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BERNSTEIN-TYPE OPERATORS ON TETRAHEDRONS

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Abstract. The aim of the paper is to construct Bernstein-type operators on tetrahedron with all straight edges and on tetrahedron with three curved edges defined by some given functions. We study the interpolation properties, the approximation accuracy (degree of exactness, precision set) and the remainder of the corresponding approximation formulas. The accuracy is also illustrated by numerical examples.

1. **Introduction**

In some previous papers were constructed and applied some interpolation operators on triangle with one curved edge respectively on tetrahedron with straight edges ([1, 6, 7, 8, 9, 12]), as well as Bernstein-type operators on triangle with all straight edges, respectively on triangle with one curved edge ([4, 5]). There were studied the interpolation properties and the accuracy of these operators respectively the remainders of the corresponding approximation formulas.

The order of an approximation operator P is given by the degree of exactness ($\operatorname{dex}(P)$) and by the precision set ($\operatorname{pres}(P)$). Remind that $\operatorname{dex}(P) = r$ if $Pf = f$ for all $f \in \mathcal{P}_r^n$ and there exists $g \in \mathcal{P}_{r+1}^n$ such that $Pg \neq g$, where $\mathcal{P}_r^n$ denotes the space of polynomials in n variables of global degree at most r. The precision set of an approximation operator is the set of all monomials for which the approximation is exact [2].

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The goal of this paper is to study Bernstein-type operators on tetrahedrons with straight edges respectively with three curved edges given by functions.

2. Bernstein-type operators on tetrahedrons with straight edges

By affine invariance it is sufficient to consider only the standard tetrahedron $\mathcal{T}_h$ with vertices $V_0 = (0,0,0)$, $V_1 = (h,0,0)$, $V_2 = (0,h,0)$ and $V_3 = (0,0,h)$, with three edges τ_1, τ_2, τ_3 along the coordinate axes and with the edges Γ_1, Γ_2, Γ_3 (opposite to the vertex V_0). Also, one denotes by σ_{012}, σ_{013}, σ_{023} and σ_{123} the tetrahedron faces from the planes $V_0V_1V_2$, $V_0V_1V_3$, $V_0V_2V_3$ and $V_1V_2V_3$ respectively (see the left side of Figure 1).

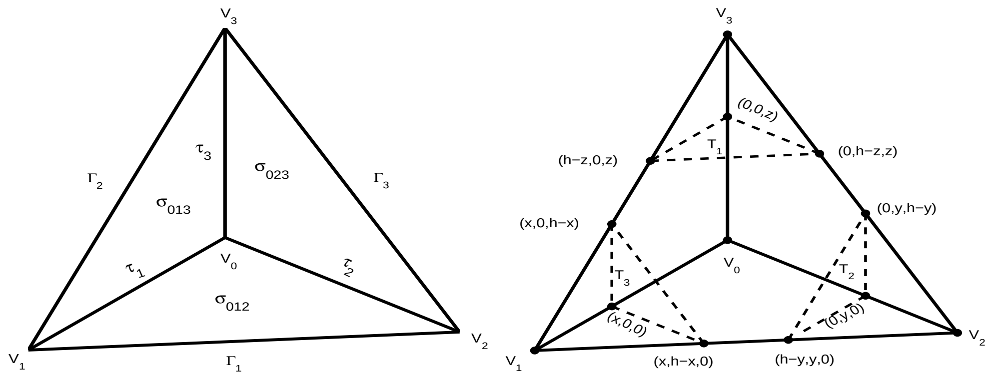

FIGURE 1. Tetrahedron with straight edges

Let Π_i, $i = 1,2,3$, be the parallel planes to the tetrahedron faces that intersect the tetrahedron edges in three points and T_i, $i = 1,2,3$, be the triangles in which the planes Π_i, $i = 1,2,3$, intersect the tetrahedron faces respectively (see the right side of Figure 1).

2.1. Univariate operators.

On each triangle one defines two Bernstein-type operators.

Remark 1. *We shall study, in detail, only the Bernstein-type operators on the triangle T_1. For the triangles T_2 and T_3 there are obtained analogous results.*

Let us consider the triangle T_1 (see Figure 2).

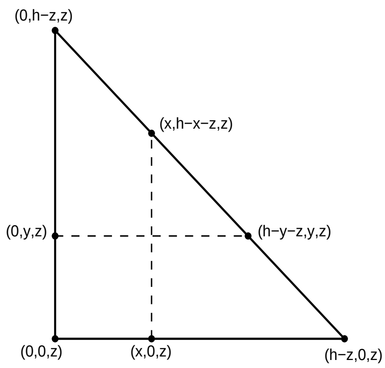

FIGURE 2. Triangle T_1

For the uniform partitions

$$\Delta_m^x = \left\{ \left(i\frac{h-y-z}{m}, y, z \right) \;\middle|\; i = \overline{0,m} \right\}$$

and

$$\Delta_n^y = \left\{ \left(x, j\frac{h-x-z}{n}, z \right) \;\middle|\; j = \overline{0,n} \right\},$$

of the intervals $[(0,y,z),(h-y-z,y,z)]$ $[(x,0,z),(x,h-x-z,z)]$ respectively, one considers the Bernstein-type operators B_m^{xy} and B_n^{yx} defined by

$$\left(B_m^{xy}F\right)(x,y,z) = \sum_{i=0}^{m} p_{m,i}(x,y,z)\, F\left(i\frac{h-y-z}{m}, y, z \right)$$

with

$$p_{m,i}(x,y,z) = \binom{m}{i} \left(\frac{x}{h-y-z} \right)^i \left(1 - \frac{x}{h-y-z} \right)^{m-i}$$

and

$$\left(B_n^{yx}F\right)(x,y,z) = \sum_{j=0}^{n} q_{n,j}(x,y,z)\, F\left(x, j\frac{h-x-z}{n}, z \right)$$

with

$$q_{n,j}\left(x,y,z\right)=\binom{n}{j}\left(\frac{y}{h-x-z}\right)^{j}\left(1-\frac{y}{h-x-z}\right)^{n-j},$$

where F is a real-valued function defined on $\mathcal{T}_h$.

Theorem 2.1. *If $F\colon \mathcal{T}_h\to\mathbb{R}$ then:*

(i) $B_m^{xy}F=F$ *on* $\sigma_{023}\cup\sigma_{123}$,

$B_n^{yx}F=F$ *on* $\sigma_{013}\cup\sigma_{123}$;

(ii) $\operatorname{dex}\left(B_m^{xy}\right)=\operatorname{dex}\left(B_n^{yx}\right)=1$;

(iii) $\operatorname{pres}\left(B_m^{xy}\right)=\left\{x^iy^jz^k \mid i=0,1; j,k\in\mathbb{N}\right\}$,

$\operatorname{pres}\left(B_n^{yx}\right)=\left\{x^iy^jz^k \mid j=0,1; i,k\in\mathbb{N}\right\}$;

(iv) $\left(B_m^{xy}e_{2jk}\right)(x,y,z)=\left[x^2+\dfrac{x\left(h-x-y-z\right)}{m}\right]y^jz^k$,

$\left(B_n^{yx}e_{i2k}\right)(x,y,z)=\left[y^2+\dfrac{y\left(h-x-y-z\right)}{n}\right]x^iz^k,\quad i,j,k\in\mathbb{N}.$

Proof. The relations

$$\begin{aligned}
p_{m,i}\left(0,y,z\right)&=\begin{cases}1, & \text{for } i=0,\\[2ex] 0, & \text{for } i>0;\end{cases}\\
p_{m,i}\left(h-y-z,y,z\right)&=\begin{cases}1, & \text{for } i=m,\\[2ex] 0, & \text{for } i<m;\end{cases}
\end{aligned}\tag{1}$$

respectively

$$\begin{aligned}
q_{n,j}\left(x,0,z\right)&=\begin{cases}1, & \text{for } j=0,\\[2ex] 0, & \text{for } j>0;\end{cases}\\
q_{n,j}\left(x,h-x-z,z\right)&=\begin{cases}1, & \text{for } j=n,\\[2ex] 0, & \text{for } j<n;\end{cases}
\end{aligned}\tag{2}$$

imply that

$$\left(B_m^{xy}F\right)\left(0,y,z\right)=F\left(0,y,z\right),$$

$$\left(B_m^{xy}F\right)\left(h-y-z,y,z\right)=F\left(h-y-z,y,z\right),$$

and

$$\begin{aligned}
\left(B_n^{yx}F\right)(x,0,y) &= F(x,0,z),\\
\left(B_n^{yx}F\right)(x,h-x-z,z) &= F(x,h-x-z,z),
\end{aligned}$$

i.e., the interpolation properties (i). Regarding the approximation accuracy, we have

$$\begin{aligned}
\left(B_m^{xy}e_{000}\right)(x,y,z) &= \sum_{i=0}^{m} p_{m,i}(x,y,z) = 1,\\
\left(B_m^{xy}e_{100}\right)(x,y,z) &= \sum_{i=0}^{m} \binom{m}{i} \left(\frac{x}{h-y-z}\right)^{i} \left(1-\frac{x}{h-y-z}\right)^{m-i} \frac{i\,(h-y-z)}{m}\\
&= x\sum_{i=0}^{m-1} \binom{m-1}{i} \left(\frac{x}{h-y-z}\right)^{i} \left(1-\frac{x}{h-y-z}\right)^{m-1-i} = x,\\
\left(B_m^{xy}e_{200}\right)(x,y,z) &= \sum_{i=0}^{m} \binom{m}{i} \left(\frac{x}{h-y-z}\right)^{i} \left(1-\frac{x}{h-y-z}\right)^{m-i} i^2 \left(\frac{h-y-z}{m}\right)^2\\
= \left(\frac{h-y-z}{m}\right)^2 & \sum_{i=2}^{m} \binom{m}{i} \left(\frac{x}{h-y-z}\right)^{i} \left(1-\frac{x}{h-y-z}\right)^{m-i} + \frac{x\,(h-y-z)}{m}\\
&= \frac{m-1}{m}x^2 + \frac{x\,(h-y-z)}{m} = x^2 + \frac{x\,(h-x-y-z)}{m},\\
\left(B_m^{xy}e_{ijk}\right)(x,y,z) &= y^j z^k \left(B_m^{xy}e_{i00}\right)(x,y,z), \quad i=0,1,2,\ j,k\in\mathbb{N},
\end{aligned}$$

respectively

$$\begin{aligned}
\left(B_n^{yx}e_{000}\right)(x,y,z) &= \sum_{j=0}^{n} q_{n,j}(x,y,z) = 1,\\
\left(B_n^{yx}e_{010}\right)(x,y,z) &= y,\\
\left(B_n^{yx}e_{i2k}\right)(x,y,z) &= y^2 + \frac{y\,(h-x-y-z)}{n},\\
\left(B_n^{yx}e_{ijk}\right)(x,y,z) &= x^i z^k \left(B_n^{yx}e_{0j0}\right)(x,y,z), \quad j=0,1,2,\ i,k\in\mathbb{N},
\end{aligned}$$

that are proved in the same way, which imply (ii)-(iv). $\square$

Let

$$F = B_m^{xy}F + R_m^{xy}F$$

be the approximation formula generated by the operator B_m^{xy}.

Theorem 2.2. *If* $F(\cdot, y, z) \in C[0, h-y-z]$ *then*

$$\left|(R_m^{xy}F)(x,y,z)\right| \leqslant \left(1+\frac{h}{2\delta\sqrt{m}}\right)\omega\left(F(\cdot,y,z);\delta\right), \quad y+z \leqslant h,$$

where $\omega(F(\cdot,y,z);\delta)$ *is the modulus of continuity of the function* F *with regard to the variable* x.

Moreover, if $\delta = 1/\sqrt{m}$ *then*

$$\left|(R_m^{xy}F)(x,y,z)\right| \leqslant \left(1+\frac{h}{2}\right)\omega\left(F(\cdot,y,z);\frac{1}{\sqrt{m}}\right). \tag{3}$$

Proof. We have

$$\begin{aligned}\left|(R_m^{xy}F)(x,y,z)\right| &\leqslant \sum_{i=0}^{m} p_{m,i}(x,y,z)\left|F(x,y,z)-F\left(i\frac{h-y-z}{m},y,z\right)\right| \\ &\leqslant \sum_{i=0}^{m} p_{m,i}(x,y,z)\left(\frac{1}{\delta}\left|x-i\frac{h-y-z}{m}\right|+1\right)\omega(F(\cdot,y,z);\delta) \\ &\leqslant \left[1+\frac{1}{\delta}\left(\sum_{i=0}^{m} p_{m,i}(x,y,z)\left(x-i\frac{h-y-z}{m}\right)^2\right)^{1/2}\right]\omega(F(\cdot,y,z);\delta) \\ &\leqslant \left[1+\frac{1}{\delta}\sqrt{\frac{x(h-x-y-z)}{m}}\right]\omega(F(\cdot,y,z);\delta).\end{aligned}$$

Since,

$$\max_{T_1}\Big[x(h-x-y-z)\Big] \leqslant \frac{h^2}{4}, \quad z \in [0,h], \tag{4}$$

we have

$$\left|(R_m^{xy}F)(x,y,z)\right| \leqslant \left(1+\frac{h}{2\delta\sqrt{m}}\right)\omega(F(\cdot,y,z);\delta)$$

respectively (for $\delta = 1/\sqrt{m}$)

$$\left|(R_m^{xy}F)(x,y,z)\right| \leqslant \left(1+\frac{h}{2}\right)\omega\left(F(\cdot,y,z);\frac{1}{\sqrt{m}}\right). \qquad \square$$

We also have

$$\left|(R_n^{yx}F)(x,y,z)\right| \leqslant \left(1+\frac{h}{2}\right)\omega\left(F(x,\cdot,z);\frac{1}{\sqrt{n}}\right). \tag{5}$$

Theorem 2.3. *If* $F(\cdot,y,z) \in C^2[0,h]$ *then*

$$(R_m^{xy}F)(x,y,z) = -\frac{x(h-x-y-z)}{2m}F^{(2,0,0)}(\xi,y,z),$$

for $0 \leqslant \xi \leqslant h-y-z;\ y,z \in [0,h]$, *and*

$$\left|(R_m^{xy}F)(x,y,z)\right| \leqslant \frac{h^2}{8m} M_{200}F, \tag{6}$$

where

$$M_{ijk}F = \max_{\mathcal{T}_h}\left|F^{(i,j,k)}(x,y,z)\right|.$$

Proof. Since $\operatorname{dex}(B_m^{xy}) = 1$, by Peano's kernel theorem, follows that

$$(R_m^{xy}F)(x,y,z) = \int_0^{h-y-z} K_{200}(x,y,z;s)\, F^{(2,0,0)}(s,y,z)\, ds,$$

where the kernel

$$K_{200}(x,y,z;s) = (x-s)_+ - \sum_{i=0}^{m} p_{m,i}(x,y,z)\left(i\frac{h-y-z}{m} - s\right)_+$$

does not change the sign $\big(K_{200}(x,y,z;s) \leqslant 0,\ s \in [0,h-y-z]\big)$. By mean value theorem, one obtains

$$\begin{aligned}(R_m^{xy}F)(x,y,z) &= F^{(2,0,0)}(\xi,y,z)\int_0^{h-y-z} K_{200}(x,y,z;s)\, ds \\ &= -\frac{x(h-x-y-z)}{2m} F^{(2,0,0)}(\xi,y,z),\ 0 \leqslant \xi \leqslant h-y-z.\end{aligned}$$

Now, the inequality of (4) implies (6). □

Remark 2. *On the same way it is proved the evaluations of the remainder in the formula*

$$F = B_n^{yx}F + R_n^{yx}F$$

i.e., for $F(x,\cdot,z) \in C[0,h-x-z]$

$$\left|(R_n^{yx}F)(x,y,z)\right| \leqslant \left(1+\frac{h}{2}\right)\omega\left(F(x,\cdot,z);\frac{1}{\sqrt{n}}\right) \tag{7}$$

respectively, for $F(x,\cdot,z) \in C^2[0,h]$

$$\left|(R_n^{yx}F)(x,y,z)\right| \leqslant \frac{h^2}{8n} M_{020}F \tag{8}$$

on $\mathcal{T}_h$.

2.2. **Product operators.** Let $P^1_{mn} = B^{xy}_m B^{yx}_n$ and $Q^1_{nm} = B^{yx}_n B^{xy}_m$ be the products of the operators B^{xy}_m and B^{yx}_n.

We have

$$\left(P^1_{mn}F\right)(x,y,z) = \sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y,z)\, q_{n,j}\left(i\frac{h-y-z}{m}, y, z\right) \times$$
$$\times F\left(i\frac{h-y-z}{m}, j\frac{(m-i)(h-z)+iy}{mn}, z\right),$$

respectively

$$\left(Q^1_{nm}F\right)(x,y,z) = \sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}\left(x, j\frac{h-x-z}{n}, z\right) q_{n,j}(x,y,z) \times$$
$$\times F\left(i\frac{(n-j)(h-z)+jx}{mn}, j\frac{h-x-z}{n}, z\right).$$

Theorem 2.4. *If F is a real-valued function defined on $\mathcal{T}_h$ then*

$$P^1_{mn}F = F \tag{9}$$

and

$$Q^1_{nm}F = F \tag{10}$$

on $\tau_3 \cup \sigma_{123}$.

Proof. Taking into account (1) and (2), one obtains

$$\left(P^1_{mn}F\right)(0,0,z) = F(0,0,z),$$
$$\left(P^1_{mn}F\right)(h-y-z,y,z) = F(h-y-z,y,z),$$

respectively

$$\left(Q^1_{nm}F\right)(0,0,z) = F(0,0,z),$$
$$\left(Q^1_{nm}F\right)(h-y-z,y,z) = F(h-y-z,y,z),$$

for all $y, z \in [0,h]$. □

For the approximation error of the operators P^1_{mn} and Q^1_{nm}, we have the following theorem.

Theorem 2.5. *If* $F(\cdot,\cdot,z) \in C([0,h]\times[0,h])$ *then*

$$\left|\left(F - P_{mn}^{1}F\right)(x,y,z)\right| \leqslant (1+h)\,\omega\left(F(\cdot,\cdot,z);\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right) \tag{11}$$

and

$$\left|\left(F - Q_{nm}^{1}F\right)(x,y,z)\right| \leqslant (1+h)\,\omega\left(F(\cdot,\cdot,z);\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right) \tag{12}$$

on $\mathcal{T}_h$.

Proof. We have

$$\begin{aligned}\left|\left(F - P_{mn}^{1}F\right)(x,y,z)\right| \leqslant & \Bigg[\frac{1}{\delta_1}\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y,z)\,q_{n,j}\left(i\frac{h-y-z}{m},y,z\right)\times \\ & \qquad\times\left|x - i\frac{h-y-z}{m}\right| \\ & + \frac{1}{\delta_1}\sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y,z)\,q_{n,j}\left(i\frac{h-y-z}{m},y,z\right)\times \\ & \qquad\times\left|y - j\frac{(m-i)(h-z)+iy}{mn}\right| \\ & + \sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y,z)\,q_{n,j}\left(i\frac{h-y-z}{m},y,z\right)\Bigg]\omega\left(F(\cdot,\cdot,z);\delta_1,\delta_2\right) \\ \leqslant & \left(\frac{1}{\delta_1}\sqrt{\frac{x(h-x-y-z)}{m}} + \frac{1}{\delta_2}\sqrt{\frac{y(h-x-y-z)}{n}} + 1\right)\omega\left(F(\cdot,\cdot,z);\delta_1,\delta_2\right).\end{aligned}$$

As,

$$\begin{aligned} x(h-x-y-z) &\leqslant \frac{(h-y-z)^2}{4} \quad \text{on} \quad [0,h-y-z], \\ y(h-x-y-z) &\leqslant \frac{(h-x-z)^2}{4} \quad \text{on} \quad [0,h-x-z], \end{aligned}$$

one obtains

$$\begin{aligned}\left|\left(F - P_{mn}^{1}F\right)(x,y,z)\right| &\leqslant \left(\frac{1}{\delta_1}\frac{h-y-z}{2\sqrt{m}} + \frac{1}{\delta_2}\frac{h-x-z}{2\sqrt{n}} + 1\right)\omega\left(F(\cdot,\cdot,z);\delta_1,\delta_2\right) \\ &\leqslant \left(\frac{1}{\delta_1}\frac{h}{2\sqrt{m}} + \frac{1}{\delta_2}\frac{h}{2\sqrt{n}} + 1\right)\omega\left(F(\cdot,\cdot,z);\delta_1,\delta_2\right).\end{aligned}$$

Now, for $\delta_1 = 1/\sqrt{m}$ and $\delta_2 = 1/\sqrt{n}$, one obtains (11). The inequality (12) is proved in the same way. □

2.3. **Boolean sum operators.** Let

$$S_{mn}^{1} := B_{m}^{xy} \oplus B_{n}^{yx} = B_{m}^{xy} + B_{n}^{yx} - B_{m}^{xy} B_{n}^{yx} \tag{13}$$

and

$$T_{nm}^{1} := B_{n}^{yx} \oplus B_{m}^{xy} = B_{n}^{yx} + B_{m}^{xy} - B_{n}^{yx} B_{m}^{xy} \tag{14}$$

be the Boolean sums of the operators B_{m}^{xy} and B_{n}^{yx}.

Theorem 2.6. *If F is a real-valued function defined on $\mathcal{T}_h$ then*

$$S_{mn}^{1} F = F \quad \text{and} \quad T_{nm}^{1} F = F$$

on $\sigma_{013} \cup \sigma_{023} \cup \sigma_{123}$.

Proof. We have:

$$\left(B_{m}^{xy} F\right)(0, y, z) = F(0, y, z), \quad \left(P_{mn}^{1} F\right)(0, y, z) = \left(B_{n}^{yx} F\right)(0, y, z)$$

which imply that

$$S_{mn}^{1} F = F \quad \text{on} \quad \sigma_{023};$$

$$\left(B_{n}^{yx} F\right)(x, 0, z) = F(x, 0, z), \quad \left(P_{mn}^{1} F\right)(x, 0, z) = \left(B_{m}^{xy} F\right)(x, 0, z)$$

which imply that

$$S_{mn}^{1} F = F \quad \text{on} \quad \sigma_{013};$$

and

$$B_{m}^{xy} F = F, \quad B_{n}^{yx} F = F, \quad P_{mn}^{1} F = F, \quad \text{on} \quad \sigma_{123}$$

which imply that

$$S_{mn}^{1} F = F \quad \text{on} \quad \sigma_{123}.$$

Analogously, it is proved that $T_{nm}^{1} F = F$ on $\sigma_{013} \cup \sigma_{023} \cup \sigma_{123}$. $\square$

Theorem 2.7. *If $F \in C(\mathcal{T}_h)$ then*

$$\left|\left(F - S_{mn}^{1} F\right)(x, y, z)\right| \leqslant \left(1 + \frac{h}{2}\right) \omega\left(F(\cdot, y, z); \frac{1}{\sqrt{m}}\right) + \\ + \left(1 + \frac{h}{2}\right) \omega\left(F(x, \cdot, z); \frac{1}{\sqrt{n}}\right) + (1 + h)\, \omega\left(F(\cdot, \cdot, z); \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}}\right)$$

on $\mathcal{T}_h$.

Proof. From the identity

$$F - S^1_{mn}F = (F - B^{xy}_m F) + (F - B^{yx}_n F) - \big(F - P^1_{mn}F\big)$$

one obtains

$$\begin{aligned}\left|\big(F - S^1_{mn}F\big)(x,y,z)\right| \leqslant \left|(R^{xy}_m F)(x,y,z)\right| + \left|(R^{yx}_n F)(x,y,z)\right| \\ + \left|\big(F - P^1_{mn}F\big)(x,y,z)\right|\end{aligned}$$

and from (3), (5), (11), the proof follows. □

Remark 3. *The same inequality is obtained for the error* $\big(F - T^1_{nm}F\big)(x,y,z)$ *using instead of* (11) *the inequality* (12).

3. **Bernstein-type operators on tetrahedrons with three curved edges**

One considers, also, the standard tetrahedron $\mathcal{T}_h$ with vertices $V_0 = (0,0,0)$, $V_1 = (h,0,0)$, $V_2 = (0,h,0)$ and $V_3 = (0,0,h)$, with three straight edges τ_1, τ_2, τ_3 along the coordinate axes and with three curved edges γ_1, γ_2, γ_3 (opposite to the vertex V_0), defined, respectively, by the one-to-one functions f_i and g_i, where g_i is the inverse of the function f_i, $i = 1,2,3$. Also, one denotes by s_{012}, s_{013}, s_{023} and the tetrahedron faces from the planes $V_0V_1V_2$, $V_0V_1V_3$, $V_0V_2V_3$ and $V_1V_2V_3$ respectively, by s_{123} the curved faced (opposite to the vertex V_0) (see the left side of Figure 3) and by t_i, $i = 1,2,3$, the triangles with one curved edge in which the planes Π_i, $i = 1,2,3$, intersect the faces of the tetrahedron $\mathcal{T}_h$, respectively (see left side of Figure 3).

Next, one considers the particular case when the face s_{123} is on the sphere $x^2 + y^2 + z^2 = h^2$, i.e., $f_i(u) = \sqrt{h^2 - u^2}$ and $g_i(v) = \sqrt{h^2 - v^2}$, $i = 1,2,3$ (see right side of Figure 3)

3.1. **Univariate operators.**

One each triangle t_i, $i = 1,2,3$, one defines two Bernstein-type operators.

We discuss here only on the triangle t_1 (Figure 4).

We have

$$(B^{xy}_m F)(x,y,z) = \sum_{i=0}^{m} p_{m,i}(x,y,z)\, F\left(i\frac{\sqrt{h^2 - y^2 - z^2}}{m}, y, z\right)$$

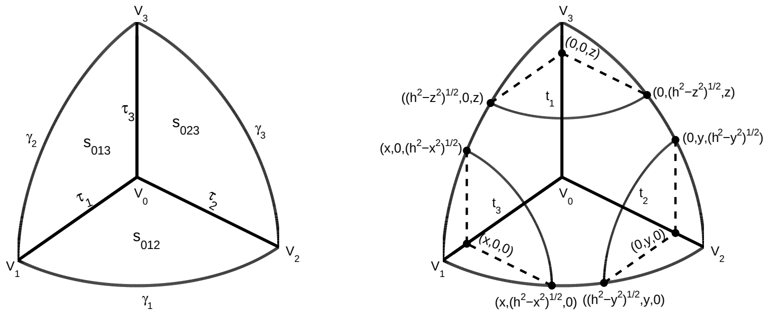

FIGURE 3. Tetrahedron with three curved edges

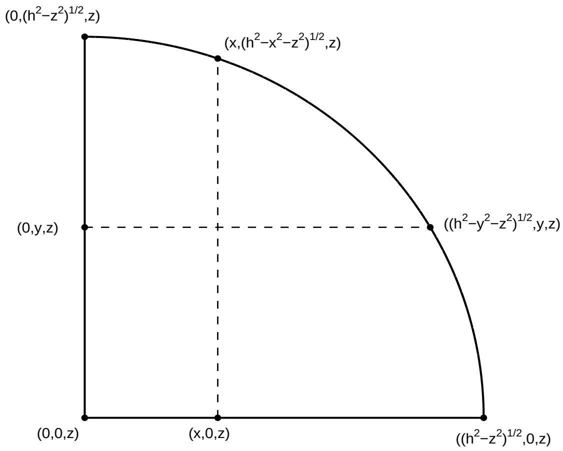

FIGURE 4. Triangle t_1

and

$$\left(B_n^{yx}F\right)(x,y,z)=\sum_{j=0}^{n}q_{n,j}\left(x,y,z\right)F\left(x,j\frac{\sqrt{h^2-x^2-z^2}}{n},z\right)$$

with

$$p_{m,i}\left(x,y,z\right)=\binom{m}{i}\left(\frac{x}{\sqrt{h^2-y^2-z^2}}\right)^{i}\left(1-\frac{x}{\sqrt{h^2-y^2-z^2}}\right)^{m-i},$$

respectively

$$q_{n,j}(x,y,z)=\binom{n}{j}\left(\frac{y}{\sqrt{h^2-x^2-z^2}}\right)^j\left(1-\frac{x}{\sqrt{h^2-x^2-z^2}}\right)^{n-j},$$

where F is a real-valued function defined on $\mathcal{T}_h$.

Following the way used in the Section 2 one can prove the corresponding theorems:

Theorem 3.1. *If $F:\mathcal{T}_h\to\mathbb{R}$ then:*

(i') $B_m^{xy}F=F$ *on* $s_{023}\cup s_{123}$,
$B_n^{yx}F=F$ *on* $s_{013}\cup s_{123}$;

(ii') $\operatorname{dex}(B_m^{xy})=\operatorname{dex}(B_n^{yx})=1$;

(iii') $\operatorname{pres}(B_m^{xy})=\left\{x^iy^jz^k \mid i=0,1;j,k\in\mathbb{N}\right\}$,
$\operatorname{pres}(B_n^{yx})=\left\{x^iy^jz^k \mid j=0,1;i,k\in\mathbb{N}\right\}$;

(iv') $$(B_m^{xy}e_{2jk})(x,y,z)=\left[x^2+\frac{x\left(\sqrt{h^2-y^2-z^2}-x\right)}{m}\right]x^iz^k,$$
$$(B_n^{yx}e_{i2k})(x,y,z)=\left[y^2+\frac{y\left(\sqrt{h^2-x^2-z^2}-y\right)}{n}\right]x^iz^k,\ i,j,k\in\mathbb{N}.$$

Let

$$F=B_m^{xy}F+R_m^{xy}F$$

be the approximation formula generated by the operator B_m^{xy}.

Theorem 3.2. *If $F(\cdot,y,z)\in C\left[0,\sqrt{h^2-y^2-z^2}\right]$ then*

$$\left|(R_m^{xy}F)(x,y,z)\right|\leqslant\left(1+\frac{h}{2\delta\sqrt{m}}\right)\omega\left(F(\cdot,y,z);\delta\right),\quad y+z\leqslant h,$$

respectively

$$\left|(R_m^{xy}F)(x,y,z)\right|\leqslant\left(1+\frac{h}{2}\right)\omega\left(F(\cdot,y,z);\frac{1}{\sqrt{m}}\right).$$

Theorem 3.3. *If $F(\cdot,y,z)\in C^2[0,h]$ then*

$$\left|(R_m^{xy}F)(x,y,z)\right|=-\frac{x\left(\sqrt{h^2-y^2-z^2}-x\right)}{2m}F^{(2,0,0)}(\xi,y,z),$$

for $0\leqslant\xi\leqslant\sqrt{h^2-y^2-z^2}$, $y,z\in[0,h]$, and

$$\left|(R_m^{xy}F)(x,y,z)\right|\leqslant\frac{h^2}{8m}M_{200}F.$$

Remark 4. *Analogous results take place for the remainder in the approximation formula*

$$F = B_n^{yx} F + R_n^{yx} F.$$

3.2. **Product operators.** Let $P_{mn} = B_m^{xy} B_n^{yx}$ and $Q_{nm} = B_n^{yx} B_m^{xy}$ be the products of the operators B_m^{xy} and B_n^{yx}, i.e.,

$$(P_{mn}F)(x,y,z) = \sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}(x,y,z)\, q_{n,j}\left(i\frac{\sqrt{h^2-y^2-z^2}}{m}, y, z\right)\times$$
$$\times F\left(i\frac{\sqrt{h^2-y^2-z^2}}{m}, j\frac{\sqrt{(m^2-i^2)(h^2-z^2)+i^2y^2}}{mn}, z\right),$$

respectively

$$(Q_{nm}F)(x,y,z) = \sum_{i=0}^{m}\sum_{j=0}^{n} p_{m,i}\left(x, j\frac{\sqrt{h^2-x^2-z^2}}{n}, z\right) q_{n,j}(x,y,z)\times$$
$$\times F\left(i\frac{\sqrt{(n^2-j^2)(h^2-z^2)+j^2x^2}}{mn}, j\frac{\sqrt{h^2-x^2-z^2}}{n}, z\right).$$

Theorem 3.4. *If* $F : \mathcal{T}_h \to \mathbb{R}$ *then*

$$P_{mn}F = F \quad \textit{and} \quad Q_{nm}F = F \quad \textit{on} \quad \tau_3 \cup s_{123}.$$

Theorem 3.5. *If* $F(\cdot,\cdot,z) \in C([0,h]\times[0,h])$*, then*

$$\left|(F - P_{mn}F)(x,y,z)\right| \leqslant (1+h)\,\omega\left(F(\cdot,\cdot,z); \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}}\right)$$

and

$$\left|(F - Q_{nm}F)(x,y,z)\right| \leqslant (1+h)\,\omega\left(F(\cdot,\cdot,z); \frac{1}{\sqrt{m}}, \frac{1}{\sqrt{n}}\right).$$

3.3. **Boolean sum operators.** If $S_{mn} = B_m^{xy} \oplus B_n^{yx}$ and $T_{nm} = B_n^{yx} \oplus B_m^{xy}$ are the Boolean sums of the operators B_m^{xy} and B_n^{yx}, then we have:

Theorem 3.6. *If* $F : \mathcal{T}_h \to \mathbb{R}$ *then*

$$S_{mn}F = F \quad \textit{and} \quad T_{nm}F = F \quad \textit{on} \quad s_{013} \cup s_{023} \cup s_{123}.$$

Theorem 3.7. *If $F \in C(\mathcal{T}_h)$ then*

$$\begin{aligned}\left|(F - S_{mn}F)(x,y,z)\right| \leqslant{} & \left(1+\frac{h}{2}\right)\omega\left(F(\cdot,y,z);\frac{1}{\sqrt{m}}\right)\\ &+\left(1+\frac{h}{2}\right)\omega\left(F(x,\cdot,z);\frac{1}{\sqrt{n}}\right)\\ &+\Big(1+h\Big)\omega\left(F(\cdot,\cdot,z);\frac{1}{\sqrt{m}},\frac{1}{\sqrt{n}}\right),\end{aligned}$$

and a similar inequality holds for the error $F - T_{nm}F$.

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