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# MULTIPLE SOLUTIONS FOR A HOMOGENEOUS SEMILINEAR ELLIPTIC PROBLEM IN DOUBLE WEIGHTED SOBOLEV SPACES

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**Abstract**. In this paper we obtain multiple solutions in double weighted Sobolev spaces for an elliptic semilinear eigenvalue problem on unbounded domain, with sublinear growth of the nonlinear term. In the proofs of the main results we use variational methods and some recent theorems from the theory of best approximation in Banach spaces, established by Ricceri in [11] and Tsar'kov in [12].

## 1. **Introduction**

A link between the critical point theory and the theory of best approximation was established recently by Ricceri in [11] and Tsar'kov in [12]. In the latter it is proved that, given a continuously Gâteaux differentiable functional $J$ defined over a real Hilbert space $X$, for each real $\sigma$ within the range of $J$ and $x_0 \in J^{-1}(]-\infty,\sigma[)$ either there exists $\lambda > 0$ such that the energy functional $\mathcal{E}_\lambda(x) = \frac{||x-x_0||^2}{2} - \lambda J(x)$ admits at least three critical points, or the set $J^{-1}([\sigma,+\infty[)$ has a unique point minimizing the distance from $x_0$. The alternative is then resolved. Supposing that $J$ admits non-convex superlevel set, and applying the results of [12], yields that the energy functional $\mathcal{E}_\lambda$ has at least three critical points for suitable $x_0 \in X$ and $\lambda > 0$. This abstract result has a natural application in the field of differential equations.

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The result of Ricceri was applied and extended by several authors: Kristály in [4] study a Schrödinger equation in $\mathbb{R}^N$, Faracci and Iannizzotto in [2] study boundary value problems involving the $p$-Laplacian on unbounded domain, Faracci, Iannizzotto, Lisei, Varga in [3] give a multiplicity result in alternative form for a class of locally Lipschitz functionals, defined on Banach spaces and applied to hemivariational inequalities on unbounded domain.

In this paper we consider a semilinear elliptic eigenvalue problem on unbounded domain and we apply a topological minimax result of Ricceri [10] to obtain a similar theorem (in alternative form) with the result of Ricceri presented above. Then, as a consequence of the obtained theorem, using the results of Tsar'kov [12], we obtain three different solutions of the considered problem.

The main problem we are confronting, is the lack of compact embeddings of Sobolev spaces. In general, if $\Omega$ is unbounded, $W^{1,p}(\Omega)$ (the space of all functions $u \in L^p(\Omega)$, such that $|\nabla u| \in L^p(\Omega)$) is not compactly embedded in any $L^r(\Omega)$. We will overcome this difficulty by using the double weighted Sobolev space $W^{1,2}(\Omega; v_0, v_1)$ with such weight functions $v_0,\ v_1,\ w$ that $W^{1,2}(\Omega; v_0, v_1)$ can be embedded compactly in $L^p(\Omega; w)$ (for $p \in [2, 2^*[$).

## 2. The problem and preliminaries

Let $\Omega \subset \mathbb{R}^N,\ (N \geq 2)$ be an unbounded domain with smooth boundary $\partial\Omega$. For the positive measurable functions $u$ and $w$, both defined in $\Omega$, we define the weighted $p$-norm $(1 \leq p < \infty)$ as

$$||u||_{p,\Omega,w} = \left( \int_\Omega |u(x)|^p w(x) dx \right)^{\frac{1}{p}}$$

and denote by $L^p(\Omega; w)$ the space of all measurable functions $u$ such that $||u||_{p,\Omega,w}$ is finite. If $p = +\infty$ we consider the Sobolev space

$$L^\infty(\Omega) = \{u : \Omega \to \mathbb{R} \mid u \text{ is measurable, } \exists C > 0 \text{ such that } |u(x)| \leq C \text{ a.e. in } \Omega\}$$

endowed with the norm

$$||u||_\infty = \inf\{C : |u(x)| \leq C \text{ for a.e. } x \in \Omega\}.$$

The double weighted Sobolev space

$$W^{1,p}(\Omega; v_0, v_1)$$

is defined as the space of all functions $u \in L^p(\Omega; v_0)$ such that all derivatives $\frac{\partial u}{\partial x_i}$ belong to $L^p(\Omega; v_1)$. The corresponding norm is defined by

$$||u||_{p,\Omega,v_0,v_1} = \left( \int_\Omega |\nabla u(x)|^p v_1(x) + |u(x)|^p v_0(x) dx \right)^{\frac{1}{p}}.$$

We are choosing our weight functions from the so-called Muckenhoupt class $A_p$, which is defined as the set of all positive functions $v$ in $\mathbb{R}^N$ satisfying

$$\frac{1}{|Q|} \left( \int_\Omega v \ dx \right)^{\frac{1}{p}} \left( \int_\Omega v^{-\frac{1}{p-1}} \ dx \right)^{\frac{p-1}{p}} \leq \bar{C}, \text{ if } 1 < p < \infty$$

$$\frac{1}{|Q|} \int_\Omega v \ dx \leq \bar{C} \ \operatorname{ess} \inf_{x \in Q} v(x), \text{ if } p = 1,$$

for all cubes $Q \in \mathbb{R}^N$ and some $\bar{C} > 0$.

In this paper we always assume that the weight functions $v_0, \ v_1, \ w$ are defined on $\Omega$, belong to $A_p$ and are chosen such that the following condition holds:
$(E)$ for $p \in [2, 2^*[$ the embedding $W^{1,2}(\Omega; v_0, v_1) \hookrightarrow L^p(\Omega; w)$ is compact.

Such weight functions there exist, see for example [7], [8].

The best embedding constant is denoted by $C_{p,\Omega}$, i.e. we have the inequality

$$||u||_{p,\Omega,w} \leq C_{p,\Omega} ||u||_{v_0,v_1}, \quad \text{for all } u \in W^{1,2}(\Omega; v_0, v_1) \tag{1}$$

where we used the abbreviation $||u||_{v_0,v_1} = ||u||_{2,\Omega,v_0,v_1}$.
We define on $W^{1,2}(\Omega; v_0, v_1)$ a continuous bilinear form associated with the operator $A(u) = -\Delta u + b(x)u$ as

$$\langle u, v \rangle_A = \int_\Omega (\nabla u \nabla v + b(x)uv) dx \tag{2}$$

and the corresponding norm with

$$||u||_A^2 = \langle u, u \rangle_A = \int_\Omega (|\nabla u(x)|^2 + b(x)|u(x)|^2) dx. \tag{3}$$

Now, we define the Banach space

$$X_A = \{u \in W^{1,2}(\Omega; v_0, v_1) : ||u||_A < \infty\}, \tag{4}$$

endowed with the norm $||\cdot||_A$.

We consider the following problem

*For a given* $u_0 \in X_A$ *and* $\lambda > 0$ *find* $u \in X_A$ *such that*

$$(P_\lambda) \qquad \begin{cases} -\Delta(u-u_0) + b(x)(u-u_0) = \lambda\alpha(x)f(u) \ \textit{in } \Omega \\ \\ \qquad\qquad\qquad\qquad u = 0 \qquad \textit{on } \partial\Omega, \end{cases}$$

*where* $f : \mathbb{R} \to \mathbb{R}$ *is a continuous function,* $\alpha : \Omega \to \mathbb{R}$ *and* $b : \Omega \to \mathbb{R}$ *are a positive and measurable functions.*

By the *weak solution* to this problem we mean a function $u \in X_A$, such that for every $v \in X_A$ we have

$$\langle u-u_0, v\rangle_A - \lambda \int_\Omega \alpha(x)f(u(x))v(x)dx = 0.$$

We will study the problem $(P_\lambda)$ assuming that $f$ is *sublinear* at the origin, that is

($f$) $f(0) = 0$ and there is a positive measurable function $f_0 : \Omega \to \mathbb{R}$ satisfying $f_0 \in L^{\frac{p}{p-1}}(\Omega, w^{\frac{1}{1-p}})$, $f_0(x) \leq C_f w(x)$ for a.e. $x \in \Omega$, where $C_f$ is a positive constant and there exists $q \in ]0,1[$ such that

$$|f(s)| \leq f_0(x)|s|^q, \text{ for every } s \in \mathbb{R} \text{ and every } x \in \Omega;$$

Furthermore we consider the following assumptions:

(K) ellipticity condition: there is a positive constant $K$, such that

$$||u||_A^2 \geq 2K||u||_{v_0,v_1}^2, \text{ for every } u \in W^{1,2}(\Omega; v_0, v_1);$$

($\alpha$) $\alpha \in L^1(\Omega, w) \cap L^\infty(\Omega)$.

In the sequel we prove several lemmas needed later.

**Lemma 2.1.** $L^1(\Omega; w) \cap L^\infty(\Omega) \subseteq L^r(\Omega; w)$, *for every* $r \geq 1$.

**Proof.** Let $u \in L^1(\Omega; w) \cap L^\infty(\Omega)$. Then, for every $r \geq 1$ we have

$$\begin{aligned} ||u||_{r,\Omega,w} &= \left(\int_\Omega |u(x)|^r w(x)dx\right)^{\frac{1}{r}} = \left(\int_\Omega |u(x)|^{r-1}|u(x)|w(x)dx\right)^{\frac{1}{r}} \leq \\ &\leq \left(\int_\Omega ||u||_\infty^{r-1}|u(x)|w(x)dx\right)^{\frac{1}{r}} = ||u||_\infty^{\frac{r-1}{r}} ||u||_{1,w}^{\frac{1}{r}}, \end{aligned}$$

which means that $||u||_{r,\Omega,w}$ is finite, so $u \in L^r(\Omega; w)$. □

**Notation.** Let $\nu = \dfrac{p}{p-(q+1)}$ and we denote by $\nu' = \frac{p}{q+1}$ its conjugate, that is $\frac{1}{\nu} + \frac{1}{\nu'} = 1$. Using the Lemma 2.1 we have that

$$L^1(\Omega; w) \cap L^\infty(\Omega) \subseteq L^\nu(\Omega; w),$$

so $\alpha \in L^\nu(\Omega; w)$.

We define the funtional $J : X_A \to \mathbb{R}$ by

$$J(u) = \int_\Omega \alpha(x)F(u(x))dx,$$

where $F(t) = \int_0^t f(s)ds$.

The next lemma summarize the properties of the functional $J$.

**Lemma 2.2.** *Let conditions $(f)$, $(K)$, $(\alpha)$ be satisfied.Then, the functional $J$ is well defined and it is sequentially weakly continuous.*

**Proof.** From the assumption $(f)$ we have

$$|F(u(x))| \leq \int_0^{u(x)} |f(s)|ds \leq f_0(x)\int_0^{u(x)} |s|^q ds \leq f_0(x)|u(x)|^{q+1}. \tag{5}$$

Then, using the conditions $(f)$, $(E)$ and the Hölder's inequality, we get

$$\begin{gathered} |J(u)| = \left|\int_\Omega \alpha(x)F(u(x))dx\right| \leq \int_\Omega \alpha(x)f_0(x)|u(x)|^{q+1}dx \leq \\ \leq C_f \int_\Omega \alpha(x)|u(x)|^{q+1}w(x)dx \leq C_f \int_\Omega \alpha(x)w(x)^{\frac{1}{\nu}}|u(x)|^{q+1}w(x)^{\frac{1}{\nu'}}dx \leq \\ \leq C_f\left(\int_\Omega \alpha(x)^\nu w(x)dx\right)^{\frac{1}{\nu}}\left(\int_\Omega |u(x)|^{\nu'(q+1)}w(x)dx\right)^{\frac{1}{\nu'}} = \\ = C_f||\alpha||_{\nu,\Omega,w}\left(\int_\Omega |u(x)|^p w(x)dx\right)^{\frac{q+1}{p}} = C_f||\alpha||_{\nu,\Omega,w}||u||_{p,\Omega,w}^{q+1} \leq \\ \leq C_f||\alpha||_{\nu,\Omega,w}C_{p,w}^{q+1}||u||_{v_0,v_1}^{q+1} \leq C_f||\alpha||_{\nu,\Omega,w}C_{p,w}^{q+1}(2K)^{-\frac{q+1}{2}}||u||_A^{q+1} = \end{gathered}$$

$$= C||u||_A^{q+1},$$

which means that the functional $J$ is well defined over $X_A$.

We prove now, that $J$ is sequentially weakly continuous. Let $\{u_n\}$ be a sequence in $X_A$, weakly convergent to some $u \in X_A$. Then, by the embedding $(E)$, it follows that $||u_n - u||_{p,\Omega,w} \to 0$.

We use the following result: for all $s \in (0, \infty)$ there is a constant $C_s > 0$ such that

$$(x + y)^s \leq C_s(x^s + y^s), \quad \text{for any } x, y \in (0, \infty). \tag{6}$$

Applying the (6), the Hölder inequalities and the Mean Value Theorem, we obtain

$$|J(u_n) - J(u)| = \left| \int_\Omega \alpha(x)F(u_n(x))dx - \int_\Omega \alpha(x)F(u(x))dx \right| \leq$$

$$\leq \int_\Omega \alpha(x)|F(u_n(x)) - F(u(x))|dx =$$

$$= \int_\Omega \alpha(x)|f((1-\theta)u_n(x) + \theta u(x))||u_n(x) - u(x)|dx \leq$$

$$\leq \int_\Omega \alpha(x)f_0(x)\,|(1-\theta)u_n(x) + \theta u(x)|^q\,|u_n(x) - u(x)|dx \leq$$

$$\leq \int_\Omega \alpha(x)f_0(x)\,((1-\theta)|u_n(x)|^q + \theta|u(x)|^q)\,|u_n(x) - u(x)|dx \leq$$

$$\leq C_f \int_\Omega \alpha(x)w(x)^{\frac{1}{\nu}}\,(|u_n(x)|^q + |u(x)|^q)\,|u_n(x) - u(x)|w(x)^{\frac{1}{\nu'}}dx \leq$$

$$\leq C_f||\alpha||_{\nu,\Omega,w}\left(\int_\Omega (|u_n(x)|^q + |u(x)|^q)^{\nu'}\,|u_n(x) - u(x)|^{\nu'}w(x)dx\right)^{\frac{1}{\nu'}} =$$

$$= C_f||\alpha||_{\nu,\Omega,w}C_1 \cdot$$

$$\cdot \left[\int_\Omega \left(|u_n(x)|^{\frac{pq}{q+1}} + |u(x)|^{\frac{pq}{q+1}}\right) w(x)^{\frac{q}{q+1}}\,(|u_n(x) - u(x)|^p)^{\frac{1}{q+1}}\,w(x)^{\frac{1}{q+1}}dx\right]^{\frac{1}{\nu'}} \leq$$

$$\leq C_f||\alpha||_{\nu,\Omega,w}C_1\left[\left(\int_\Omega |u_n(x)|^p w(x)dx\right)^{\frac{q}{q+1}} + \left(\int_\Omega |u(x)|^p w(x)dx\right)^{\frac{q}{q+1}}\right]^{\frac{1}{\nu'}} \cdot$$

$$\cdot\left(\int_\Omega |u_n(x) - u(x)|^p w(x)dx\right)^{\frac{1}{q+1}\cdot\frac{1}{\nu'}} =$$

$$= C_f||\alpha||_{\nu,\Omega,w}C_1\left(||u_n||_{p,\Omega,w}^{\frac{pq}{q+1}} + ||u||_{p,\Omega,w}^{\frac{pq}{q+1}}\right)^{\frac{q+1}{p}}||u_n - u||_{p,\Omega,w} \leq$$

$$\leq C_f||\alpha||_{\nu,\Omega,w}C_1C_2\left(||u_n||_{p,\Omega,w}^q + ||u||_{p,\Omega,w}^q\right)||u_n - u||_{p,\Omega,w} \leq$$

$$\leq C_f||\alpha||_{\nu,\Omega,w}C_1C_2C_{p,w}^q\left(||u_n||_{v_0,v_1}^q + ||u||_{v_0,v_1}^q\right)||u_n - u||_{p,\Omega,w},$$

$$\leq C_f||\alpha||_{\nu,\Omega,w}C_1C_2C_{p,w}^q(2K)^{\frac{-q}{2}}\left(||u_n||_A^q + ||u||_A^q\right)||u_n - u||_{p,\Omega,w},$$

where $\theta \in ]0,1[$ is the constant from the Mean Value Theorem, $C_1, C_2$ are the constants from the inequality (6) and $K$ is the constant from the ellipticity condition $(K)$.

Since $\{u_n\}$ is weakly convergent to $u \in X_A$, we can assume without loss of generality that there exist a constant $M > 0$ such that

$$||u_n||_A \leq M \text{ and } ||u_n - u||_A \leq M, \text{ for all } n \in \mathbb{N}.$$

Then we have

$$|J(u_n) - J(u)| \leq ||\alpha||_{\nu,\Omega,w}C_fC_1C_2C_{p,w}^q(2K)^{\frac{-q}{2}}2M^q \cdot ||u_n - u||_{p,\Omega,w},$$

concluding that $J(u_n) \to J(u)$, whenever $n \to \infty$. □

Now, for a given $u_0 \in X_A$ and for $\lambda > 0$, we can define the energy functional $\mathcal{E}_\lambda : X_A \to \mathbb{R}$ related to the problem $(P_\lambda)$ by

$$\mathcal{E}_\lambda(u) = \frac{1}{2}||u - u_0||_A^2 - \lambda J(u).$$

We observe, that for every $v \in X_A$, we have

$$\langle \mathcal{E}_\lambda'(u), v\rangle_A = \langle u - u_0, v\rangle_A - \lambda\int_\Omega \alpha(x)f(u(x))v(x)dx. \tag{7}$$

Hence the critical points of the energy functional $\mathcal{E}_\lambda$ are exactly the weak solutions of the problem $(P_\lambda)$. Therefore, instead of looking for solutions of the problem $(P_\lambda)$, we are seeking for the critical points of $\mathcal{E}_\lambda$.

In the next lemmas we prove two properties of the energy functional, namely that $\mathcal{E}_\lambda$ is coercive and it satisfies the Palais-Smale condition, for every $\lambda > 0$.

**Lemma 2.3.** *Let the conditions $(f)$, $(K)$, $(\alpha)$ be satisfied. Then the functional $\mathcal{E}_\lambda$ is coercive, for every $\lambda > 0$.*

**Proof.** Using again the Hölder's inequality combined with the conditions $(f)$ and $(E)$, we obtain

$$\begin{aligned}
&\mathcal{E}_\lambda(u) = \frac{1}{2}||u-u_0||_A^2 - \lambda\int_\Omega \alpha(x)F(u(x))v(x)dx \geq \\
\geq\ & \frac{1}{2}||u-u_0||_A^2 - \lambda\int_\Omega \alpha(x)f_0(x)|u(x)|^{q+1}dx \geq \\
\geq\ & \frac{1}{2}||u-u_0||_A^2 - \lambda C_f\int_\Omega \alpha(x)w(x)^{\frac{1}{\nu}}|u(x)|^{q+1}w(x)^{\frac{1}{\nu'}}dx \geq \\
\geq\ & \frac{1}{2}||u-u_0||_A^2 - \lambda C_f\left(\int_\Omega \alpha(x)^\nu w(x)dx\right)^{\frac{1}{\nu}}\left(\int_\Omega |u(x)|^{(q+1)\nu'}w(x)dx\right)^{\frac{1}{\nu'}} = \\
=\ & \frac{1}{2}||u-u_0||_A^2 - \lambda C_f||\alpha||_{\nu,\Omega,w}||u||_{p,\Omega,w}^{q+1} \geq \\
\geq\ & \frac{1}{2}||u-u_0||_A^2 - \lambda C_f C_{p,w}^{q+1}||\alpha||_{\nu,\Omega,w}||u||_{v_0,v_1}^{q+1} \geq \\
\geq\ & \frac{1}{2}||u-u_0||_A^2 - \lambda C_f C_{p,w}^{q+1}||\alpha||_{\nu,\Omega,w}(2K)^{\frac{-q-1}{2}}||u||_A^{q+1}.
\end{aligned}$$

Therefore $\mathcal{E}_\lambda(u) \to \infty$, whenever $||u||_A \to \infty$, since $q+1 < 2$. □

**Lemma 2.4.** *Assume that $(f)$, $(K)$, $(\alpha)$ are satisfied. Then $\mathcal{E}_\lambda$ satisfies the Palais-Smale condition for every $\lambda > 0$.*

**Proof.** Let $\{u_n\} \subset X_A$ be an arbitrary Palais-Smale sequence for $\mathcal{E}_\lambda$, i.e.

(a) $\{\mathcal{E}_\lambda(u_n)\}$ is bounded;

(b) $\mathcal{E}'_\lambda(u_n) \to 0$, as $n \to \infty$.

We will prove that $\{u_n\}$ contains a strongly convergent subsequence in $X_A$. From the coercivity of $\mathcal{E}_\lambda$, it follows that $\{u_n\}$ is bounded, hence we can find a subsequence, which we still denote by $\{u_n\}$, weakly convergent to a point $u \in X_A$. Then by the embedding condition $(E)$, $\{u_n\}$ tends strongly to $u$ in $L^p(\Omega; w)$, so $||u_n - u||_{p,\Omega,w} \to 0$, as $n \to \infty$.

Since the sequence from (b) tends to 0, for $n \in \mathbb{N}$ big enough, we have

$$\left|\langle \mathcal{E}'_\lambda(u_n), \frac{u_n}{||u_n||_A}\rangle_A\right| \leq \varepsilon,$$

or equivalently

$$|\langle \mathcal{E}'_\lambda(u_n), u_n\rangle_A| \leq \varepsilon||u_n||_A.$$

Then, by (7) we get

$$\langle u_n - u_0, u_n\rangle_A - \lambda \int_\Omega \alpha(x) f(u_n(x)) u_n(x) dx \le \varepsilon ||u_n||_A.$$

Rearranging the inequality and taking the absolute value, we obtain

$$|\langle u_n - u_0, u_n\rangle_A| \le \varepsilon ||u_n||_A + \lambda \int_\Omega \alpha(x) |f(u_n(x)) u_n(x)| dx.$$

After simple computations this inequality gives us

$$||u_n - u||_A^2 \le |\langle u_n - u_0, u_n - u\rangle_A| + |\langle u_0 - u, u_n - u\rangle_A| \le$$

$$\le |\langle u_n, u_n - u\rangle_A| + |\langle u, u_n - u\rangle_A| + 2|\langle u_0, u_n - u\rangle_A| \le$$

$$\le 4\varepsilon ||u_n - u||_A + \lambda \int_\Omega \alpha(x) |f(u_n(x))(u_n(x) - u(x))| dx +$$

$$+\lambda \int_\Omega \alpha(x) |f(u(x))(u_n(x) - u(x))| dx + \lambda \int_\Omega \alpha(x) |f(u_0(x))(u_n(x) - u(x))| dx.$$

Now, we will estimate the integrals from the above inequality using the inequalities of Hölder, the ellipticity condition $(K)$ and the embedding condition $(E)$. The first integral can be estimated as follows

$$\begin{aligned}
& \int_\Omega \alpha(x) |f(u_n(x))(u_n(x) - u(x))| dx \le \\
\le \quad & C_f \int_\Omega \alpha(x) w(x)^{\frac{1}{\nu}} |u_n(x)|^q |u_n(x) - u(x)| w(x)^{\frac{1}{\nu'}} dx \le \\
\le \quad & C_f ||\alpha||_{\nu,\Omega,w} \left( \int_\Omega |u_n(x)|^{q\nu'} |u_n(x) - u(x)|^{\nu'} w(x) dx \right)^{\frac{1}{\nu'}} = \\
= \quad & C_f ||\alpha||_{\nu,\Omega,w} \left( \int_\Omega (|u_n(x)|^p w(x))^{\frac{q}{q+1}} (|u_n(x) - u(x)|^p w(x))^{\frac{1}{q+1}} dx \right)^{\frac{1}{\nu'}} \le \\
\le \quad & C_f ||\alpha||_{\nu,\Omega,w} \left[ \left( \int_\Omega |u_n(x)|^p w(x) dx \right)^{\frac{q}{q+1}} \left( \int_\Omega |u_n(x) - u(x)|^p w(x) dx \right)^{\frac{1}{q+1}} \right]^{\frac{q+1}{p}} = \\
= \quad & C_f ||\alpha||_{\nu,\Omega,w} ||u_n||_{p,\Omega,w}^q ||u_n - u||_{p,\Omega,w} \le \\
\le \quad & C_f ||\alpha||_{\nu,\Omega,w} C_{p,w}^q ||u_n||_{v_0,v_1}^q ||u_n - u||_{p,\Omega,w} \le \\
\le \quad & C_f ||\alpha||_{\nu,\Omega,w} C_{p,w}^q (2K)^{\frac{-q}{2}} M^q ||u_n - u||_{p,\Omega,w}
\end{aligned}$$

where in the last inequality we used that $\{u_n\}$ is bounded, hence there is a constant $M > 0$ such that $||u_n||_A < M$, $||u||_A < M$.

Proceeding in the same manner for the other two integrals, we obtain:

$$\int_\Omega \alpha(x)|f(u(x))(u_n(x)-u(x))|dx \le C_f||\alpha||_{\nu,\Omega,w}C^q_{p,w}(2K)^{\frac{-q}{2}}M^q||u_n-u||_{p,\Omega,w}$$

$$\int_\Omega \alpha(x)|f(u_0(x))(u_n(x)-u(x))|dx \le C_f||\alpha||_{\nu,\Omega,w}C^q_{p,w}(2K)^{\frac{-q}{2}}||u_0||^q_A||u_n-u||_{p,\Omega,w}.$$

Then, we have

$$\begin{aligned}||u_n-u||^2_A \quad &\le 4\varepsilon||u_n-u||_A+\\ &+\lambda C_f||\alpha||_{\nu,\Omega,w}C^q_{p,w}(2K)^{\frac{-q}{2}}(2M^q+||u_0||^q_A)||u_n-u||_{p,\Omega,w}.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrarily choosen, $||u_n-u||_A$ is bounded, $||u_0||_{p,\Omega,w}$ is finite ($u_0$ being given) and $||u_n-u||_{p,\Omega,w}$ tends to 0 as $n\to\infty$, we conclude that $||u_n-u||_A\to 0$, whenever $n\to\infty$. □

We conclude this section by recalling two results which will be used in proofs of the next section. The first one is a topological minimax theorem due to B. Ricceri:

**Theorem 2.1.** [10, Theorem 1 and Remark 1] *Let $X$ be a topological space, $\Gamma$ a real interval, and $f : X\times\Gamma\to\mathbb{R}$ a function satisfying the following conditions:*

($A1$) *for every $x\in X$, the function $f(x,\cdot)$ is quasi-concave and continuous;*

($A2$) *for every $\lambda\in\Gamma$, the function $f(\cdot,\lambda)$ is lower semicontinuous and each of its local minima is a global minimum;*

($A3$) *there exist $\rho_0 > \sup_\Gamma \inf_X f$ and $\lambda_0\in\Gamma$ such that $\{x\in X : f(x,\lambda_0)\le\rho_0\}$ is compact.*

*Then,*

$$\sup_\Gamma \inf_X f = \inf_X \sup_\Gamma f.$$

The next result of Tsar'kov is from the theory of best approximation in Banach spaces.

**Theorem 2.2.** [12, Theorem 2] *Let $X$ be an uniformly convex Banach space, with strictly convex topological dual, $M$ a sequentially weakly closed, non-convex subset of $X$. Then, for any convex, dense subset $S$ of $X$, there exists $x_0\in S$ such that the set*

$$\{y\in M : ||y-x_0|| = d(x_0,M)\}$$

*contains at least two distinct points.*

## 3. **Main result**

The main theorem of our paper is the following

**Theorem 3.1.** *Let $\Omega \subseteq \mathbb{R}^N$ be an unbounded domain with smooth boundary $\partial\Omega$ or $\Omega = \mathbb{R}^N (N \geq 2)$. Suppose that $W^{1,2}(\Omega; v_0, v_1)$ satisfies the embedding property $(E)$ and $X_A$ is the space defined by (4). Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the condition $(f)$ and let $\alpha : \Omega \to \mathbb{R}$ be a strictly positive function satisfying $(\alpha)$.*

*Then for every $\sigma \in ]\inf_{X_A} J, \sup_{X_A} J[$ and every $u_0 \in J^{-1}(]-\infty, \sigma[)$, one of the following assertions is true:*

- $(B1)$ *there exists $\lambda > 0$ such that the problem $(P_\lambda)$ has at least three solutions in $X_A$;*
- $(B2)$ *there exists $v \in J^{-1}(\sigma)$ such that for all $u \in J^{-1}([\sigma, \infty[)$, $u \neq v$,*

$$||u - u_0||_A > ||v - u_0||_A.$$

**Proof.** Fix $\lambda$ and $u_0$ as in the statement of the theorem and assume that (B1) does not hold. We shall prove that (B2) is true.

Choosing $\Lambda = [0, \infty)$ and endowing $X_A$ with the weak topology, we define the function $g : X_A \times \Lambda \to \mathbb{R}$ by

$$g(u, \lambda) = \frac{||u - u_0||_A^2}{2} + \lambda(\sigma - J(u)).$$

We show that all the hypotheses of Theorem 2.1 are satisfied.

(A1): It is trivial.

(A2): Let $\lambda > 0$ be fixed. By Lemma 2.2, the functional $g(\cdot, \lambda)$ is sequentially weakly continuous. Moreover, $g(\cdot, \lambda)$ is coercive. Indeed, using Lemma 2.3, we have the following inequality for all $u \in X_A$

$$g(u, \lambda) \geq \frac{1}{2}||u - u_0||_A^2 - \lambda C_f C_{p,w}^{q+1} (2K)^{\frac{-q-1}{2}} ||\alpha||_{\nu,\Omega,w} ||u||_A^{q+1} + \lambda\sigma.$$

Since $q+1 < 2$, the right-hand side of the above inequality goes to $+\infty$ as $||u||_A \to \infty$.

Then, as a consequence of the Eberlain-Smulian theorem, $g(\cdot, \lambda)$ is weakly continuous.

It remains to check that every local minima of $g(\cdot, \lambda)$ is a global minimum. Arguing by contradiction, we suppose that $g(\cdot, \lambda)$ has a local minimum, which is not global minimum. Besides, $g(\cdot,\lambda)$ being coercive and satisfying the Palais-Smale condition (which results from Lemma 2.4), it has a global minimum too. Then using the Eberlain-Smulian theorem, it follows that it has two strong local minima. Hence, by the Mountain-Pass theorem (see [9]) results that $g(\cdot,\lambda)$ (or equivalently the energy functional $\mathcal{E}_\lambda$) admits a third critical point. Therefore the problem $(P_\lambda)$ should have at least three solutions in $X_A$, against our assumption, that $(B1)$ does not hold. Thus, the condition (A2) is fulfilled.

(A3): We observe that there exists some $u_1 \in X_A$ such that $J(u_1) > \sigma$, so

$$\sup_{\lambda\in\Lambda}\inf_{u\in X_A} g(u,\lambda) \le \sup_{\lambda\in\Lambda} g(u_1,\lambda) = \frac{||u_1-u_0||_A}{2} < \infty,$$

hence (A3) is satisfied.

Now, Theorem 2.1 assures that

$$\sup_{\lambda\in\Lambda}\inf_{u\in X_A} g(u,\lambda) = \inf_{u\in X_A}\sup_{\lambda\in\Lambda} g(u,\lambda) := \alpha. \tag{8}$$

We observe, that the function $\lambda \mapsto \inf_{u\in X_A} g(u,\lambda)$ tends to $-\infty$ as $\lambda\to\infty$ (since $\sigma < \sup_{u\in X_A} J(u)$) and it is upper semicontinuous in $\Lambda$. Hence, it attains its supremum in some $\bar\lambda \in \Lambda$, that is,

$$\alpha = \inf_{u\in X_A} g(u,\bar\lambda) = \inf_{u\in X_A}\left(\frac{||u-u_0||_A^2}{2} + \bar\lambda(\sigma - J(u))\right). \tag{9}$$

We will determine the infimum in the right-hand side of (8). Since for any $u \in J^{-1}(]-\infty,\sigma[)$ we have $\sup_{\lambda\in\Lambda} g(u,\lambda) = \infty$, it follows that

$$\alpha = \inf_{u\in J^{-1}([\sigma,\infty[)} \frac{||u-u_0||_A^2}{2}.$$

Then, since the functional $u \mapsto \frac{||u-u_0||_A^2}{2}$ is coercive and sequentially weakly lower semicontinuous while the set $J^{-1}([\sigma,\infty[)$ is sequentially weakly closed, there exists $v \in J^{-1}([\sigma,\infty[)$ such that it attains its infimum in $v$, that is

$$\alpha = \frac{||v-u_0||_A^2}{2}.$$

We can observe that $v$ is actually belonging to $J^{-1}(\sigma)$, so we can write

$$\alpha = \inf_{u\in J^{-1}(\sigma)} \frac{||u-u_0||_A^2}{2} > 0, \tag{10}$$

where the inequality is motivated by the choice of $u_0$ in the assertion of the theorem.

Combining (9) and (10) yields that

$$\inf_{u\in X_A}\left(\frac{||u-u_0||_A^2}{2}+\bar\lambda(\sigma-J(u))\right)=\inf_{u\in J^{-1}(\sigma)}\frac{||u-u_0||_A^2}{2}, \tag{11}$$

which became after a rearrangment of the equation

$$\inf_{u\in X_A}\left(\frac{||u-u_0||_A^2}{2}-J(u)\right)=\inf_{u\in J^{-1}(\sigma)}\left(\frac{||u-u_0||_A^2}{2}-\bar\lambda\sigma\right). \tag{12}$$

Now, we prove that $\bar\lambda>0$. Arguing by contradiction, we suppose that $\bar\lambda=0$. Then by (9) we get, that $\alpha=0$, against (10).

Finally, we prove $(B2)$, namely we prove that $v$ defined above is the only point of $J^{-1}([\sigma,+\infty[)$ minimizing the distance from $u_0$. We argue by contradiction.

Let $w\in J^{-1}([\sigma,+\infty[)$ be such that $||w-u_0||_A=||v-u_0||_A$ and $w$ is different from $v$. As above, we have that $w\in J^{-1}(\sigma)$, so $w$ and $v$ are global minima of the functional $\mathcal{E}_\lambda$ over $J^{-1}(\sigma)$ for $\lambda=\bar\lambda$. Hence, by (12) both $w$ and $v$ are global minima for $\mathcal{E}_\lambda$ over the all space $X_A$. Thus, applying the mountain pass theorem again (see [9]), we obtain that $\mathcal{E}_\lambda$ has at least three critical points, against the assumption that (B1) does not hold (recall $\bar\lambda$ is positive). This concludes the proof. □

In the next corollary the alternative of Theorem 3.1 is resolved, so we obtain a multiplicity result for the problem $(P_\lambda)$.

**Corollary 1.** *Let $\Omega$, $f$, $\alpha$, $X_A$ be as in the Theorem 3.1 and let $S$ be a convex, dense subset of $X_A$. Moreover, let $J^{-1}([\sigma,+\infty[)$ be not convex for some $\sigma\in]\inf_{X_A}J,\sup_{X_A}J[$.*

*Then there exist $u_0\in J^{-1}(]-\infty,\sigma[)\cap S$ and $\lambda>0$ such that the problem $(P_\lambda)$ admits at least tree solutions.*

**Proof.** From Lemma 2.2, it follows that $J$ is sequentially weakly continuous, hence the set $M=J^{-1}(]\sigma,+\infty[)$ is sequentially weakly closed. Since $M$ is not convex, we can apply the Theorem 2.2, which assures the existence of some $u_0\in S$, such that the set $\{y\in M:||y-u_0||_A=d(u_0,M)\}$ contains at least two distinct points. So, there exist two different points $v_1,v_2\in M$ such that

$$||v_1-u_0||_A=||v_2-u_0||_A=d(u_0,M).$$

Clearly $u_0 \notin M$, so $u_0 \in J^{-1}(]-\infty, \sigma[)$. Then the condition (B2) in Theorem 3.1 is false, so (B1) must be true, which means that there exist $\lambda > 0$ such that $(P_\lambda)$ has at least three solutions in $X_A$. $\square$

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