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# ON THE DIVERGENCE OF THE PRODUCT QUADRATURE PROCEDURES

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**Abstract**. The main result of this paper emphasizes the phenomenon of the double condensation of singularities with respect to the product-quadrature procedures associated to the spaces $C$ and $L^1$; some estimations concerning the error of these procedures are given, too.

## 1. **Introduction**

Let us consider the Banach space $C$ of all continuous functions $f : [-1, 1] \to \mathbb{R}$, endowed with the uniform norm $\|\cdot\|$. Denote by $L^1$ the Banach space of all measurable functions (classes of functions) $g : [-1, 1] \to \mathbb{R}$ so that $|g|$ is Lebesgue integrable on $[-1, 1]$, endowed with the norm:

$$\|g\|_1 = \int_{-1}^{1} |g(x)|dx, \quad g \in L^1.$$

Let $\mathcal{M} = \{x_n^k : \ n \geq 1; \ 1 \leq k \leq n\}$ be a triangular node matrix, with $-1 \leq x_n^1 < x_n^2 < x_n^3 < \cdots < x_n^n \leq 1$, $\forall\ n \geq 1$. For each integer $n \geq 1$, denote by $\Lambda_n : [-1, 1] \to \mathbb{R}$ the *Lebesgue function* associated to the $n$-th row of $\mathcal{M}$, i.e.

$$\Lambda_n(x) = \Lambda_n(\mathcal{M}; x) = \sum_{k=1}^{n} |l_n^k(x)|, \quad |x| \leq 1,$$

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where $l_n^k = l_n^k(\mathcal{M};\cdot)$, $1 \le k \le n$, are the fundamental polynomials of Lagrange interpolation with respect to the nodes $x_n^k$, $1 \le k \le n$. The real numbers

$$\lambda_n = \lambda_n(\mathcal{M}) = \|\Lambda_n\|, \quad n \ge 1$$

are known as *Lebesgue constants*.

Starting from these data, let us consider the *product-quadrature procedures* described by the formulas

$$\int_{-1}^{1} g(x)f(x)dx = \int_{-1}^{1} g(x)L_n(f;x)dx + R_n(f;g), \quad f \in C,\ g \in L^1,\ n \ge 1 \tag{1.1}$$

where

$$L_n(f;x) = L_n(\mathcal{M},f;x) = \sum_{k=1}^{n} f(x_n^k)l_n^k(x), \quad n \ge 1 \tag{1.2}$$

are the *Lagrange interpolation polynomials* associated to the node matrix $\mathcal{M}$ and to the function $f$, while $R_n(f;g)$, $n \ge 1$, will be refered to as the *errors* of the product-quadrature procedures described by (1.1).

Denoting by

$$a_n^k : L^1 \to \mathbb{R}, \quad a_n^k(g) = \int_{-1}^{1} g(x)l_n^k(x)dx, \quad n \ge 1,\ 1 \le k \le n \tag{1.3}$$

$$D_n : C \times L^1 \to \mathbb{R}, \quad D_n(f;g) = \sum_{k=1}^{n} f(x_n^k)a_n^k(g), \quad n \ge 1 \tag{1.4}$$

$$A : C \times L^1 \to \mathbb{R}, \quad A(f;g) = \int_{-1}^{1} g(x)f(x)dx, \tag{1.5}$$

the product quadrature formulas (1.1) become:

$$A(f;g) = D_n(f;g) + R_n(f;g), \quad f \in C,\ g \in L^1,\ n \ge 1. \tag{1.6}$$

Remark that the product-quadrature procedures described by (1.1) or (1.6) are of interpolatory type with respect to the space $C$, i.e.:

$$A(P,g) = D_n(P,g), \quad n \ge 1,\ P \in \mathcal{P}_{n-1},\ g \in L^1 \tag{1.7}$$

where $\mathcal{P}_m$ is the space of all polynomials of degree at most $m \in \mathbb{N}$.

I.H. Sloan and W.E. Smith, [7], have established important results concerning the convergence of the product-quadrature procedures (1.6), for some node matrices

$\mathcal{M}$ whose $n$-th rows consist of the roots of the orthogonal polynomials associated to a weight-function $w(x)$ satisfying given integral inequalities, particularly for some Jacobi matrices $\mathcal{M}^{(\alpha,\beta)}$, $\alpha > -1$, $\beta > -1$. Moreover, these authors proved the existence of a pair $(f_0, g_0) \in C \times L^1$ so that the sequence $(D_n(f_0; g_0))_{n\geq 1}$ does not converge to $A(f_0, g_0)$ in (1.6).

The aim of this paper is to establish the topological structure of the sets of unbounded divergence in $C$ and $L^1$, corresponding to the product-quadrature procedures described by (1.6). On this subject, remark the results obtained by I. Muntean and S. Cobzaş for $g(x) = 1$, [1], [2].

## 2. Estimations concerning the norm of the functionals and operators involved in the product quadrature procedures

**2.1.** Firstly, let us consider the functionals $a_n^k$ given by (1.3). It is clear that $a_n^k$ are linear functionals for each $n \geq 1$ and $k \in \{1, 2, 3, \ldots, n\}$. On the other hand, the inequality

$$|a_n^k(g)| \leq \|l_n^k\| \cdot \|g\|_1 \tag{2.1}$$

proves the continuity of $a_n^k$ and leads to the inequality

$$\|a_n^k\| \leq \|l_n^k\| \tag{2.2}$$

Conversely, let $u \in [-1, 1]$ and $h > 0$ be given real numbers so that $u + h \in [-1, 1]$. Defining the function $g_0 \in L^1$ with $\|g_0\|_1 = 1$ by:

$$g_0(x) = \begin{cases} 1/h; & u \leq x \leq u + h \\ 0, & \text{otherwise} \end{cases} \tag{2.3}$$

we deduce:

$$\|a_n^k\| = \sup\{|a_n^k(g)| : \ g \in L^1, \ \|g\|_1 \leq 1\} \geq |a_n^k(g_0)|$$
$$= \left|\frac{1}{h}\int_u^{u+h} l_n^k(x)dx\right|, \ \forall\ h > 0, \ \forall\ u \in [-1, 1] \text{ with } u + h \in [-1, 1],$$

which implies:

$$\|a_n^k\| \geq \lim_{h\searrow 0} \left|\frac{1}{h}\int_u^{u+h} l_n^x(x)dx\right| = |l_n^k(u)|, \ \forall\ u \in [-1, 1],$$

so:

$$\|a_n^k\| \geq \|l_n^k\| \tag{2.4}$$

The relations (2.2) and (2.4) give:

$$\|a_n^k\| = \|l_n^k\| \tag{2.5}$$

**2.2.** Further, let $C^*$ be the Banach space of all linear and continuous functionals defined on $C$. Let us introduce the operators $T_n : L^1 \to C^*$, $g \mapsto T_n g$, $g \in L^1$, $n \geq 1$, where

$$(T_n g)(f) = \sum_{k=1}^{n} a_n^k(g) f(x_n^k), \quad f \in C \tag{2.6}$$

The linearity of the operators $T_n$, $n \geq 1$, follows from the corresponding property of the functionals $a_n^k$, $1 \leq k \leq n$. For each given $n \geq 1$, $T_n$ is a continuous operator, too; indeed, the inequality

$$|(T_n g)(f)| \leq \left( \sum_{k=1}^{n} |a_n^k(g)| \right) \|f\|$$

is valid for all $f \in C$ and it implies:

$$\|T_n g\| \leq \sum_{k=1}^{n} |a_n^k(g)|, \ \forall\, n \geq 1, \ \forall\, g \in L^1 \tag{2.7}$$

Now, the relations (2.7) and (2.5) give:

$$\|T_n g\| \leq \left( \sum_{k=1}^{n} \|l_n^k\| \right) \|g\|_1,$$

which proves the continuity of $T_n$, $n \geq 1$.

Now, let us establish the equality:

$$\|T_n g\| = \sum_{k=1}^{n} |a_n^k(g)|, \quad n \geq 1. \tag{2.8}$$

It remains to prove the converse inequality of (2.7). To this end, let consider for each $n \geq 1$, the function $f_n \in C$, $\|f_n\| = 1$, defined by:

$$f_n(x) = \begin{cases} \operatorname{sign} a_n^k(g), & \text{if } x \in \{x_n^k : \ 1 \leq k \leq n\} \\ 1, & \text{if } x \in \{-1, 1\} \setminus \{x_n^k : \ 1 \leq k \leq n\} \\ \text{linear}, & \text{otherwise} \end{cases}$$

We obtain, in accordance with (2.6):

$$\|T_n g\| = \sup\{|(T_n g)(f)| : \ f \in C, \ \|f\| \leq 1\} \geq |(T_n g)(f_n)| = \sum_{k=1}^{n} |a_n^k(g)|;$$

so, the equality (2.8) is true.

**2.3.** Finally, let us deduce the norm of the operator $T_n$, $n \geq 1$. Taking into account the relations (2.8) and (2.3), we have:

$$\|T_n\| = \sup\left\{\sum_{k=1}^{n} |a_n^k(g)| : \ g \in L^1, \ \|g\|_1 \leq 1\right\} \geq \sum_{k=1}^{n} |a_n^k(g_0)|$$

$$= \sum_{k=1}^{n} \left|\frac{1}{h}\int_u^{u+h} l_n^k(x)dx\right|, \ \forall \ h > 0,$$

therefore:

$$\|T_n\| \geq \lim_{h \searrow 0} \sum_{k=1}^{n} \left|\frac{1}{h}\int_u^{u+h} l_n^k(x)dx\right| = \sum_{k=1}^{n} |l_n^k(u)|, \ \forall \ u \in [-1,1]$$

which leads to the inequality

$$\|T_n\| \geq \lambda_n, \ \forall \ n \geq 1 \tag{2.9}$$

Conversely, we obtain from (2.6) and (1.3), by using the classic equality

$$\lambda_n = \sup\{\|L_n(f;\cdot)\| : \ f \in C, \ \|f\| \leq 1\}, \quad n \geq 1, \quad [6],[8],[3]:$$

$$\|T_n g\| = \sup\left\{\left|\int_{-1}^{1} g(x)L_n(f;x)dx\right| : \ f \in C, \ \|f\| \leq 1\right\}$$
$$\leq \|g\|_1 \cdot \sup\{\|L_n(f;\cdot)\| : \ f \in C, \ \|f\| \leq 1\} = \lambda_n \|g\|_1,$$

which shows that the opposite inequality of (2.9) is also true; so, we have:

$$\|T_n\| = \lambda_n, \ \forall \ n \geq 1. \tag{2.10}$$

A lower bound of the Lebesgue constants $\lambda_n$, $n \geq 1$, is given by Theorem of Lozinski-Harsiladze, [6], [8], [3]:

$$\lambda_n \geq \frac{2}{\pi^2} \ln n, \ \forall \ n \geq 1. \tag{2.11}$$

## 3. Superdense unbounded divergence of the product quadrature procedures

The main result of this paper is the following:

**Theorem 3.1.** *Given a node matrix $\mathcal{M}$ in the interval $[-1,1]$, there exists a superdense set $X_0$ in $C$ so that for each $f$ in $X_0$ the set*

$$Y_0(f) = \{g \in L^1 :\ \sup\{|D_n(f;g)| :\ n \geq 1\} = \infty\}$$

*is superdense in $L^1$.*

**Proof.** Firstly, we shall use the following *principle of condensation of the singularities*, deduced from [1, Theorem 5.4]:

*If $X$ is a Banach space, $Y$ is a normed space and $(A_n)_{n\geq 1}$ is a sequence of continuous linear operators from $X$ into $Y$ so that the set of norms $\{\|A_n\| :\ n \geq 1\}$ is unbounded, then the set of singularities of the family $\{A_n :\ n \geq 1\}$, i.e.*

$$\mathcal{S}(A_n) = \{x \in X :\ \sup\{\|A_n(x)\| :\ n \geq 1\} = \infty\},$$

*is superdense in $X$.*

Take $X = L^1$, $Y = C^*$ and $A_n = T_n : L^1 \to C^*$. The set $\{\|T_n\| :\ n \geq 1\}$ is unbounded, in accordance with (2.10) and (2.11); consequently, the set

$$\mathcal{S}(T_n) = \{g \in L^1 :\ \sup\{\|T_n g\| :\ n \geq 1\} = \infty\} \tag{3.1}$$

is superdense in $L^1$.

Next, let us apply the following *principle of the double condensation of singularities* [1], [2]:

*Suppose that $X$ is a Banach space, $Y$ is a normed space and $T$ is a nonvoid separable complete metric space without isolated points.*

*Let $\{A_n :\ n \geq 1\}$ be a family of mappings of $X \times T$ into $Y$ satisfying the following conditions:*

*(i) For each $t \in T$ and $n \geq 1$, the operator $A_n^t : X \to Y$, $A_n^t(x) = A_n(x,t)$, is linear and continuous.*

*(ii) For each $x \in X$ and $n \geq 1$, the operator $A_n^x : T \to Y$, $A_n^x(t) = A_n(x,t)$, is continuous.*

*(iii) There exists a dense set $\mathcal{T}_0$ in $T$ so that*

$$\sup\{\|A_n^t\| : \ n \geq 1\} = \infty, \ \forall \ t \in \mathcal{T}_0.$$

*Then, there exists a superdense set $X_0$ in $X$ so that for each $x \in X$ the set*

$$Y_0(x) = \{t \in T : \ \sup\{\|A_n(x,t)\| : \ n \geq 1\} = \infty\}$$

*is superdense in $T$.*

Take $X = (C, \|\cdot\|)$, $T = (L^1, \|g\|_1)$, $Y = \mathbb{R}$ and $A_n = D_n : C \times L^1 \to \mathbb{R}$, $n \geq 1$, see (1.4). Let us verify the validity of the previous hypotheses.

**(i)** We have:

$$D_n^g = T_n g, \quad g \in L^1, \ n \geq 1 \tag{3.2}$$

The linearity of $D_n^g$ follows from (2.6) and (1.3), while its continuity is a consequence of (2.7).

**(ii)** Taking into account (2.1), we deduce

$$|D_n^f| = \left|\sum_{k=1}^{n} a_n^k(g) f(x_n^k)\right| \leq \|g\|_1 \cdot \|f\| \cdot \sum_{k=1}^{n} \|l_n^k\|,$$

which proves the continuity of the linear functional $D_n^f$.

**(iii)** In accordance with (3.1) and (3.2) and taking $\mathcal{T}_0 = \mathcal{S}(T_n)$ we have:

$$\sup\{\|D_n^g\| : \ n \geq 1\} = \sup\{\|T_n g\| : \ n \geq 1\} = \infty, \ \forall \ g \in \mathcal{T}_0.$$

Now, let us apply the previous principle of the double condensation of singularities, which completes the proof of this theorem.

**Remark 3.2.** A dual-type result with respect to the Theorem 3.1 is also true [3]:

*Given a node matrix $\mathcal{M}$ in the interval $[-1,1]$, there exists a superdense set $X_1$ in $L^1$ so that for each $g \in X_1$ there exists a superdense set $Y_1(g)$ in $C$ satisfying the equality*

$$\limsup_{n\to\infty} |D_n(f;g)| = \infty, \ \textit{for each } g \in X_1 \textit{ and } f \in Y_1(g).$$

## 4. Estimations for the error of the product-quadrature procedures

In accordance with (1.6) and (1.7), writing

$$|R_n(f;g) = A(f-P;g) + D_n(P-f;g),$$

with an arbitrary $P \in \mathcal{P}_{n-1}$, we deduce:

$$|R_n(f;g)| \leq |A(f-P;g)| + |D_n(f-P;g)|. \tag{4.1}$$

Let $s \geq 0$ be an integer and denote by $C^s$ the Banach space of all functions $f : [-1,1] \to \mathbb{R}$ which are continuous together with their derivatives up to the order $s$, endowed with the norm:

$$\|f\|^{(s)} = \|f^{(s)}\| + \sum_{i=0}^{s-1} \|f^{(i)}(0)|, \text{ if } s \geq 1$$

and $\|f\|^{(0)} = \|f\|$.

It follows from the Theorem of Jackson [6], [8], [9], that there exist a polynomial $P \in \mathcal{P}_{n-1}$ and a positive number $M$ which does not depend on $n$ so that:

$$\|f^{(j)} - P^{(j)}\| \leq \frac{M}{n^{s-j}} \omega\left(f^{(s)}; \frac{1}{n}\right), \quad 0 \leq j \leq s, \tag{4.2}$$

for sufficient large $n \geq 1$, where $\omega(h;\cdot)$ is the modulus of continuity of a function $h \in C$.

Now, we deduce for each $i \in \{0,1,2,3,\ldots,s\}$, see also [4]:

$$\|f-P\|^{(i)} \leq \sum_{j=0}^{i} \|f^{(j)} - P^{(j)}\| \leq 2Mn^{i-s}\omega\left(f^{(s)}; \frac{1}{n}\right) \tag{4.3}$$

Taking $A_g = A(\cdot, g) : C^s \to \mathbb{R}$, with a given $g \in L^1$, we deduce from (4.1):

$$|R_n(f;g)| \leq \|A_g\| \cdot \|f-P\|^{(s)} + \left(\sum_{k=1}^{n} |a_n^k(g)|\right) \cdot \|f-P\| \tag{4.4}$$

Next, combining (4.2), (4.3) and (4.4), we obtain:

$$|R_n(f,g)| \leq M\left(2\|A_g\| + n^{-s}\sum_{k=1}^{n} |a_n^k(g)|\right)\omega\left(f^{(s)}; \frac{1}{n}\right) \tag{4.5}$$

A simple exercise leads to the inequalities:

$$\|A_g\| \leq \|g\|_1, \quad \sum_{k=1}^{n} |a_n^k(g)| \leq \lambda_n \|g\|_1,$$

which, together with (4.5), give for each $f \in C^s$ and $g \in L^1$:

$$|R_n(f;g)| \leq M(2 + \lambda_n \cdot n^{-s})\|g\|_1 \cdot \omega\left(f^{(s)}; \frac{1}{n}\right). \tag{4.6}$$

Denote by $DL(C)$ the subset of $C$ which consists of all functions $f \in C$ satisfying a Dini-Lipschitz condition

$$\lim_{\delta \searrow 0} \omega(f;\delta) \ln \delta = 0.$$

We are in a position to prove the following statement.

**Theorem 4.1.** *Suppose that $\mathcal{M} = \mathcal{M}^T$ is the Chebyshev node matrix, namely its $n$-th row consists of the roots of the Chebyshev polynomial*

$$P_n(x) = \cos(n \arccos x), \quad n \geq 1.$$

*The product-quadrature procedures described by (1.6) are convergent for each pair $(f,g) \in DL(C) \times L^1$ and for each pair $(f,g) \in C^s \times L^1$, if $s \geq 1$.*

**Proof.** If $s = 0$, we obtain from (4.6) and $\lambda_n \sim \ln n$, [5], [8], [3]:

$$|R_n(f;g)| \leq M(2 + \ln n)\|g\|_1 \cdot \omega\left(f; \frac{1}{n}\right),$$

so

$$\lim_{n\to\infty} R_n(f;g) = 0$$

for each $f \in DL(C)$ and $g \in L^1$. If $s \geq 1$, remark that $\lambda_n n^{-s} \sim n^{-s} \ln n$ and use again (4.6).

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