is true.

In order to prove the first inequality from the saddle point definition, let $v \in \mathbb{R}^m_+$. Then

$$L_\eta\left(x^0, v^0\right) - L_\eta\left(x^0, v\right) =$$

$$= \left\langle v^0, g\left(x^0\right)\right\rangle - \left\langle v, g\left(x^0\right)\right\rangle + \left\langle v^0, \left[\nabla g\left(x^0\right)\right]\left(\eta\left(x^0\right)\right)\right\rangle - \left\langle v, \left[\nabla g\left(x^0\right)\right]\left(\eta\left(x^0\right)\right)\right\rangle =$$

$$= -\left\langle v, g\left(x^0\right)\right\rangle + \left\langle \eta\left(x^0\right), \left[\nabla g\left(x^0\right)\right]^{\mathrm{T}}\left(v^0\right)\right\rangle - \left\langle v, \left[\nabla g\left(x^0\right)\right]\left(\eta\left(x^0\right)\right)\right\rangle =$$

$$= -\left\langle v, g\left(x^0\right)\right\rangle - \left\langle \nabla f\left(x^0\right), \eta\left(x^0\right)\right\rangle - \left\langle v, \left[\nabla g\left(x^0\right)\right]\left(\eta\left(x^0\right)\right)\right\rangle =$$

$$= -\left\langle \nabla f\left(x^0\right), \eta\left(x^0\right)\right\rangle - \left\langle v, g\left(x^0\right) + \left[\nabla g\left(x^0\right)\right]\left(\eta\left(x^0\right)\right)\right\rangle \geqq$$

$$\geqq -\left\langle \nabla f\left(x^0\right), \eta\left(x^0\right)\right\rangle \geqq$$

$$\geqq 0.$$

Consequently, $\left(x^0, v^0\right)$ is a saddle point of the lagrangian of Problem $\left(P_\eta\left(x^0\right)\right)$. □

Remark 18. *If $\eta\left(x^0\right) = 0$, then the hypotheses (i) and (ii) from Theorem 17 are satisfied.*

Remark 19. *If $f, g_1, ..., g_m$ are invex at x^0 with respect to η, then the hypotheses (i) and (ii) from Theorem 17 are satisfied.*

Remark 20. *The hypothesis that the original problem (P) satisfies a suitable constraint qualification at x^0 is essential. Indeed, for the problem*

$$\begin{aligned} \min\ & f(x) = x_2 \\ s.t.\ & x \in X = \mathbb{R}^2 \\ & g_1(x) = x_1 + x_2^2 \leqq 0, \\ & g_2(x) = -x_1 + x_2^2 \leqq 0, \end{aligned} \qquad (\widehat{P})$$

we have the set of all feasible solutions $\mathfrak{F}\left(\widehat{P}\right) = \{(0,0)\}$, and hence $x^0 = (0,0)$ is the unique optimal solution. Let us remark that Problem $\left(\widehat{P}\right)$ is convex, and then the functions f, g_1, g_2 are invex at $x^0 = (0,0)$ with respect to $\eta : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\eta(x) = x, \text{ for all } x \in \mathbb{R}^2.$$

In this case, the η-approximated optimization problem is

$$\begin{aligned} \min\ & x_2 \\ s.t.\ & (x_1, x_2) \in \mathbb{R}^2 \\ & -x_1 \leqq 0, \\ & x_1 \leqq 0. \end{aligned} \qquad (\widehat{P}_\eta(x^0))$$

Thus, $\widehat{L}_\eta : \mathbb{R}^2 \times \mathbb{R}^2_+ \to \mathbb{R}$ is defined by

$$\widehat{L}_\eta(x,v) = x_2 - v_1x_1 + v_2x_1, \text{ for all } (x,v) = ((x_1,x_2),(v_1,v_2)) \in \mathbb{R}^2 \times \mathbb{R}^2_+,$$

and $\left(x^0, v^0\right)$, where $v^0 = \left(v_1^0, v_2^0\right) \geqq 0$, is not a saddle point of the lagrangian of Problem $\left(\widehat{P}_\eta(x^0)\right)$.

4. Conclusions

In this paper one shows that the invexity hypotheses from paper [3] can be weaker.

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