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THE CHARACTERS OF THE BLASCHKE-GROUP OF THE ARITHMETIC FIELD

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Abstract. We consider a locally compact metric space, $\mathbb{B}$ with arithmetic addition and multiplication, which is closely related to the usual multiplication of real numbers in the dyadic system. This results a non-Archimedian local field, the so-called 2-adic local field. Some orthogonal series are studied with respect the inner product defined with the Haar-measure μ. The Blaschke-functions defined on the 2-adic field, $B_a(x) = \frac{x \overset{\bullet}{+} a}{e \overset{\bullet}{+} a \bullet x}$ form a commutative group with respect to the function composition, the so-called Blaschke-group. We shall determine the characters of this group. By means of the exponential and tangent functions on the 2-adic field and the characters of its additive group we can identify the desired characters. We consider Fourier-series with respect to these characters and summability questions are examined. A simple recursion leads to the FFT-algorithm, the so-called Fast-Fourier Transform.

1. **Introduction**

According to Volovich[4] some non-Archimedean normed fields must be used for a global space-time theory in order to unify both microscopic and macroscopic physics. Some problems occured with the practical applications of the classical fields $\mathbb{R}$ and $\mathbb{C}$, because in sciences there are absolute limitations on measurements like Plank time, Plank length, Plank mass, and also there is a problem with the Archimedean axiom on the microscopic level. Volovich proposes to base physics on a coalition of

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non-Archimedean normed fields and classical fields as $\mathbb{R}$ or $\mathbb{C}$. The so-called p-adic field is a suitable non-Archimedean normed field. As $p \to \infty$, many of the fundamental functions of p-adic analysis approach their counterparts in classical analysis. Thus p-adic analysis could provide a bridge from microscopic to macroscopic physics. The simplest example of a p-adic field is the 2-adic field used in this paper.

Characters are very useful in numerous branches of mathematics, for example in many cases are used Fourier-series with respect to characters.

Denote by $\mathbb{A} := \{0, 1\}$ the set of bits and by

$$\mathbb{B} := \{a = (a_j, j \in \mathbb{Z}) \mid a_j \in \mathbb{A} \text{ and } \lim_{j \to -\infty} a_j = 0\} \tag{1}$$

the set of bytes. The numbers a_j are called the additive digits of $a \in \mathbb{B}$. The zero element of $\mathbb{B}$ is $\theta := (x_j, j \in \mathbb{Z})$ where $x_j = 0$ for $j \in \mathbb{Z}$, that is, $\theta = (\cdots, 0, 0, 0, \cdots)$. The order of a byte $x \in \mathbb{B}$ is defined in the following way: For $x \neq \theta$ let $\pi(x) = n$ if and only if $x_n = 1$ and $x_j = 0$ for all $j < n$, furthermore set $\pi(\theta) = +\infty$. The norm of a byte x is defined by

$$\|x\| := 2^{-\pi(x)} \text{ for } x \in \mathbb{B} \setminus \{\theta\}, \qquad \text{and } \|\theta\| := 0. \tag{2}$$

The sets $I_n(x) := \{y \in \mathbb{B} : y_k = x_k \text{ for } k < n\}$, the so-called intervals in $\mathbb{B}$ of rank $n \in \mathbb{Z}$ and center x are of basic importance. Set $\mathbb{I}_n := I_n(\theta) = \{x \in \mathbb{B} : \|x\| \leqq 2^{-n}\}$ for any $n \in \mathbb{Z}$. The unit ball $\mathbb{I} := \mathbb{I}_0$ can be identified with the set of sequences $\mathbb{I} = \{a = (a_j, j \in \mathbb{N}) \mid a_j \in \mathbb{A}\}$ via the map $(\ldots, 0, 0, a_0, a_1, \ldots) \mapsto (a_0, a_1, \ldots)$. Furthermore $\mathbb{S} := \{x \in \mathbb{B} : \|x\| = 1\} = \{x \in \mathbb{B} : \pi(x) = 0\} = \{x \in \mathbb{I} : x_0 = 1\}$ is the unit sphere of the field.

We will use the normalized Haar-measure on $\mathbb{B}$, which satisfies $\mu(I_n(a)) := 2^{-n}$. Some orthogonal series are studied with respect the inner product defined with the Haar-measure μ by

$$\langle f, g \rangle := \int_{\mathbb{I}} f(x)\overline{g(x)} d\mu(x).$$

Now, consider the 2-adic (or arithmetical) sum $a \overset{\bullet}{+} b$ of elements $a = (a_n, n \in \mathbb{Z}), b = (b_n, n \in \mathbb{Z}) \in \mathbb{B}$, defined by

$$a \overset{\bullet}{+} b := (s_n, n \in \mathbb{Z})$$

where the bits $q_n, s_n \in \mathbb{A}$ $(n \in \mathbb{Z})$ are obtained recursively as follows:

$$\begin{aligned} &q_n = s_n = 0 \quad \text{for } n < m := \min\{\pi(a), \pi(b)\}, \\ &\text{and } \ a_n + b_n + q_{n-1} = 2q_n + s_n \quad \text{for } n \geq m. \end{aligned} \tag{3}$$

The 2-adic (or arithmetical) product of $a, b \in \mathbb{B}$ is $a \bullet b := (p_n, n \in \mathbb{Z})$, where the sequences $q_n \in \mathbb{N}$ and $p_n \in \mathbb{A}$ $(n \in \mathbb{Z})$ are defined recursively by

$$\begin{aligned} &q_n = p_n = 0 \quad (n < m := \pi(a) + \pi(b)) \\ &\text{and } \sum_{j=-\infty}^{\infty} a_j b_{n-j} + q_{n-1} = 2q_n + p_n \quad (n \geq m). \end{aligned} \tag{4}$$

The reflection x^- of a byte $x = (x_j, j \in \mathbb{Z})$ is defined by its additive digits:

$$(x^-)_j = \begin{cases} x_j, & \text{for } j \leqq \pi(x) \\ 1 - x_j, & \text{for } j > \pi(x). \end{cases}$$

Note, that x^- is the additive inverse of an $x \in \mathbb{B}$.

The operations $\overset{\bullet}{+}, \bullet$ are commutative. Notice, that

$$\pi(a \bullet b) = \pi(a) + \pi(b). \tag{5}$$

Moreover, $(\mathbb{B}, \overset{\bullet}{+}, \bullet)$ is a non-Archimedian normed field with respect the (2) norm, that is,

$$\|x \overset{\bullet}{+} y\| \leq \max\{\|x\|, \|y\|\}, \qquad \|x \bullet y\| = \|x\| \cdot \|y\| \tag{6}$$

with equality if and only if $\|x\| \neq \|y\|$ See [2]. The operations $\overset{\bullet}{+}, \bullet$ are continuous with respect the metric introduced by the norm (2), that is, $(\mathbb{B}, \overset{\bullet}{+}, \bullet)$ is a topological field. $(\mathbb{S}, \bullet)$ is a subgroup of $(\mathbb{B}, \bullet)$.

We will use the following notation: $a \overset{\bullet}{-} b := a \overset{\bullet}{+} b^-$.

The multiplicative identity of $\mathbb{B}$ is the element $e = e_0 = (\delta_{n0}, n \in \mathbb{Z})$, where δ_{nk} is the Kronecker-symbol. Furthermore we will use the elements $e_k := (\delta_{nk}, n \in \mathbb{Z})$

for some $k \in \mathbb{Z}$. We can observe, that $e_k \bullet e_m = e_{k+m}$ for all $k, m \in \mathbb{Z}$. In general, multiplication by e_k shifts bytes: $e_k \bullet a = (a_{n-k}, n \in \mathbb{Z})$. We will represent infinite products on this field by $\prod_{n=1}^{\infty} \bullet\, \alpha_j := \lim_{n\to\infty}(\alpha_1 \bullet \alpha_2 \bullet \cdots \bullet \alpha_n)$.

A character of a topological group $(\mathbb{G}, +)$ is a continuous function $\phi : \mathbb{G} \to \mathbb{C}$ which satisfies $|\phi(x)| = 1$ and $\phi(x + y) = \phi(x)\phi(y)$ for all $x, y \in \mathbb{G}$.

2. The characters of the Blaschke-group

For $x \in \mathbb{I}$ and $a \in \mathbb{I}_1$ we have by (6) and (5) that $e \overset{\bullet}{-} a \bullet x \neq \theta$, thus $e \overset{\bullet}{-} a \bullet x$ has a multiplicative inverse in $\mathbb{B}$. For $a \in \mathbb{I}_1$ define the Blaschke function on $\mathbb{I}$:

$$B_a(x) := (x \overset{\bullet}{-} a) \bullet (e \overset{\bullet}{-} a \bullet x)^{-1} = \frac{x \overset{\bullet}{-} a}{e \overset{\bullet}{-} a \bullet x}. \qquad (x \in \mathbb{I}) \tag{7}$$

The Blaschke function $B_a : \mathbb{I} \to \mathbb{I}$ is a bijection for any $a \in \mathbb{I}_1$. The composition of two Blaschke-functions is also a Blaschke-function: $B_a \circ B_b = B_c$ where $c = \frac{a \overset{\bullet}{+} b}{e \overset{\bullet}{+} a \bullet b}$ is also in $\mathbb{I}_1$ for $a, b \in \mathbb{I}_1$. Thus the maps B_a $(a \in \mathbb{I}_1)$ form a commutative group with respect to the function composition. See[3]. We will call

$$\mathcal{B} := \{B_a, a \in \mathbb{I}_1\} \tag{8}$$

the Blashke-group of the field $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$.

We will determine the characters of the Blashke-group $(\mathcal{B}, \circ)$, where $\circ$ denotes the function composition.

Using the notation $x \lhd y := \frac{x \overset{\bullet}{+} y}{e \overset{\bullet}{+} x \bullet y}$ $(x, y \in \mathbb{I}_1)$, the map

$$B : (\mathbb{I}_1, \lhd) \to (\mathcal{B}, \circ),\ a \mapsto B_a$$

is an isomorphism, which is continuous, consequently it is useful if we define the character group of $(\mathbb{I}_1, \lhd)$.

We already know the characters of $(\mathbb{I}_1, \overset{\bullet}{+})$ and for this reason it is suitable to find a continuous isomorphism from $(\mathbb{I}_1, \overset{\bullet}{+})$ onto $(\mathbb{I}_1, \lhd)$, that is a function γ satisfying the equation

$$\gamma(x \overset{\bullet}{+} y) = \frac{\gamma(x) \overset{\bullet}{+} \gamma(y)}{e \overset{\bullet}{+} \gamma(x) \bullet \gamma(y)}. \ (x, y \in \mathbb{I}_1) \tag{9}$$

This equation is the analogue of the function equation of the tangent function on $\mathbb{C}$, where the tangent function can be expressed by the exponential function in the following way:

$$\tan(x) = \frac{\exp(ix) - \exp(-ix)}{i(\exp(ix) + \exp(-ix))} = \frac{\exp(2ix) - 1}{i(\exp(2ix) + 1)}. \quad (x \in \mathbb{C})$$

Furthermore, we will use the function ζ, expressed in the following infinite product form:

$$\zeta(x) := \prod_{j=1}^{\infty} \bullet \, b_j^{x_j} \qquad (x = (x_j, j \in \mathbb{Z}) \in \mathbb{I}_1) \tag{10}$$

where

$$b_1 := e \overset{\bullet}{+} e_2, \ b_n := b_{n-1} \bullet b_{n-1} \qquad (n \geq 2). \tag{11}$$

We will call the function ζ the $(\mathbb{S}, \bullet)$-valued exponential function on $\mathbb{I}_1$, which is a continuous function satisfying the function-equation

$$\zeta(x \overset{\bullet}{+} y) = \zeta(x) \bullet \zeta(y) \qquad (x, y \in \mathbb{I}_1). \tag{12}$$

This function ζ satisfies indeed (12) on $\mathbb{I}_1$, which can be easily seen analogous to [2], pp 59-60, where we find in a way different basis $(b_n, n \geq 1)$. Since $b_n = e \overset{\bullet}{+} c_n$ $(n \geq 1)$ with $\pi(c_n) = n + 1$, the function ζ has the following representation:

$$\zeta(x) = \prod_{j=1}^{\infty} \bullet \, (e \overset{\bullet}{+} c_j)^{x_j} = \prod_{j=1}^{\infty} \bullet \, (e \overset{\bullet}{+} x_j c_j). \tag{13}$$

Let us denote $\tilde{\mathbb{S}} := \{x \in \mathbb{S} : x_1 = 0\}$. We can see as in Theorem 2 in [2] that ζ is 1-1 and continuous from $\mathbb{I}_1$ onto $\tilde{\mathbb{S}}$.

Now, we will call the function

$$\gamma(x) := \frac{\zeta(x) \overset{\bullet}{-} e}{\zeta(x) \overset{\bullet}{+} e} \quad (x \in \mathbb{I}_1) \tag{14}$$

the tangent-like function on $(\mathbb{I}_1, \stackrel{\bullet}{+})$ and

$$\tan(x) := \frac{\zeta^2(x) \stackrel{\bullet}{-} e}{\zeta^2(x) \stackrel{\bullet}{+} e} \quad (x \in \mathbb{I}_1) \tag{15}$$

the tangent function on $(\mathbb{I}_1, \stackrel{\bullet}{+})$.

Lemma 1. *For any $a, b \in \mathbb{B}, x \in \mathbb{I}_1$, and $y \in \mathbb{I}_1$ holds*

$$\begin{aligned} &a)\ \frac{a \stackrel{\bullet}{+} a}{b \stackrel{\bullet}{+} b} = \frac{a}{b} \\ &b)\ a \stackrel{\bullet}{+} a = e_1 \bullet a \\ &c)\ \zeta^2(x) = \zeta(e_1 \bullet x) \\ &d)\ \frac{e \stackrel{\bullet}{+} y}{e \stackrel{\bullet}{-} y} \in \tilde{\mathbb{S}}, \end{aligned} \tag{16}$$

where $\zeta^2(x) = \zeta(x) \bullet \zeta(x)$.

Proof. a) The relation holds, because $a \bullet (b \stackrel{\bullet}{+} b) = b \bullet (a \stackrel{\bullet}{+} a)$ is satisfied by the commutativity and distributivity of the operations.

b) Using the notations of the recursive definition for the addition $\stackrel{\bullet}{+}$, we have $(a \stackrel{\bullet}{+} a)_n = 0$ if and only if $q_{n-1} = 0$. But $q_{n-1} = 0$ is equivalent with $a_{n-1} = 0$, which holds exactly when $(e_1 \bullet a)_n = 0$, because multiplication by e_1 shifts a. Similarly $(a \stackrel{\bullet}{+} a)_n = 1 \Leftrightarrow q_{n-1} = 1 \Leftrightarrow a_{n-1} = 1 \Leftrightarrow (e_1 \bullet a)_n = 1$.

c) It is a simple consequence of b) or directly: $b_j \bullet b_j = b_{j+1}$ $(j \geq 1)$, thus using the commutativity and associativity of $\bullet$ we have $\zeta^2(x) = \left(\prod_{j=1}^{\infty} \bullet\, b_j^{x_j}\right) \bullet \left(\prod_{j=1}^{\infty} \bullet\, b_j^{x_j}\right) = \prod_{j=1}^{\infty} \bullet\, b_{j+1}^{x_j} = \zeta(e_1 \bullet x)$ $(x \in \mathbb{I}_1)$

d) It can be easily established, that for $y = (0, y_1, y_2 \ldots) \in \mathbb{I}_1$ holds:

$$e \stackrel{\bullet}{+} y = (1, y_1, y_2, y_3, \ldots)$$

and

$$e \stackrel{\bullet}{-} y = (1, y_1, (y^-)_2, \ldots) = e \stackrel{\bullet}{+} y^-.$$

Applying the notation

$$\frac{e \dot{+} y}{e \dot{-} y} = z,$$

we can state first, that $\pi(z) = \pi(e \dot{+} y) - \pi(e \dot{-} y) = 0$ that is, $z \in \mathbb{S}$, and then

$$e \dot{+} y = z \bullet (e \dot{-} y).$$

Now, examining the 0th and the 1-st digits of the right and left side, we find that:

$$\begin{cases} 1 = z_0 \cdot 1 \\ y_1 = z_0 \cdot y_1 + z_1 \cdot 1 \pmod 2 \end{cases}$$

which means, that $z_0 = 1$ and $z_1 = 0$, and so $z \in \tilde{\mathbb{S}}$. Note, that $z = e \Leftrightarrow y = \theta$.

□

With Lemma 1 c) we can see, that the the tangent-like function γ is closely related to tan: namely $\gamma(x) = \tan(e_{-1} \bullet x)$ $(x \in \mathbb{I}_1)$.

Theorem 1. *The function γ is a continuous isomorphism from $(\mathbb{I}_1, \dot{+})$ onto $(\mathbb{I}_1, \triangleleft)$.*

Proof.

$$\gamma(x) \triangleleft \gamma(y) = \frac{\gamma(x) \dot{+} \gamma(y)}{e \dot{+} \gamma(x) \bullet \gamma(y)} = \frac{\frac{\zeta(x) \dot{-} e}{\zeta(x) \dot{+} e} \dot{+} \frac{\zeta(y) \dot{-} e}{\zeta(y) \dot{+} e}}{e \dot{+} \frac{\zeta(x) \dot{-} e}{\zeta(x) \dot{+} e} \bullet \frac{\zeta(y) \dot{-} e}{\zeta(y) \dot{+} e}}$$

$$= \frac{\zeta(x) \bullet \zeta(y) \dot{+} \zeta(x) \bullet \zeta(y) \dot{-} e \dot{-} e}{\zeta(x) \bullet \zeta(y) \dot{+} \zeta(x) \bullet \zeta(y) \dot{+} e \dot{+} e} = \frac{\zeta(x) \bullet \zeta(y) \dot{-} e}{\zeta(x) \bullet \zeta(y) \dot{+} e} = \gamma(x \dot{+} y)$$

where we used Lemma 1 a).

The function γ is a 1-1 map from $(\mathbb{I}_1, \dot{+})$ onto $(\mathbb{I}_1, \triangleleft)$. To see, that γ is a 1-1 map, we have from

$$\frac{\zeta(x) \dot{-} e}{\zeta(x) \dot{+} e} = \frac{\zeta(y) \dot{-} e}{\zeta(y) \dot{+} e}$$

the equation

$$\zeta(x) \dot{+} \zeta(x) = \zeta(y) \dot{+} \zeta(y).$$

Taking in consideration, that $f(a) := a \overset{\bullet}{+} a$ is 1-1, satisfying $a \overset{\bullet}{+} a = e_1 \bullet a$, we have

$$\zeta(x) = \zeta(y),$$

which gives that $x = y$.

To see, that for any $y \in \mathbb{I}_1$ there is an $x \in \mathbb{I}_1$ such that $\gamma(x) = y$, we have to solve in x the equation:

$$\frac{\zeta(x) \overset{\bullet}{-} e}{\zeta(x) \overset{\bullet}{+} e} = y,$$

thus

$$\zeta(x) = \frac{e \overset{\bullet}{+} y}{e \overset{\bullet}{-} y}.$$

Now,

$$x = \zeta^{-1}\left(\frac{e \overset{\bullet}{+} y}{e \overset{\bullet}{-} y}\right).$$

Thus we proved that γ is onto if $\zeta^{-1}\left(\frac{e \overset{\bullet}{+} y}{e \overset{\bullet}{-} y}\right) \in \mathbb{I}_1$ which holds in consequence of Lemma 1 d). Thus we proved that γ is an isomorphism from $(\mathbb{I}_1, \overset{\bullet}{+})$ onto $(\mathbb{I}_1, \triangleleft)$.

□

We consider $\varepsilon(t) := \exp(2\pi i t)$ $(t \in \mathbb{R})$. The characters of the group $(\mathbb{I}_1, \overset{\bullet}{+})$ are given by the product system $(v_m, m \in \mathbb{P})$ generated by the functions

$$v_{2^n}(x) := \varepsilon\left(\frac{x_n}{2} + \frac{x_{n-1}}{2^2} + \cdots + \frac{x_1}{2^n}\right) \qquad (x = (0, x_1, x_2 \ldots) \in \mathbb{I}_1, n \in \mathbb{P}),$$

that is, the functions $v_m(x) = \prod_{j=1}^{\infty}(v_{2^j}(x))^{m_j}$ $(m \in \mathbb{P})$. [2] Recall, that $\mathbb{P}$ is the set of positive numbers, $\mathbb{P} := \mathbb{N} \setminus \{0\}$.

Theorem 2. *The characters of the group* $(\mathbb{I}_1, \triangleleft)$ *are the functions*

$$v_n \circ \gamma^{-1} (n \in \mathbb{P}).$$

Corollary 1. *The characters of* $(\mathcal{B}, \circ)$ *are the functions*

$$v_n \circ \gamma^{-1} \circ B^{-1} (n \in \mathbb{P}),$$

where $(\mathcal{B}, \circ)$ *denotes the Blaschke-group of the arithmetic field* $(\mathbb{I}, \dot{+}, \bullet)$*, and* $B : (\mathbb{I}_1, \triangleleft) \to (\mathcal{B}, \circ)$ *is the function* $a \mapsto B_a$*.*

3. Recursion

In (13) we used the notation $b_n = e \dot{+} c_n$ $(n \geq 1)$ where $\pi(c_n) = n+1$, now consider $b_n = e \dot{+} e_{n+1} \dot{+} d_n$ $(n \geq 1)$ where $\pi(d_n) \geq n+2$. Now the function ζ has the following representation:

$$\zeta(x) = \prod_{j=1}^{\infty} \bullet (e \dot{+} e_{j+1} \dot{+} d_j)^{x_j} = \prod_{j=1}^{\infty} \bullet (e \dot{+} x_j e_{j+1} \dot{+} x_j d_j). \ (d_j \in \mathbb{I}_{n+2})$$

Easy inductive arguments establish that $\zeta(x)$ is a simple recursion:

$$(\zeta(x))_n = x_{n-1} + f(x_1, \ldots, x_{n-2}) \qquad (n \geq 1) \tag{17}$$

and $(\zeta(x))_0 = 1$. Thus $z := \zeta(x) \dot{-} e = (\zeta(x) \dot{+} e^-) = (1, 0, \zeta_2, \zeta_3, \ldots) \dot{+} (1, 1, 1, 1, \ldots) = (0, 0, \zeta_2, \zeta_3, \zeta_4, \ldots)$ can also be written as a simple recursion:

$$z_n = x_{n-1} + f(x_1, \ldots, x_{n-2}) \qquad (n \geq 2).$$

Analogous, $t := \zeta(x) \dot{+} e = (1, 0, \zeta_2, \zeta_3, \ldots) \dot{+} (1, 0, 0, 0, \ldots) = (0, 1, \zeta_2, \zeta_3, \zeta_4, \ldots)$ as a simple recursion:

$$t_n = x_{n-1} + f(x_1, \ldots, x_{n-2}) \qquad (n \geq 2).$$

The multiplicative inverse element of $t \in \mathbb{I}_1$ is also a simple recursion:

$$(t^{-1})_n = x_{n+1} + f(x_1, \ldots, x_n)$$

for some function f. See[[2], pp. 39-40.]

Using $(t^{-1})_{-1} = 1$ and $(\gamma(x))_n = z_2(t^{-1})_{n-2} + \ldots + z_{n+1}(t^{-1})_{-1} + q_{n-1}$ (mod 2), follows that

$$(\gamma(x))_n = x_n + f(x_1, \ldots, x_{n-1}). \tag{18}$$

Denote with $\mathcal{A}$ the σ-algebra generated by the intervals $I_n(a)$ $(a \in \mathbb{I}, n \in \mathbb{N})$. Let $\mu(I_n(a)) \doteq 2^{-n}$ be the measure of $I_n(a)$. Extending this measure to $\mathcal{A}$ we get a probability measure space $(\mathbb{I}, \mathcal{A}, \mu)$. Let $\mathcal{A}_n$ be the sub-σ-algebra of $\mathcal{A}$ generated by the intervals $I_n(a)$ $(a \in \mathbb{I})$. Let $L(\mathcal{A}_n)$ denote the set of $\mathcal{A}_n$-measurable functions on $\mathbb{I}$. The conditional expectation of an $f \in L^1(\mathbb{I})$ with respect to $\mathcal{A}_n$ is of the form

$$(\mathcal{E}_n f)(x) = \frac{1}{\mu(I_n(x))} \int_{I_n(x)} f d\mu.$$

A sequence of functions $(f_n, n \in \mathbb{N})$ is called a dyadic martingale if each f_n is $\mathcal{A}_n$-measurable and

$$(\mathcal{E}_n f_{n+1}) = f_n \qquad (n \in \mathbb{N}).$$

The sequence of martingale differences of f_n $(n \in \mathbb{N})$ is the sequence

$$\phi_n \doteq f_{n+1} - f_n \qquad (n \in \mathbb{N}).$$

We notice that every dyadic martingale difference sequence has the form $\phi_n = r_n g_n$ $(n \in \mathbb{N})$ where $(g_n, n \in \mathbb{N})$ is a sequence of functions such that each g_n is $\mathcal{A}_n$-measurable and $(r_n, n \in \mathbb{N})$ denotes the Rademacher system on $\mathbb{I}$:

$$r_n(x) \doteq (-1)^{x_n} \ (n \in \mathbb{N}).$$

The martingale difference sequence $(\phi_n, n \in \mathbb{N})$ is called a unitary dyadic martingale difference sequence or a UDMD sequence if $|\phi_n(x)| = 1$ $(n \in \mathbb{N})$. Thus $(\phi_n, n \in \mathbb{N})$ is a UDMD sequence if and only if

$$\phi_n = r_n g_n, \ g_n \in L(\mathcal{A}_n), \ |g_n| = 1 \ \ (n \in \mathbb{N}). \tag{19}$$

Let us call a system $\psi = (\psi_m, m \in \mathbb{N})$ a UDMD product system if it is a product system generated by a UDMD system, i.e., there is a UDMD system $(\phi_n, n \in \mathbb{N})$ such that for each $m \in \mathbb{N}$, with binary expansion is given by $m = \sum_{j=0}^{\infty} m_j 2^j$ $(m_j \in \mathbb{A}, j \in \mathbb{N})$, the function ψ_m satisfies

$$\psi_m = \prod_{j=0}^{\infty} \phi_j^{m_j} \qquad (m \in \mathbb{N}).$$

By (18) the byte $\gamma^{-1}(x)$ can also be written by a simple recursion for any $x \in \mathbb{I}_1$, therefore we have the following:

Corollary 2. *The functions $v_n \circ \gamma^{-1} (n \in \mathbb{P})$, the characters of $(\mathbb{I}_1, \triangleleft)$ form a UDMD product system.*

Proof. The $(v_{2^n} \circ \gamma^{-1}, n \in \mathbb{P})$ functions satisfy the requirements of a UDMD-system: $v_{2^n}(\gamma(x)) = \varepsilon\left(\frac{x_n}{2}\right) g(x_1, \ldots, x_{n-1}) = (-1)^{x_n} g(x_1, \ldots, x_{n-1})$, with some $g \in L(\mathcal{A}_n)$, and $|g(x_1, \ldots, x_{n-1})| = 1$.

□

As $(v_n \circ \gamma^{-1}, n \in \mathbb{P})$ is a UDMD product system, the discrete Fourier coefficients with respect this system can be computed with the Fast Fourier Algorithm.

4. (C,1) summability

By (18) $\gamma : I_n(x) \to I_n(\gamma(x))$ is a bijection $(x \in \mathbb{I}_1, n \in \mathbb{N})$, thus for any dyadic interval E holds $\mu(t \in \mathbb{I}_1 : \gamma(t) \in E) = \mu(E)$ and this follows for any E measurable sets also. Therefore the variable transformation $\gamma(x)$ is measure preserving. Consequently, it holds

$$\int_{\mathbb{I}_1} f \circ \gamma d\mu = \int_{\mathbb{I}_1} f d\mu. \tag{20}$$

The Gamma-Fourier coefficients of an $f \in L^1(\mathbb{I}_1)$ are defined by

$$\widehat{f^\gamma}(n) \doteq \int_{\mathbb{I}_1} f(x) v_n(\gamma(x)^{-1}) d\mu(x) \qquad (n \in \mathbb{P}).$$

We have by (20):

$$\widehat{f^\gamma}(n) = \widehat{f \circ \gamma}(n), \tag{21}$$

where $\widehat{f}(n)$ are the well-known Fourier coefficients of an $f \in L^1(\mathbb{I})$. [1]

The Gamma-Fourier series of an $f \in L^1(\mathbb{I}_1)$ is the series

$$S^\gamma f \doteq \sum_{k=0}^{\infty} \widehat{f^\gamma}(k) v_k \circ \gamma^{-1},$$

and the n-th partial sums of the Gamma-Fourier series S^γ is

$$S_n^\gamma f \doteq \sum_{k=0}^{n-1} \widehat{f^\gamma}(k) v_k \circ \gamma^{-1} \ (n \in \mathbb{P}).$$

It follows by (21) that

$$S_n^\gamma f = [S_n(f \circ \gamma)] \circ \gamma^{-1} \tag{22}$$

where S_n is the well-known n-th partial sum of the Walsh-Fourier series. See[1] .

If the Gamma-Cesaro (or $(G-C,1)$) means of $S^\gamma f$ are defined by $\sigma_0 f \doteq 0$ and

$$\sigma_n^\gamma f \doteq \frac{1}{n}\sum_{k=1}^{n} S_k^\gamma f, \qquad (n \in \mathbb{P})$$

then it follows by (22) that

$$\sigma_n^\gamma f(x) = \frac{1}{n}\sum_{k=1}^{n} \left[S_k(f\circ\gamma)\right](\gamma^{-1}(x)) = \sigma_n(f\circ\gamma)(\gamma^{-1}(x)). \tag{23}$$

where σ_n means the well known n-th Cesaro mean of Sf. [1]

Now, we use the theorem of the $(C,1)$-summability of the Walsh-Fourier series on the field $(\mathbb{I}, \overset{\bullet}{+}, \bullet)$ due to Gy. Gát [5]: $\lim\limits_{m\to\infty}(\sigma_m f)(x) = f(x)$ a.e. for any $f \in L^1(\mathbb{I})$.

Thus with (23) we have $\lim\limits_{n\to\infty} \sigma_n^\gamma f(x) = \lim\limits_{n\to\infty} \sigma_n(f\circ\gamma)(\gamma^{-1}(x)) = (f\circ\gamma\circ\gamma^{-1})(x) = f(x)$ a.e. for any $f \in L^1(\mathbb{I}_1)$.

Theorem 3. *On the field* $(\mathbb{I}_1, \overset{\bullet}{+}, \bullet)$ *holds* $\lim\limits_{n\to\infty} \sigma_n^\gamma f(x) = f(x)$ *a.e. for any* $f \in L^1(\mathbb{I}_1)$.

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