STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 3, September 2009

ABOUT CANONICAL FORMS OF THE NOMOGRAPHIC FUNCTIONS

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Abstract. This paper carries on the study of the functions of four variables in order to find the canonical forms (analogous to those of three variables) as well as nomograms in space with coplanar points on which the functions can be nomographically represented. We also build the nomograms in space for the canonical forms found out by Kazangapov respectively Wojtowicz. The factors of anamorphosis are examined.

The data in many physical, chemical, biological, tehnical experiments presented in tables with several entrances can by analyzed by performed computers. However the reading of the data and the subsequent results can be difficult. Therefore the use some simple ,,drawings", in order to analyses the relations between these data renders efficient employment of nomograms.

In [5]and [6] we are concerned with the study of nomographic functions of four variables $F(z_1, z_2, z_3, z_4)$, $F : D \subset \mathbb{R}^4 \to \mathbb{R}, D = D_1 \times D_2 \times D_3 \times D_4$, $D_i : a_i \le z_i \le b_i$, $i = \overline{1, 4}$.

In the first one we have in view a classification of these functions according to their rank with respect to the variables they depend on. In the second one we analyze the nomograms in space with coplanar points, on which the functions can be nomographically represented. These functions of four variables are of rank two with respect to each of their variables.

Received by the editors: 22.10.2008.

2000 *Mathematics Subject Classification.* 65S05.

Key words and phrases. nomogram, nomographic functions, canonical forms.

This paper was presented at the 7-th Joint Conference on Mathematics and Computer Science, July 3-6, 2008, Cluj-Napoca, Romania.

The study of these functions and of their corresponding equations (i.e. of the forms $F(z_1, z_2, z_3, z_4) = 0$) is further performed by looking for their canonical forms (analogous to those with three variables). We also have in view the nomographic representation of these canonical forms, both by the compound plane nomograms and the nomograms in space with coplanar points. We provide a classification of these functions according to the genus of nomogram on which the equation (and also the corresponding function) can be nomographically represented.

This study is also extended to the case of the functions of several variables (of five-eight variables).We also attempt a study of the canonical forms for the equations with four variables, founded by N.Kazangapov [3] and J. Wojtowics [8]. We will study the anamorphosis factors which permit the writing of the function of several variables by a determinant Massau of fourth order. We shell also analyze the number of these determinants and, consequently, the corresponding nomograms in space with coplanar points.

Definition 1. [5] *The function $F \equiv F(z_1, z_2, z_3, z_4)$ is called nomographic in space if:*

a) the rank of the function F with respect to each of its variables is at least two;

b) there exist the functions $X_i(z_1), Y_i(z_2), Z_i(z_3), T_i(z_4)$, $i = \overline{1,4}$ so that:

$$F(z_1, z_2, z_3, z_4) \equiv \begin{vmatrix} X_1(z_1) & X_2(z_1) & X_3(z_1) & X_4(z_1) \\ Y_1(z_2) & Y_2(z_2) & Y_3(z_2) & Y_4(z_2) \\ Z_1(z_3) & Z_2(z_3) & Z_3(z_3) & Z_4(z_3) \\ T_1(z_4) & T_2(z_4) & T_3(z_4) & T_4(z_4) \end{vmatrix}, \tag{1}$$

(i.e. F may be written in the form of determinant Massau of fourth order).

The definition of the rank of a function of four variables [5] is further generalized for the functions of eight variables in order to nomographically represent this function (as well of the equation which is attached to it) by a nomogram in space with coplanar points and with binary nets.

Let us consider the function of eight variables $F(z_1, z_2, z_3, ..., z_8)$ where $F : E \subset \mathbb{R}^8 \to \mathbb{R}$, $E = E_1 \times E_2 \times ... \times E_8$, $a_i \leq z_i \leq b_i$, $i = \overline{1, 8.}$

Definition 2. *The function with eighth variables $F = F(z_1, z_2, ..., z_8)$ is of rank n with respect to the variables z_1 and z_5 if there exist the real functions of two variables, $U_i(z_1, z_5)$, $i = \overline{1, n}$, as well as, the real functions of six variables $V_i(z_2, z_3, z_4, z_6, z_7, z_8)$, $i = \overline{1, n}$ so as to have:*

$$F(z_1, z_2, ..., z_8) \equiv \sum_{i=1}^{n} U_i(z_1, z_5) V_i(z_2, z_3, z_4, z_6, z_7, z_8), \tag{2}$$

where n is the greatest possible natural number for which the relation (2) occurs.

In a similar way, we can define the rank of the function $F(z_1, z_2, ..., z_8)$ with respect to any pair of two variables z_i, z_j; $i, j = \overline{1, 8}$, $i < j$.

The nomographic function of eight variables can also be defined according to Definition l.

Definition 3. *The function of eight variables $F(z_1, z_2, ..., z_8)$ is called nomographic in space if:*

1) the rank of the function F with respect to each pair of two variables z_i, z_j; $i, j = \overline{1, 8}$, $i < j$, is at least two;

2) there exist the functions $X_i(z_1, z_5), Y_i(z_2, z_6), Z_i(z_3, z_7), T_i(z_4, z_8)$, $i = \overline{1, 4}$, so that:

$$F(z_1, z_2, ..., z_8) \equiv \begin{vmatrix} X_1(z_1, z_5) & X_2(z_1, z_5) & X_3(z_1, z_5) & X_4(z_1, z_5) \\ Y_1(z_2, z_6) & Y_2(z_2, z_6) & Y_3(z_2, z_6) & Y_4(z_2, z_6) \\ Z_1(z_3, z_7) & Z_2(z_3, z_7) & Z_3(z_3, z_7) & Z_4(z_3, z_7) \\ T_1(z_4, z_8) & T_2(z_4, z_8) & T_3(z_4, z_8) & T_4(z_4, z_8) \end{vmatrix}. \tag{3}$$

This definition calls for writing the function F as a determinant Massau of fourth order. This determinant contains only the functions of two variables in each of its lines.

In particular, if z_5, z_6, z_7, z_8 are real constants, we obtain from (3) the form (1). It is obvious that, if the number of variables in the pairs z_i, z_j; $i, j = \overline{1, 8}$, $i < j$

from (2) is reduced by a unit we can obtain the rank of function under study with respect only to one variable (the remaining variable). Considering that one, two or three from variables z_5, z_6, z_7, z_8 become real constants we can also obtain other particular cases of the determinant (3).

The equation of Soreau can be solved by a nomogram, thanks to geometrically imposed conditions; i.e. the condition of coplanarity of four points in space.

Definition 4. *By a nomogram associated to a nomographic function we understand the equation's nomogram which is obtained by the equalization of respective function with zero.*

According to this definition, the nomographic representation of the function F (which has been brought to the form (3)) is equivalent to the nomographic representation of the Soreau equation associated to this function.

The equation Soreau associated to the function (3)

$$\begin{vmatrix} X_1(z_1,z_5) & X_2(z_1,z_5) & X_3(z_1,z_5) & X_4(z_1,z_5) \\ Y_1(z_2,z_6) & Y_2(z_2,z_6) & Y_3(z_2,z_6) & Y_4(z_2,z_6) \\ Z_1(z_3,z_7) & Z_2(z_3,z_7) & Z_3(z_3,z_7) & Z_4(z_3,z_7) \\ T_1(z_4,z_8) & T_2(z_4,z_8) & T_3(z_4,z_8) & T_4(z_4,z_8) \end{vmatrix} = 0 \tag{4}$$

represents (after elementary transformations) the condition that the four points, from the space $\mathbb{R}^3$, are situated in the same plane. This is possible, since according to the Definition 2, a nomographic function of eight variables has at least the rank two with respect to any from the pairs of the variables z_i, z_j; $i, j = \overline{1,8}$, $i < j$.

From the relation (4) we obtain:

$$\begin{vmatrix} \frac{X_1(z_1,z_5)}{X_5(z_1,z_5)} & \frac{X_2(z_1,z_5)}{X_5(z_1,z_5)} & \frac{X_3(z_1,z_5)}{X_5(z_1,z_5)} & 1 \\ \frac{Y_1(z_2,z_6)}{Y_5(z_2,z_6)} & \frac{Y_2(z_2,z_6)}{Y_5(z_2,z_6)} & \frac{Y_3(z_2,z_6)}{Y_5(z_2,z_6)} & 1 \\ \frac{Z_1(z_3,z_7)}{Z_5(z_3,z_7)} & \frac{Z_2(z_3,z_7)}{Z_5(z_3,z_7)} & \frac{Z_3(z_3,z_7)}{Z_5(z_3,z_7)} & 1 \\ \frac{T_1(z_4,z_8)}{T_5(z_4,z_8)} & \frac{T_2(z_4,z_8)}{T_5(z_4,z_8)} & \frac{T_3(z_4,z_8)}{T_5(z_4,z_8)} & 1 \end{vmatrix} = 0, \tag{5}$$

where the functions $X_5(z_1,z_5), Y_5(z_2,z_6), Z_5(z_3,z_7), T_5(z_4,z_8)$ are the linear combinations of the functions X_i, Y_i, Z_i, T_i, $i=\overline{1,4}$ and $a,b,c,d \in \mathbb{R}$, i.e.:

$$\begin{aligned}
X_5(z_1,z_5) \equiv \quad & aX_1(z_1,z_5)+bX_2(z_1,z_5)+cX_3(z_1,z_5)+dX_4(z_1,z_5)\\
Y_5(z_2,z_6) \equiv \quad & aY_1(z_2,z_6)+bY_2(z_2,z_6)+cY_3(z_2,z_6)+dY_4(z_2,z_6)\\
Z_5(z_3,z_7) \equiv \quad & aZ_1(z_3,z_7)+bZ_2(z_3,z_7)+cZ_3(z_3,z_7)+dZ_4(z_3,z_7)\\
T_5(z_4,z_8) \equiv \quad & aT_1(z_4,z_8)+bT_2(z_4,z_8)+cT_3(z_4,z_8)+dT_4(z_4,z_8).
\end{aligned}$$

According to Definition 3 each of the above linear combinations has at least two terms. Subsequently those four coplanar points will have the Cartesian coordinates equal to:

$$(P_i): \quad x=\frac{A_1(z_i,z_{i+4})}{A_5(z_i,z_{i+4})}; \quad y=\frac{A_2(z_i,z_{i+4})}{A_5(z_i,z_{i+4})}; \quad z=\frac{A_3(z_i,z_{i+4})}{A_5(z_i,z_{i+4})} \tag{6}$$

for $i=\overline{1,4}$; and $A_j(z_i,z_{i+4})$ for $j=1,2,3,5$ successively take the values $X_j(z_1,z_5)$, $Y_j(z_2,z_6)$, $Z_j(z_3,z_7)$, $T_j(z_4,z_8)$.

The points P_i are situated on the binary nets (z_i,z_{i+4}) for $i=\overline{1,4}$ (i.e. on a net consisting of two families of marked curves in space; one of them depends on the parameter z_i; the other on the parameters z_{i+4} (see Fig.1)

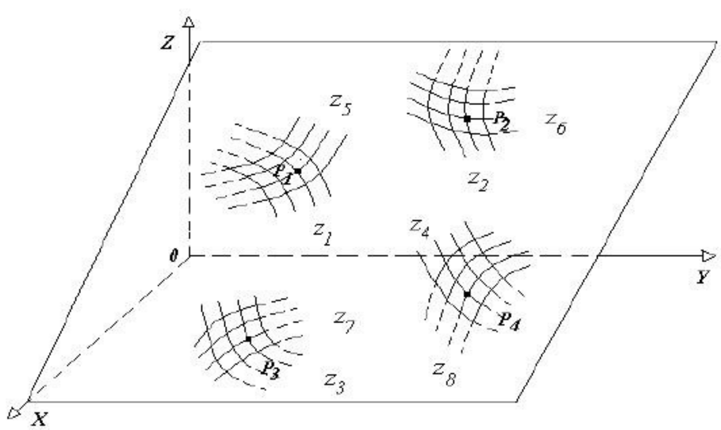

FIGURE 1

These families are obtained by the elimination, for all points P_i, $i = \overline{1,4}$, of the parameters z_{i+4}, respectively z_i,from the equation (6). This way we found the equations of two pairs of cylindrical surfaces for each $i = \overline{1,4}$:

$$S_1^i(x, y, z_i) = 0, \quad S_2^i(x, z, z_i) = 0, \tag{7}$$

$$S_3^i(x, y, z_{i+4}) = 0, \quad S_4^i(x, z, z_{i+4}) = 0. \tag{8}$$

The cylindrical surfaces (7) and(8) provide families of marked distorted curves in space with the parameter z_i (respectively z_{i+4}) from the binary net (z_i, z_{i+4}).

Thus, a function of eight variables $F(z_1, z_2, ..., z_8)$, which can be written in the form (3), is nomographically represented (like the equation associated to it, $F(z_1, z_2, ..., z_8) = 0$) by a nomogram in space with coplanar points. This nomogram contains four binary nets (i.e. both of them consisting of two families of distorted marked curves in space $\mathbb{R}^3$ (see Fig. 1).

The usage (or the ,,key" to its usage) of the nomogram from Fig.1 is simple: given the values of the first seven variables of the equation (4) we can find with the help of the first six of them, the coordinates of the three points in space, P_1, P_2, P_3, situated in three binary nets of the nomogram . The plane determinate by these three points, intersects the curves marked z_7 in a point P_4 from the binary net (z_7, z_8). The mark of the curve from the family having parameter z_8, which passes through last point P_4, will given the value of the eighth variable of the equation.

If we written the function of four variables $F(z_1, z_2, z_3, z_4)$ in the form (1) (and also the equation $F(z_1, z_2, z_3, z_4) = 0$) than it can be represented by a nomogram with coplanar points. The scales of the variables $z_i, i = \overline{1,4}$ are situated on the distorted curves C_i, $i = \overline{1,4}$ in $\mathbb{R}^3$, [4]. The usage of this nomogram is almost the same as the usage of the one above, if we replace the binary net with the marked scales.

If the number of variables of the nomographic function varies between five and eight, we also obtain a nomographic representation by a nomogram in space with coplanar points. The nomogram has four marked elements and it consists of the combinations including both marked scales and binary marked nets.

M. d'Ocagne and R. Soreau have found the canonical forms for the equation with three variables. M.Warmus [7] asserted seven main case for the nomographic functions of three variables. In [1] we studied the connection between the canonical forms and the main cases of Warmus.

A.N.Kazangapov [3] analyzed the canonical forms for the equations of four variables of four nomographic order

$$\begin{aligned} &A_0 f_1 f_2 f_3 f_4 + A_1 f_2 f_3 f_4 + A_2 f_1 f_3 f_4 + A_3 f_1 f_2 f_4 + A_4 f_1 f_2 f_3 + \\ &+ B_{12} f_1 f_2 + B_{13} f_1 f_3 + B_{14} f_1 f_4 + B_{23} f_2 f_3 + B_{24} f_2 f_4 + B_{34} f_3 f_4 + \\ &+ C_1 f_1 + C_2 f_2 + C_3 f_3 + C_4 f_4 + D = 0, \end{aligned} \tag{9}$$

where $f_i = f_i(z_i)$, $i = \overline{1,4}$ and its coefficients are real numbers.

He found three canonical equations:

$$f_1 f_2 f_3 f_4 - 1 = 0 \tag{10}$$

$$f_1 + f_2 + f_3 + f_4 = 0 \tag{11}$$

$$f_1 f_2 f_3 + f_1 f_2 f_4 + f_1 f_3 f_4 + f_2 f_3 f_4 = f_1 + f_2 + f_3 + f_4. \tag{12}$$

Consequently for the functions of four variables of the rank two with respect to each variable we have the following canonical forms :

$$F(z_1, z_2, z_3, z_4) \equiv X_1 Y_1 Z_1 T_1 - X_2 Y_2 Z_2 T_2, \tag{13}$$

$$F(z_1, z_2, z_3, z_4) \equiv X_1 Y_2 Z_2 T_2 + X_2 Y_1 Z_2 T_2 + X_2 Y_2 Z_1 T_2 + X_2 Y_2 Z_2 T_1, \tag{14}$$

$$\begin{aligned} F(z_1, z_2, z_3, z_4) &\equiv X_1 Y_1 Z_1 T_2 + X_1 Y_1 Z_2 T_1 + X_1 Y_2 Z_1 T_1 + X_2 Y_1 Z_1 T_1 \\ &- X_1 Y_2 Z_2 T_2 - X_2 Y_1 Z_2 T_2 - X_2 Y_2 Z_1 T_2 - X_2 Y_2 Z_2 T_1, \end{aligned} \tag{15}$$

where $X_i = X_i(z_1)$, $Y_i = Y_i(z_2)$, $Z_i = Z_i(z_3)$, $T_i = T_i(z_4)$, $i = 1, 2$.

We will study the nomograms in space by which the canonical equations (10)-(12) (as well as the functions (13)-(15)) are represented.

1. a) In [6] we analysed the six distinct projective nomograms that correspond to the equation (10) (and to function (13)). These nomograms are of genus zero (all its scales are rectilinear). We also built the nomograms in space, which is a compound nomogram of two plane nomograms with alignment points. One of them is situated in the plane X0Y, and another in Y0Z.

Since the variables of (10) are separated (so the Goursat condition is satisfied) for this equation we can build a plane compound nomogram from two nomograms with alignment points [4].

b) We can increase the genus of a nomogram that corresponds to (10) with two-four units if we multiply this equation with an anamorphosis factor. In this way, if the factor is $f_i - f_j$, $i, j = \overline{1,4}$, $i < j$, we can build a space nomogram with coplanar points of genus two, whose scales of variables z_i and z_j are situated on a quadratic curve (a conic) and the other two scales on the straight lines (see Fig. 2).

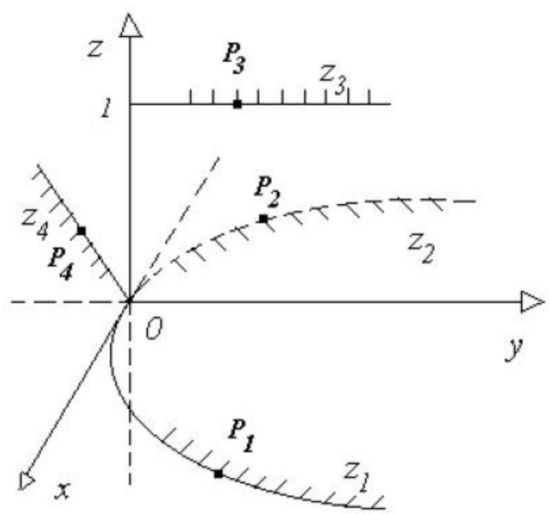

FIGURE 2

For example, if $i = 1$ and $j = 2$, we have:

$$(f_1 - f_2)(f_1 f_2 f_3 f_4 - 1) = 0 \tag{16}$$

or

$$f_1 \left[f_2 + \frac{1}{f_2 f_3 f_4} \right] - \frac{1}{f_3 f_4} - f_1^2 = 0. \tag{17}$$

With the notations

$$x = f_2 + \frac{1}{f_2 f_3 f_4}, \quad y = \frac{1}{f_3 f_4}, \quad z = \frac{1}{f_4}, \tag{18}$$

we obtain the disjunction equations

$$\begin{cases} f_i x \quad -y \qquad\quad -f_i^2 & = 0, \quad i = 1,2 \\ \qquad f_3 y \quad -z & = 0 \\ \qquad\qquad f_4 z \quad -1 & = 0, \end{cases} \tag{19}$$

and, after the elementary transformations, the equation Soreau

$$\begin{vmatrix} \frac{1}{f_1} & \frac{1}{f_1^2} & 0 & 1 \\ \frac{1}{f_2} & \frac{1}{f_2^2} & 0 & 1 \\ 0 & -f_3 & 1 & 1 \\ 0 & 0 & \frac{-f_4}{1-f_4} & 1 \end{vmatrix} = 0 \tag{20}$$

In the case of building the nomogram for a concrete equation the modulus of the scales and the dimensions of nomogram must necessarily appear.

Remark 1. *By a convenient permutation of the variables in the anamorphosis factor we can build other two scales on the curvilinear support.*

In this case the canonical form of the function of four variables is

$$F(z_1, z_2, z_3, z_4) \quad \equiv \quad X_1^2 Y_1 Y_2 Z_1 T_1 + X_1 X_2 [Y_2^2 Z_2 T_2 - Y_1^2 Z_1 T_1] - X_2^2 Y_1 Y_2 Z_2 T_2. \tag{21}$$

The function is of rank three with respect to variables z_1 and z_2, and respectively of rank two with z_3 and z_4.

c) For obtain the nomogram in space with coplanar points of genus three subsequently, we must multiply the equation (10) by the anamorphosis factor $(f_1 - f_2)(f_1 - f_3)(f_2 - f_3)$ and get

$$f_1^3 - f_1^2 \left[f_2 + f_3 + \frac{1}{f_2 f_3 f_4} \right] + f_1 \left[f_2 f_3 + \frac{1}{f_3 f_4} + \frac{1}{f_2 f_4} \right] - \frac{1}{f_4} = 0 \tag{22}$$

By notations

$$x = f_2 + f_3 + \frac{z}{f_2 f_3}; \ y = f_2 f_3 + \frac{f_2 + f_3}{f_2 f_3} z; \ z = \frac{1}{f_4}, \tag{23}$$

we find, from (22), the equation of disjunction of the variables

$$f_1^3 - f_1^2 x + f_1 y - z = 0. \tag{24}$$

By removing the f_3 and f_2 from (23), we find other two equations of disjunction, which together with the last one from (23) and with (24) give:

$$\begin{vmatrix} f_1 & f_1^2 & f_1^3 & 1 \\ f_2 & f_2^2 & f_2^3 & 1 \\ f_3 & f_3^2 & f_3^3 & 1 \\ 0 & 0 & \dfrac{1}{f_4} & 1 \end{vmatrix} = 0, \tag{25}$$

where we multiply by -1 the second and fourth columns, then interchange the first and third columns, and finally divide each of its lines by the elements of the last column.

In this way the equation (10) (respectively (22)) is nomographically represented by a nomogram in space with coplanar points of genus three. The scales of the variables z_i, $i = \overline{1,3}$ are on the curve (curve distorted in space) of the equations: $Y = X^2$ and $Z = X^3$; while for the variable z_4 there exists a straight line support.

Remark 2. *We can choose scales on the curved support for any three variables of (10) by a convenient change of the anamorphosis factor.*

Therefore another canonical form for the function of four variables is:

$$\begin{aligned} F(z_1, z_2, z_3, z_4) \quad & \equiv X_1 X_2^2 [Y_1^2 Y_2 (Z_1^3 T_1 - Z_2^3 T_2) + Z_1^2 Z_2 (Y_2^3 T_2 - Y_1^3 T_1)] - \\ & - X_1^2 X_2 [Y_1 Y_2^2 (Z_1^3 T_1 - Z_2^3 T_2) + Z_1 Z_2^2 (Y_2^3 T_2 - Y_1^3 T_1)] + \\ & + Y_1 Y_2 Z_1 Z_2 (Y_2 Z_1 - Y_1 Z_2)(X_1^3 T_1 - X_2^3 T_2) \end{aligned} \tag{26}$$

This function is of rank four with respect to variables z_i, $i = \overline{1,3}$ and of rank two with respect to z_4.

d) The nomogram in space with coplanar points of genus four for the equation (10) is obtained by multiplying it by the factor $(f_1 - f_2)(f_1 - f_3)(f_1 - f_4)(f_2 - f_3)(f_2 - f_4)(f_3 - f_4)$.

We obtain

$$\begin{aligned} &f_1^4 + 1 - \left[f_2 + f_3 + f_4 + \frac{1}{f_2 f_3 f_4}\right] f_1^3 + [f_2 f_3 + f_2 f_4 + f_3 f_4 + \\ &+ \frac{1}{f_2 f_3} + \frac{1}{f_2 f_4} + \frac{1}{f_3 f_4}\bigg] f_1^2 - \left[\frac{1}{f_2} + \frac{1}{f_3} + \frac{1}{f_4} + f_2 f_3 f_4\right] f_1 = 0. \end{aligned} \tag{27}$$

With the substitutions

$$\begin{aligned} x = &\quad \frac{1}{f_2} + \frac{1}{f_3} + \frac{1}{f_4} + f_2 f_3 f_4, \; z = f_2 + f_3 + f_4 + \frac{1}{f_2 f_3 f_4} \\ y = &\quad f_2 f_3 + f_2 f_4 + f_3 f_4 + \frac{1}{f_2 f_3} + \frac{1}{f_2 f_4} + \frac{1}{f_3 f_4}, \end{aligned} \tag{28}$$

we find the disjunction equations:

$$-f_i \; x + f_i^2 \; y - f_i^3 \; z + f_i^4 + 1 = 0, \; i = \overline{1,4} \tag{29}$$

and the equation Soreau

$$\begin{vmatrix} f_1 & f_1^2 & f_1^3 & 1 + f_1^4 \\ f_2 & f_2^2 & f_2^3 & 1 + f_2^4 \\ f_3 & f_3^2 & f_3^3 & 1 + f_3^4 \\ f_4 & f_4^2 & f_4^3 & 1 + f_4^4 \end{vmatrix} = 0. \tag{30}$$

Therefore the equation (10) (respectively (27)) is nomographically represented by a nomogram in space with all its scales situated on a distorted curve in space $X^4 + Y^4 - X^2 Y = 0$ and $Z(X^4 + Y^4) - XY^3 = 0$.

The corresponding nomographic function is of rank four with respect to all its variables and have the canonical form

$$F(z_1, z_2, z_3, z_4) \equiv \begin{vmatrix} X_1 X_2^3 & X_1^2 X_2^2 & X_1^3 X_2 & X_1^4 + X_2^4 \\ Y_1 Y_2^3 & Y_1^2 Y_2^2 & Y_1^3 Y_2 & Y_1^4 + Y_2^4 \\ Z_1 Z_2^3 & Z_1^2 Z_2^2 & Z_1^3 Z_2 & Z_1^4 + Z_2^4 \\ T_1 T_2^3 & T_1^2 T_2^2 & T_1^3 T_2 & T_1^4 + T_2^4 \end{vmatrix} \tag{31}$$

2. The equation (11) is obtain from (10) using the logarithmic function. The function (14) will be nomographically represented by the same kind of nomograms in space like as equation (10) (respectively the function (11)).

3. For the equation (12) brought to the form

$$(f_1f_2 + f_1f_3 + f_2f_3 - 1)f_4 + f_1f_2f_3 - f_1 - f_2 - f_3 = 0 \tag{32}$$

and with $x = f_1 + f_2 + f_3$, $y = f_1f_2 + f_1f_3 + f_2f_3$, $z = f_1f_2f_3$, we obtain the equation Soreau

$$\begin{vmatrix} f_1^2 & f_1 & \frac{1}{f_1} & 1 \\ f_2^2 & f_2 & \frac{1}{f_2} & 1 \\ f_3^2 & f_3 & \frac{1}{f_3} & 1 \\ -1 & \frac{1}{f_4} & -\frac{1}{f_4} & 1 \end{vmatrix} = 0 \tag{33}$$

and the nomogram in space with three scales which lie on the distorted curve and one scale on the straight line support.

The corresponding nomographic function is of rank four with respect to the variable z_i, $i = \overline{1,3}$ and two with respect to z_4. Its canonical form can be obtained from (33)

4. J. Wojtovicz [8] found another canonical form

$$f_i + f_j = f_kf_m, \quad i,\ j,\ k,\ m = \overline{1,4}, \quad i \neq j \neq k \neq m. \tag{34}$$

With notation $f_i = x$; $f_j = y$; $f_k = z$, we obtain the equation Soreau

$$\begin{vmatrix} 0 & 0 & f_i & 1 \\ 1 & 0 & f_j & 1 \\ 0 & 1 & f_k & 1 \\ \frac{1}{2 - f_m} & \frac{f_m}{f_m - 2} & 0 & 1 \end{vmatrix} = 0 \tag{35}$$

and also the corresponding nomogram in space of genus zero (in fact the scale of the variable z_m has as support a degenerate quadratic curve: two straight lines of equations $X = 0$ and $2X + Y - 1 = 0$; only the last one of these being considerate as a support).

The canonical forms is of rank two with respect to all its variables

$$F(z_i, z_j, z_k, z_m) \equiv X_1Y_2Z_2T_2 - X_2(Y_2Z_1T_1 - Y_1Z_2T_2). \tag{36}$$

We can obtain other canonical forms for the equations (respectively functions) of several variables by generalization those of four variables i.e.

$$F_{12}(z_1,z_5)F_{26}(z_2,z_6)F_{37}(z_3,z_7)F_{48}(z_4,z_8)-1=0, \tag{37}$$

$$\begin{aligned} F(z_1,z_2,...,z_8) \quad &\equiv X_1(z_1,z_5)Y_1(z_2,z_6)Z_1(z_3,z_7)T_1(z_4,z_8)- \\ &-X_2(z_1,z_5)Y_2(z_2,z_6)Z_2(z_3,z_7)T_2(z_4,z_8). \end{aligned} \tag{38}$$

They are nomographically represented by a nomogram in space with coplanar points. These points are situated in four binary marked nets.

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