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CLOSEDNESS OF THE SOLUTION MAP FOR PARAMETRIC VECTOR EQUILIBRIUM PROBLEMS

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Abstract. The objective of this paper is to study the parametric vector equilibrium problems governed by vector topologically pseudomonotone maps. The main result gives sufficient conditions for closedness of the solution map defined on the set of parameters.

1. **Introduction**

M. Bogdan and J. Kolumbán [5] gave sufficient conditions for closedness of the solution map. They considered the parametric equilibrium problems governed by topologically pseudomonotone maps depending on a parameter. In this paper we generalize their result for parametric vector equilibrium problems.

Let (X, σ) be a Hausdorff topological space and let P (the set of parameters) be another Hausdorff topological space. Let $\mathcal{Z}$ be a real topological vector space with an ordering cone C, where C is a closed convex cone in $\mathcal{Z}$ with $\operatorname{Int} C \neq \emptyset$ and $C \neq \mathcal{Z}$.

We consider the following parametric vector equilibrium problem, in short $(VEP)_p$:

Find $a_p \in D_p$ such that

$$f_p\left(a_p, b\right) \in \left(-\operatorname{Int} C\right)^c, \ \forall b \in D_p,$$

where D_p is a nonempty subset of X and $f_p : X \times X \to \mathcal{Z}$ is a given function.

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Denote by $S(p)$ the set of the solutions for a fixed p. Suppose that $S(p) \neq \emptyset$, for all $p \in P$. (For sufficient conditions for the existence of solutions see [6].)

The paper is organized as follows. In Section 2, we recall the notions of the vector topological pseudomonotonicity and the Mosco convergence of the sets. Section 3 is devoted to the closedness of the solution map for parametric vector equilibrium problems. In the final section, we investigate the generalized Hadamard well-posedness of parametric vector equilibrium problems.

2. Preliminaries

In this section, the notion of vector topologically pseudomonotone bifunctions with values in $\mathcal{Z}$ is used. First the definition of the suprema and the infima of subsets of $\mathcal{Z}$ are given. Following [1], for a subset A of $\mathcal{Z}$ the superior of A with respect to C is defined by

$$\operatorname{Sup} A = \left\{ z \in \bar{A} : A \cap (z + \operatorname{Int} C) = \emptyset \right\}$$

and the inferior of A with respect to C is defined by

$$\operatorname{Inf} A = \left\{ z \in \bar{A} : A \cap (z - \operatorname{Int} C) = \emptyset \right\}.$$

Let $(z_i)_{i \in I}$ be a net in $\mathcal{Z}$. Let $A_i = \{z_j : j \geq i\}$ for every i in the index set I. The limit inferior of (z_i) is given by

$$\operatorname{Liminf} z_i := \operatorname{Sup} \left(\bigcup_{i \in I} \operatorname{Inf} A_i \right).$$

Similarly, the limit superior of (z_i) is defined as

$$\operatorname{Limsup} z_i := \operatorname{Inf} \left(\bigcup_{i \in I} \operatorname{Sup} A_i \right).$$

The next definition is a generalization of the vector topological pseudomonotonicity given by Chadli, Chiang and Huang in [6].

Definition 2.1. *Let (X, σ) be a Hausdorff topological space, and let D be a nonempty subset of X. A function $f : D \times D \to \mathcal{Z}$ is called vector topologically pseudomonotone if for every $b \in D$, $v \in \operatorname{Int} C$ and for each net $(a_i)_{i \in I}$ in D satisfying*

$$a_i \xrightarrow{\sigma} a \in D \text{ and } \operatorname{Liminf} f(a_i, a) \cap (-\operatorname{Int} C) = \emptyset,$$

there is i_0 *in the index set* I *such that*

$$\overline{\{f(a_j, b) : j \geq i\}} \subset f(a,b) + v - \operatorname{Int} C$$

for all $i \geq i_0$.

Let us consider σ and τ two topologies on X. Suppose that τ is stronger than σ on X.

For the parametric domains in $(VEP)_p$ we shall use the following type of convergence, which is a slight generalization of Mosco's convergence in [11].

Definition 2.2 ([5], Definition 2.2)**.** *Let* D_p *be subsets of* X *for all* $p \in P$. *The sets* D_p *converge to* D_{p_0} *in the Mosco sense* ($D_p \xrightarrow{M} D_{p_0}$) *as* $p \to p_0$ *if:*

a) *for every subnet* $(a_{p_i})_{i \in I}$ *with* $a_{p_i} \in D_{p_i}$, $p_i \to p_0$ *and* $a_{p_i} \xrightarrow{\sigma} a$ *imply* $a \in D_{p_0}$;

b) *for every* $a \in D_{p_0}$, *there exist* $a_p \in D_p$ *such that* $a_p \xrightarrow{\tau} a$ *as* $p \to p_0$.

3. **Closedness of the solution map**

This section is devoted to prove the closedness of the solution map for parametric vector equilibrium problems.

Theorem 3.1. *Let* X *be a Hausdorff topological space with* σ *and* τ *two topologies, where* τ *is stronger than* σ. *Let* D_p *be nonempty sets of* X, *and let* $p_0 \in P$ *be fixed. Suppose that* $S(p) \neq \emptyset$ *for each* $p \in P$ *and the following conditions hold:*

i) $D_p \xrightarrow{M} D_{p_0}$;

ii) *For each net of elements* $(p_i, a_{p_i}) \in GraphS$, *if* $p_i \to p_0$, $a_{p_i} \xrightarrow{\sigma} a$, $b_{p_i} \in D_{p_i}$, $b \in D_{p_0}$, *and* $b_{p_i} \xrightarrow{\tau} b$ *there exists a subnet of* $(p_i, a_{p_i})_{i \in I}$, *denoted by the same indexes, such that one of the following conditions applies*

$(C1)$ $\left(f_{p_i}(a_{p_i}, b_{p_i}) - f_{p_0}(a_{p_i}, b)\right)_{i \in I}$ *converge to an element belonging to* $-\operatorname{Int} C$, *when* $p_i \to p_0$

or

$(C2)$ $\left(f_{p_i}(a_{p_i}, b_{p_i}) - f_{p_0}(a_{p_i}, b)\right)_{i \in I}$ *converge to an element belonging to* $-\partial C$, *when* $p_i \to p_0$ *and* $\left(f_{p_i}(a_{p_i}, b_{p_i}) - f_{p_0}(a_{p_i}, b)\right)_{i \in I} \in -C$;

iii) $f_{p_0} : X \times X \to \mathcal{Z}$ *is vector topologically pseudomonotone.*

Then the solution map $p \longmapsto S(p)$ *is closed at* p_0, *i.e. for each net of elements* $(p_i, a_{p_i}) \in GraphS$, $p_i \to p_0$ *and* $a_{p_i} \xrightarrow{\sigma} a$ *imply* $(p_0, a) \in GraphS$.

Proof. Let $(p_i, a_{p_i})_{i\in I}$ be a net of elements $(p_i, a_{p_i}) \in GraphS$ i.e.

$$f_{p_i}(a_{p_i}, b) \in (-\operatorname{Int} C)^c,\ \forall b \in D_{p_i} \tag{1}$$

with $p_i \to p_0$ and $a_{p_i} \xrightarrow{\sigma} a$. By the Mosco convergence of the sets D_p we get $a \in D_{p_0}$. Moreover there exists a net $(b_{p_i})_{i\in I}$, $b_{p_i} \in D_{p_i}$ such that $b_{p_i} \xrightarrow{\tau} a$. From the assumption *ii)* we obtain that there exists a subnet of $(p_i, a_{p_i})_{i\in I}$, denoted by the same indexes, such that

$$\begin{array}{l}(f_{p_i}(a_{p_i}, b_{p_i}) - f_{p_0}(a_{p_i}, a))_{i\in I} \text{ converge to an element} \\ \text{belonging to } -\operatorname{Int} C, \text{ when } p_i \to p_0\end{array} \tag{2}$$

or

$$\begin{array}{l}(f_{p_i}(a_{p_i}, b_{p_i}) - f_{p_0}(a_{p_i}, a))_{i\in I} \text{ converge to an element} \\ \text{belonging to } -\partial C, \text{ when } p_i \to p_0 \text{ and} \\ (f_{p_i}(a_{p_i}, b_{p_i}) - f_{p_0}(a_{p_i}, a))_{i\in I} \in -C.\end{array} \tag{3}$$

Since $-\operatorname{Int} C$ is an open cone, from (2) follows that there exists an index $j_0 \in I$ such that

$$f_{p_i}(a_{p_i}, b_{p_i}) - f_{p_0}(a_{p_i}, a) \in -\operatorname{Int} C \subset -C,\ i \geq j_0. \tag{4}$$

By replacing b with b_{p_i} in (1) we get

$$f_{p_i}(a_{p_i}, b_{p_i}) \in (-\operatorname{Int} C)^c. \tag{5}$$

From (5), (3) and (4) we obtain that

$$f_{p_0}(a_{p_i}, a) \in (-\operatorname{Int} C)^c,\ \text{for } i \geq j_0,$$

since $(-\operatorname{Int} C)^c$ is closed, in both cases we have

$$\operatorname{Liminf} f_{p_0}(a_{p_i}, a) \subset (-\operatorname{Int} C)^c \ \text{ for } i \geq j_0.$$

Now we can apply $iii)$ and we obtain that for every $b \in D$ and $v \in \operatorname{Int} C$, there exists $j_1 \in I$ such that

$$\overline{\{f_{p_0}(a_{p_i}, b) : i \geq j\}} \subset f_{p_0}(a, b) + v - \operatorname{Int} C, \ \forall j \geq j_1. \tag{6}$$

We have to prove that

$$f_{p_0}(a, b) \in (-\operatorname{Int} C)^c, \ \forall b \in D_{p_0}.$$

Assume the contrary, that there exists $\overline{b} \in D_{p_0}$ such that

$$f_{p_0}(a, \overline{b}) \in -\operatorname{Int} C.$$

Let be $f_{p_0}(a, \overline{b}) = -v$ where $v \in \operatorname{Int} C$. From (6) we obtain that there exists $j_1 \in I$ such that

$$\overline{\{f_{p_0}(a_{p_i}, \overline{b}) : i \geq j\}} \subset -v + v - \operatorname{Int} C = -\operatorname{Int} C, \ \forall j \geq j_1. \tag{7}$$

Since $\overline{b} \in D_{p_0}$ from the Mosco convergence of the sets D_p, we have that there exists $(\overline{b}_{p_i})_{i \in I} \subset D_{p_i}$ such that $\overline{b}_{p_i} \xrightarrow{\tau} \overline{b}$. By using again $ii)$, it follows that there exists a subnet of $(p_i, a_{p_i})_{i \in I}$, denoted by the same indexes, such that

$$f_{p_i}(a_{p_i}, \overline{b}_{p_i}) - f_{p_0}(a_{p_i}, \overline{b}) \in -\operatorname{Int} C \subset -C, \ i \geq j_2, \tag{8}$$

where we have used the same reasoning as before.

From (7) and (8) it follows

$$f_{p_i}(a_{p_i}, \overline{b}_{p_i}) \in -\operatorname{Int} C, \ i \geq \sup\{j_1, j_2\}, \tag{9}$$

but on other side $(p_i, a_{p_i}) \in GraphS$, and

$$f_{p_i}(a_{p_i}, \overline{b}_{p_i}) \in (-\operatorname{Int} C)^c$$

which is a contradiction. Hence $(p_0, a) \in GraphS$. □

Remark 3.2. *The assumption $ii)$ of the Theorem 3.1 is weaker then the following statement*

$ii')$ For each net of elements $(p_i, a_{p_i}) \in GraphS$, if $p_i \to p_0$, $a_{p_i} \xrightarrow{\sigma} a$, $b_{p_i} \in D_{p_i}$, $b \in D_{p_0}$,and $b_{p_i} \xrightarrow{\tau} b$ then

$$\operatorname{Liminf}(f_{p_i}(a_{p_i}, b_{p_i}) - f_{p_0}(a_{p_i}, b)) \cap (-\operatorname{Int} C) \neq \emptyset.$$

Indeed, first we prove that $ii^{'}) \Rightarrow ii)$.

For simplicity, we introduce the following notation

$$u_{p_i} = f_{p_i}\left(a_{p_i}, b_{p_i}\right) - f_{p_0}\left(a_{p_i}, b\right).$$

From $ii')$ *we obtain that for every* $i_0 \in I$ *we have*

$$\operatorname{Liminf} u_{p_i} \cap (-\operatorname{Int} C) \neq \emptyset \text{ where } i \geq i_0.$$

Wherefrom it follows that there exists a point u *from the limit points of net* $\left(u_{p_i}\right)_{i \in I}$ *such that for every neighborhood* U *of* u *we have*

$$U \cap [\operatorname{Liminf} u_{p_i} \cap (-\operatorname{Int} C)] \neq \emptyset. \tag{10}$$

There are two cases to be distinguished:

Case 1. $u \in \operatorname{Liminf} u_{p_i} \cap (-\operatorname{Int} C)$. *Since* u *is a limit point of* (u_{p_i}) *there exists a subnet* $\left(u_{p_j}\right)$ *converging to* u. *So we have that* $u \in -\operatorname{Int} C$ *then the condition* $(C1)$ *in assumption* $ii)$ *holds.*

Case 2. $u \notin \operatorname{Liminf} u_{p_i} \cap (-\operatorname{Int} C)$. *In this case we must have that* $u \in -\partial C$. *From (10) it follows that for every neighborhood* U *of* u *there exists an* $u_{p_i} \in -\operatorname{Int} C \subset -C$ *such that* $u_{p_i} \in U$. *This leads to the condition* $(C2)$ *of the assumption* $ii)$.

These two assumptions are not equivalent, because there exist nets which satisfy only the assumption $ii)$. *For example, let the net* $\left(u_{p_i}\right)_{i \in I}$ *be defined by* $u_{p_i} = (2, 4 + 1/p_i)$ *for* $i \in I$, *where* $p_i \to \infty$ *and the cone* C *is given by*

$$C = \left\{(a, b) \in \mathbb{R}^2 : b \geq |2a|\right\}.$$

This net has only one limit inferior point in the $(2, 4)$ *which is located on the boundary of the* C *cone. Hence the assumption* $ii)$ *holds, but the assumption* $ii')$ *fails.*

Remark 3.3. *The Theorem 3.1 does not imply the scalar case. The only exception represents the following condition:*

For each net of elements $(p_i, a_{p_i}) \in GraphS$, *if* $p_i \to p_0$, $a_{p_i} \xrightarrow{\sigma} a$, $b_{p_i} \in D_{p_i}$, $b \in D_{p_0}$, *and* $b_{p_i} \xrightarrow{\tau} b$ *there exists a subnet of* $(p_i, a_{p_i})_{i \in I}$, *denoted by the same indexes,* *such that*

$$(C3) \quad \begin{array}{l} \left(f_{p_i}\left(a_{p_i}, b_{p_i}\right) - f_{p_0}\left(a_{p_i}, b\right)\right)_{i \in I} \text{ converge to } 0 \text{ and} \\ \left(f_{p_i}\left(a_{p_i}, b_{p_i}\right) - f_{p_0}\left(a_{p_i}, b\right)\right)_{i \in I} \notin -C. \end{array}$$

The following example confirms this statement.

Example 3.4. *Let $P = \mathbb{N} \cup \{\infty\}$, $p_0 = \infty$ (∞ means $+\infty$ from real analysis), where we consider the topology induced by the metric given by $d(m,n) = |1/m - 1/n|$, $d(n,\infty) = d(\infty,n) = 1/n$, for $m,n \in \mathbb{N}$, and $d(\infty,\infty) = 0$. Let $X = [0,1]$ where σ, τ are the natural topology, $\mathcal{Z} = \mathbb{R}^2$, $D_p = [0,1]$, $p \in P$, the real vector functions $f_p : [0,1] \times [0,1] \to \mathbb{R}^2$. The ordering cone C is the third quadrant i.e. $C = \left\{(a,b) \in \mathbb{R}^2 : a \leq 0, b \leq 0\right\}$.*

Let $f_n(a,b) = (a - b - 1/n, -2a + 1)$, $n \in \mathbb{N}$ and

$$f_\infty(a,b) = \begin{cases} (a-b, -a+1) & \text{if} \quad a > 0 \\ (b,1) & \text{if} \quad a = 0 \end{cases}.$$

The f_∞ is vector topologically pseudomonotone, and the condition $(C3)$ holds. We have $(n, 1/n) \in GraphS$ for each $n \in \mathbb{N}$, $S(\infty) = \{1\}$ so $0 \notin S(\infty)$. Hence S is not closed at ∞.

M. Bogdan and J. Kolumbán [5] showed that the topological pseudomonotonicity and the assumption $ii)$ are essential in scalar case.

If the $(VEP)_p$ is defined on constant domains, $D_p = X$ for all $p \in P$, we can omit the Mosco convergence. In this case condition $ii)$ can be weakened to:

(C) For each net of elements $(p_i, a_{p_i}) \in GraphS$, if $p_i \to p_0$, $a_{p_i} \xrightarrow{\sigma} a$, and $b \in X$, there exists a subnet of $(p_i, a_{p_i})_{i \in I}$, denoted by the same indexes, such that

$$\begin{array}{l} \left(f_{p_i}\left(a_{p_i}, b\right) - f_{p_0}\left(a_{p_i}, b\right)\right)_{i \in I} \text{ converge to an element} \\ \text{belonging to } -\operatorname{Int} C, \text{ when } p_i \to p_0 \end{array}$$

or

$$\begin{array}{l} \left(f_{p_i}\left(a_{p_i}, b\right) - f_{p_0}\left(a_{p_i}, b\right)\right)_{i \in I} \text{ converge to an element} \\ \text{belonging to } -\partial C, \text{ when } p_i \to p_0 \text{ and} \\ \left(f_{p_i}\left(a_{p_i}, b\right) - f_{p_0}\left(a_{p_i}, b\right)\right)_{i \in I} \in -C. \end{array}$$

Theorem 3.5. *Let* (X, σ) *be a Hausdorff topological space and let* $p_0 \in P$ *be fixed. Suppose that* $S(p) \neq \emptyset$*, for each* $p \in P$*, and*

i) f_p *satisfies condition* (C) *at* p_0*;*

ii) $f_{p_0} : X \times X \to \mathcal{Z}$ *is vector topologically pseudomonotone.*

Then the solution map $p \longmapsto S(p)$ *is closed at* p_0*.*

Proof. The proof is similar to the proof of the Theorem 3.1. □

4. **Hadamard well-posedness**

Let us recall some classical definitions from set-valued analysis. Let X, Y be topological spaces. The map $T : X \to 2^Y$ is said to be *upper semi-continuous* at $u_0 \in domT := \{u \in X | T(u) \neq \emptyset\}$ if for each neighborhood V of $T(u_0)$, there exists a neighborhood U of u_0 such that $T(U) \subset V$. The map T is considered to be *closed* at $u \in domT$ if for each net $(u_i)_{i \in I}$ in $domT$, $u_i \to u$ and each net $(y_i)_{i \in I}$, $y_i \in T(u_i)$, with $y_i \to y$ one has $y \in T(u)$. The map T is said to be *closed* if its graph $GraphT = \{(u, y) \in X \times Y | y \in T(u)\}$ is closed, namely if $(u_i, y_i) \in GraphT$, $(u_i, y_i) \to (u, y)$ then $(u, y) \in GraphT$.

Closedness and upper semi-continuity of a multifunction are closely related.

Proposition 4.1 ([3] Proposition 1.4.8, 1.4.9)**.**

i) *If* $T : Y \to 2^X$ *has closed values and is upper semi-continuous then* T *is closed;*

ii) *If* X *is compact and* T *is closed at* $y \in Y$ *then* T *is upper semi-continuous at* $y \in Y$*.*

Now we recall the notion of generalized Hadamard well-posedness.

Definition 4.2. *The problem* $(VEP)_p$ *is said to be Hadamard well-posed (briefly H-wp) at* $p_0 \in P$ *if* $S(p_0) = \{a_{p_0}\}$ *and for any* $a_p \in S(p)$ *one has* $a_p \xrightarrow{\sigma} a_{p_0}$*, as* $p \to p_0$*. The problem* $(VEP)_p$ *is said to be generalized Hadamard well-posed (briefly gH-wp) at* $p_0 \in P$ *if* $S(p_0) \neq \emptyset$ *and for any* $a_p \in S(p)$*, if* $p \to p_0$*,* (a_p) *must have a subsequence* $\sigma-$*converging to an element of* $S(p_0)$*.*

With the help of the next result we are able to establish the relationship between upper semi-continuity and Hadamard well-posedness.

Proposition 4.3 ([13] Theorem 2.2)**.** *Let X and Y be Hausdorff topological spaces and $T : Y \to 2^X$ be a set valued map. If T is upper semi-continuous at $y \in Y$ and $T(y)$ is compact, then T is gH-wp at y. If more, $T(y) = \{x^*\}$, then T is H-wp at y.*

In the following we prove that the solution map of $(VEP)_p$ has closed value at p_0.

Proposition 4.4. *If D_{p_0} is closed with respect to the σ topology and $f_{p_0} : X \times X \to \mathcal{Z}$ is vector topologically pseudomonotone, then $S(p_0)$ is closed with respect to the σ topology.*

Proof. Let $a_i \in S(p_0)$, with $a_i \xrightarrow{\sigma} a$. Since D_{p_0} is closed with respect to the σ topology, we have $a \in D_{p_0}$. From $a_i \in S(p_0)$ it follows that

$$f_{p_0}(a_i, a) \in (-\operatorname{Int} C)^c, \ \forall i \in I,$$

since $(-\operatorname{Int} C)^c$ is closed, we get

$$\operatorname{Liminf} f_{p_0}(a_i, a) \subset (-\operatorname{Int} C)^c.$$

By using the vector topological pseudomonotonicity we obtain that for every $b \in D$ and $v \in \operatorname{Int} C$ there is j_1 in the index set I such that

$$\overline{\{f_{p_0}(a_i, b) : i \geq j\}} \subset f_{p_0}(a, b) + v - \operatorname{Int} C, \ \forall j \geq j_1. \tag{11}$$

We have to prove that $a \in S(p_0)$, i.e.

$$f_{p_0}(a, b) \in (-\operatorname{Int} C)^c, \ \forall b \in D_{p_0}.$$

Assume the contrary, that there exists $\bar{b} \in D_{p_0}$ such that

$$f_{p_0}\left(a, \bar{b}\right) \in -\operatorname{Int} C.$$

Let $f_{p_0}\left(a, \bar{b}\right) = -v$ where $v \in \operatorname{Int} C$. From (11) we obtain that

$$\overline{\left\{f_{p_0}\left(a_i, \bar{b}\right) : i \geq j\right\}} \subset -v + v - \operatorname{Int} C = -\operatorname{Int} C, \ \forall j \geq j_1$$

which is a contradiction to $a_i \in S(p_0)$. Thus $a \in S(p_0)$. □

Now we can formulate the following result.

Corollary 4.5. *Let (X,σ) be a compact Hausdorff topological space and P be a Hausdorff topological space. Let D_p be nonempty sets of X, and D_{p_0} be a closed subset of X. Suppose that $S(p) \neq \emptyset$ for each $p \in P$ and the following conditions hold:*

i) $D_p \xrightarrow{M} D_{p_0}$;

ii) *For each net of elements $(p_i, a_{p_i}) \in GraphS$, if $p_i \to p_0$, $a_{p_i} \xrightarrow{\sigma} a$, $b_{p_i} \in D_{p_i}$, $b \in D_{p_0}$, and $b_{p_i} \xrightarrow{\tau} b$ there exists a subnet of $(p_i, a_{p_i})_{i\in I}$, denoted by the same indexes, such that*

$$\left(f_{p_i}(a_{p_i}, b_{p_i}) - f_{p_0}(a_{p_i}, b)\right)_{i\in I} \text{ converge to an element belonging to } -\operatorname{Int} C, \text{ when } p_i \to p_0$$

or

$$\left(f_{p_i}(a_{p_i}, b_{p_i}) - f_{p_0}(a_{p_i}, b)\right)_{i\in I} \text{ converge to an element belonging to } -\partial C, \text{ when } p_i \to p_0 \text{ and } \left(f_{p_i}(a_{p_i}, b_{p_i}) - f_{p_0}(a_{p_i}, b)\right)_{i\in I} \in -C;$$

iii) *$f_{p_0} : X \times X \to \mathcal{Z}$ is vector topologically pseudomonotone.*

Then $(VEP)_p$ is generalized Hadamard well-posed at p_0. Furthermore, if $S(p_0) = \{a_{p_0}\}$ (a singleton), then $(VEP)_p$ is Hadamard well-posed at p_0.

Proof. By Theorem 3.1 it follows that the solution map S is closed at p_0. We may use Proposition 4.1 *ii)* to state that S is upper semi-continuous at p_0. The set $S(p_0)$ is closed by Proposition 4.4, hence it is compact. The conclusion follows by Proposition 4.3. □

We can obtain similar result in the case of constant domains.

Corollary 4.6. *Let (X,σ) be a compact Hausdorff topological space. Let $p_0 \in P$ be fixed and $S(p) \neq \emptyset$, for each $p \in P$. If the hypotheses of Theorem 3.5 are satisfied then $(VEP)_p$ is generalized Hadamard well-posed at p_0. Furthermore, if $S(p_0) = \{a_{p_0}\}$ (a singleton), then $(VEP)_p$ is Hadamard well-posed at p_0.*

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