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FIRST ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS WITH STATE-DEPENDENT MODIFIED ARGUMENT

EDITH EGRI AND IOAN A. RUS

Abstract. The aim of our paper is to investigate the Cauchy problem constituting from the first order functional differential equation with state-dependent modified argument of the following form

$$x'(t) = f(t, x(t), x(g(t, x(t)))), \quad t \in [a, b],$$

where $x \in C([a-h, b], [a-h, b]) \cap C^1([a, b], [a-h, b])$, $h > 0$, and the associated generalized initial value $x|_{[a-h,a]} = \varphi$. We look for the solutions of the mentioned problem and deal with its properties, searching conditions for its existence and uniqueness, studying the data dependence: continuity, Lipschitz-continuity and differentiability regarding a parameter.

1. Introduction

Functional differential equations with state dependent modified argument was considered by numerous researchers, as they play an important role in applications. From the numerous works, which are related to functional differential equations, it is worth to mention V. R. Petuhov [12], R. D. Driver [3], R. J. Oberg [11], G. M. Dunkel [4], L. E. Elsgoltz and S. B. Norkin [7], B. Rzepecki [13], J. K. Hale [8], F. Hartung and J. Turi [9], V. Kalmanovskii and A. Myshkis [10], A. Buică [1]. For the application of the Picard operator's technique see I. A. Rus [14], [15], M. A. Şerban [16], E. Egri and I. A. Rus [6], E. Egri [5]. Some other results on iterative functional

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differential equations can be found in K. Wang [18], J. G. Si and S. S. Cheng [17], S. S. Cheng, J. G. Si and X. P. Wang [2].

The purpose of this paper is to study the following problem

$$x'(t) = f(t, x(t), x(g(t, x(t)))), \quad t \in [a, b], \tag{1}$$

$$x|_{[a-h,a]} = \varphi, \tag{2}$$

with $x \in C([a-h, b], [a-h, b]) \cap C^1([a, b], [a-h, b])$.

We suppose that

(C_1) $h > 0$;

(C_2) $f \in C([a, b] \times [a-h, b]^2, \mathbb{R})$;

(C_3) $g \in C([a, b] \times [a-h, b], [a-h, b])$;

(C_4) $\varphi \in C([a-h, a], [a-h, b])$;

(C_5) there exists $L_f > 0$ such that

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \leq L_f(|u_1 - v_1| + |u_2 - v_2|),$$

$\forall t \in [a, b]$, $u_i, v_i \in [a-h, b]$, $i = 1, 2$;

(C_6) there exists $L_g > 0$ such that

$$|g(t, u) - g(t, v)| \leq L_g|u - v|,$$

$\forall t \in [a, b]$, $u, v \in [a-h, b]$.

Realize that the problem $(1) + (2)$ is equivalent with the following fixed point equation

$$x(t) = \begin{cases} \varphi(t), & t \in [a-h, a], \\ \varphi(a) + \displaystyle\int_a^t f(s, x(s), x(g(s, x(s))))\,\mathrm{d}s, & t \in [a, b], \end{cases} \tag{3}$$

where $x \in C([a-h, b], [a-h, b])$.

2. Existence

Observe that the set $C([a-h,b],\mathbb{R})$ can be endowed with the Chebyshev norm

$$\|x\|_C = \max_{t\in[a-h,b]} |x(t)|.$$

Henceforth we consider on the set $C([a-h,b],[a-h,b])$ the metric induced by this norm.

Regarding our problem we define the following operator

$$A : C([a-h,b],[a-h,b]) \to C([a-h,b],\mathbb{R}),$$

where

$$A(x)(t) := \text{the right hand side of (3)}. \tag{4}$$

In this manner we obtained the fixed point equation $x = A(x)$, which hereafter will be the subject of our research. Denote by F_A the fixed point set of the operator A.

Remark that the set $C([a-h,b],\mathbb{R})$ along with the Chebyshev norm, $\|\cdot\|_C$ constitutes a Banach space.

We have our first result.

Theorem 2.1. *We suppose that*

(i) *the conditions* $(C_1)-(C_4)$ *are satisfied;*

(ii) $m_f, M_f \in \mathbb{R}$ *are such that*

(1) $m_f \le f(t,u_1,u_2) \le M_f,\ \forall\, t\in[a,b],\ u_i\in[a-h,b],\ i=1,2;$

(2) $a \le h + \varphi(a) + \min\{0, m_f(b-a)\};$

(3) $b \ge \varphi(a) + \max\{0, M_f(b-a)\}.$

Then the problem $(1)+(2)$ *has at least a solution.*

Proof. To justify the existence of the solution we will apply Schauder's theorem. For this purpose, to have a self-mapping operator, it is necessary to have satisfied the invariance property of the set $C([a-h,b],[a-h,b])$ for the operator $A : C([a-h,b],[a-h,b]) \to C([a-h,b],\mathbb{R})$. Therefore, it must hold the conclusion

$$x(t)\in[a-h,b] \Longrightarrow A(x)(t)\in[a-h,b],\ \forall t\in[a-h,b].$$

Taking into consideration the assumption (C_4), for $t \in [a-h, a]$ the condition above is realized. Moreover, from the definition of the operator A we have

$$\min_{t\in[a,b]} A(x)(t) = \varphi(a) + \min\{0, m_f(b-a)\},$$

$$\max_{t\in[a,b]} A(x)(t) = \varphi(a) + \max\{0, M_f(b-a)\}.$$

In this case we obtain

$$a - h \leq A(x)(t) \leq b, \forall t \in [a, b],$$

if the relations

$$a - h \leq \min_{t\in[a,b]} A(x)(t), \quad \max_{t\in[a,b]} A(x)(t) \leq b$$

are true. But these are fulfilled by the condition (ii). Therefore, it is right to consider the self-mapping operator

$$A : C([a-h, b], [a-h, b]) \to C([a-h, b], [a-h, b]).$$

Observe that the operator A is completely continuous, since the subset

$$C([a-h, b], [a-h, b]) \subset C([a-h, b], \mathbb{R})$$

is bounded, convex and closed, and what is more, the family of functions $A(C([a-h, b], [a-h, b]))$ is relatively compact. Consequently, it can be applied Schauder's fixed point theorem. Therefore, we have $F_A \neq \emptyset$, or equivalently, the problem $(1) + (2)$ has at least a solution. □

3. **Existence and uniqueness**

To study the existence and uniqueness of the solution of the Cauchy problem $(1) + (2)$, take an arbitrary positive number L and construct the set

$$C_L([a-h, b], [a-h, b]) :=$$
$$\{x \in C([a-h, b], [a-h, b]) \mid |x(t_1) - x(t_2)| \leq L|t_1 - t_2|, \forall\, t_1, t_2 \in [a-h, b]\}.$$

Notice that the subset $C_L([a-h,b],[a-h,b]) \subset C([a-h,b],\mathbb{R})$ can be endowed with the Chebyshev metric defined by

$$\|x-y\|_C := \max_{t\in[a-h,b]} \left(|x(t)-y(t)|\right), \tag{5}$$

and in this manner we obtain a complete metric space.

We have:

Theorem 3.1. *Consider the Cauchy problem* (1) + (2) *and suppose that*

(i) *the conditions* $(\mathrm{C}_1)-(\mathrm{C}_6)$ *are satisfied;*

(ii) $\varphi \in C_L([a-h,a],[a-h,b])$;

(iii) $m_f, M_f \in \mathbb{R}$ *are such that*

(1) $m_f \le f(t,u_1,u_2) \le M_f,\ \forall\, t\in[a,b],\ u_i\in[a-h,b],\ i=1,2;$

(2) $a \le h + \varphi(a) + \min\{0, m_f(b-a)\};$

(3) $b \ge \varphi(a) + \max\{0, M_f(b-a)\};$

(iv) $\max\{|M_f|,|m_f|\} \le L;$

(v) $L_f(b-a)(2+LL_g) < 1.$

Then the problem (1) + (2) *has in* $C_L([a-h,b],[a-h,b])$ *a unique solution.*

Proof. Consider the operator

$$A : C_L([a-h,b],[a-h,b]) \to C([a-h,b],\mathbb{R})$$

given by (4). We want to apply the contraction principle for this operator. Therefore, first admit that A is self-mapping. Since all the conditions of the existence theorem hold, we have

$$a-h \le A(x)(t) \le b, \ \text{ when } a-h \le x(t) \le b,$$

for all $t\in[a-h,b]$. Moreover, from the condition (ii), if $t_1,t_2\in[a-h,a]$, we obtain

$$|A(x)(t_1)-A(x)(t_2)| = |\varphi(t_1)-\varphi(t_2)| \le L|t_1-t_2|.$$

On the other hand, if $t_1,t_2\in[a,b]$, due to (iv), we have

$$|A(x)(t_1)-A(x)(t_2)| = \left|\int_{t_1}^{t_2} f(s,x(s),x(g(s,x(s))))\,\mathrm{d}s\right| \le$$

$$\le \max\{|m_f|,|M_f|\}|t_1-t_2| \le L|t_1-t_2|,$$

which involves that the operator A is L-Lipschitz. Accordingly, we have $A(x) \in C_L([a-h,b],[a-h,b])$ for all $x \in C_L([a-h,b],[a-h,b])$.

Henceforward concede that from the condition (v) the operator A is an L_A-contraction, with

$$L_A := L_f(b-a)(2+LL_g).$$

Indeed, for all $t \in [a-h,a]$ we have $|A(x_1)(t) - A(x_2)(t)| = 0$. Furthermore, for $t \in [a,b]$ we successively get

$$\begin{aligned}
&|A(x_1)(t) - A(x_2)(t)| = \\
&= \left| \int_a^t [f(s,x_1(s),x_1(g(s,x_1(s)))) - f(s,x_2(s),x_2(g(s,x_2(s))))] \, \mathrm{d}s \right| \le \\
&\le \int_a^t |f(s,x_1(s),x_1(g(s,x_1(s)))) - f(s,x_2(s),x_2(g(s,x_2(s))))| \, \mathrm{d}s \le \\
&\le L_f \int_a^t \big[|x_1(s)-x_2(s)| + |x_1(g(s,x_1(s))) - x_2(g(s,x_2(s)))|\big] \mathrm{d}s \le \\
&\le L_f(b-a)\|x_1-x_2\|_C + \\
&+ L_f \int_a^t \big[|x_1(g(s,x_1(s))) - x_1(g(s,x_2(s)))| + |x_1(g(s,x_2(s))) - x_2(g(s,x_2(s)))|\big] \mathrm{d}s \le \\
&\le L_f(b-a)\|x_1-x_2\|_C + L_f \int_a^t \big[L \cdot |g(s,x_1(s)) - g(s,x_2(s))| + \|x_1-x_2\|_C\big] \mathrm{d}s \le \\
&\le 2L_f(b-a)\|x_1-x_2\|_C + LL_f \int_a^t |g(s,x_1(s)) - g(s,x_2(s))| \mathrm{d}s \le \\
&\le 2L_f(b-a)\|x_1-x_2\|_C + LL_f \int_a^t L_g|x_1(s)-x_2(s)| \mathrm{d}s \le \\
&\le \big[2L_f(b-a) + LL_fL_g(b-a)\big] \|x_1-x_2\|_C,
\end{aligned}$$

and it follows that

$$||A(x_1) - A(x_2)||_C \le L_A||x_1-x_2||_C, \quad L_A = L_f(b-a)(2+LL_g).$$

From the condition (vi) we have $L_A < 1$, consequently the operator A is an L_A-contraction. By applying the contraction principle the operator A has a unique fixed point, i.e. the problem $(1)+(2)$ has in $C_L([a-h,b],[a-h,b])$ a unique solution. □

4. **Data dependence: continuity**

In order to study the continuous dependence of the fixed points we will use the following result:

Lemma 4.1. (I. A. Rus [15]) *Let (X, d) be a complete metric space and*

$$A, B : X \to X$$

two operators. We suppose that

(i) *the operator A is a γ-contraction;*
(ii) $F_B \neq \emptyset$;
(iii) *there exists $\eta > 0$ such that*

$$d(A(x), B(x)) \leq \eta, \ \forall\, x \in X.$$

Then, if $F_A = \{x_A^*\}$ and $x_B^* \in F_B$, we have

$$d(x_A^*, x_B^*) \leq \frac{\eta}{1-\gamma}.$$

Now, let f_i and φ_i as in Theorem 3.1. For $i = 1, 2$ we consider the following two Cauchy problems

$$x'(t) = f_i(t, x(t), x(g(t, x(t)))), \quad t \in [a, b], \tag{6}$$

$$x|_{[a-h,a]} = \varphi_i. \tag{7}$$

We assign to the problems (6) + (7) the operators

$$A_i : C_L([a-h, b], [a-h, b]) \to C_L([a-h, b], [a-h, b]),$$

given by

$$A_i(x)(t) := \begin{cases} \varphi_i(t), & t \in [a-h, a], \\ \varphi_i(a) + \displaystyle\int_a^t f_i(s, x(s), x(g(s, x(s))))\, \mathrm{d}s, & t \in [a, b], \end{cases} \tag{8}$$

$i = 1, 2$. From Theorem 3.1 the operators A_1 and A_2 are contractions. We will denote by x_1^*, x_2^* their unique fixed points.

Then, accordingly to Lemma 4.1 we have the result as follows.

Theorem 4.1. *We suppose the conditions of Theorem 3.1 concerning to the problems* $(6)+(7)$ *are satisfied and, moreover,*

(i) *there exists* η_1 *such that*

$$|\varphi_1(t)-\varphi_2(t)|\le \eta_1,\quad \forall\, t\in[a-h,a]$$

(ii) *there exists* $\eta_2>0$ *such that*

$$|f_1(t,u_1,u_2)-f_2(t,u_1,u_2)|\le \eta_2,\quad \forall\, t\in[a,b],\ \forall\, u_i\in[a-h,b],\ i=1,2.$$

Then the following estimation holds:

$$\|x_1^*-x_2^*\|_C\le \frac{\eta_1+\eta_2(b-a)}{1-L_f(b-a)(2+LL_g)},$$

where $L_f=\max\{L_{f_1},L_{f_2}\}$ *and* $L_g=\max\{L_{g_1},L_{g_2}\}$.

Proof. Observe that, since the assumptions of Theorem 3.1 are realized, the operators A_i $(i=1,2)$ given by (8) are L_{A_i}-contractions with

$$L_{A_i}:=L_{f_i}(b-a)(2+LL_{g_i}).$$

Consider $t\in[a-h,a]$. From the condition (ii) it follows that

$$\|A_1(x)-A_2(x)\|_C\le\eta_1\le\eta_1+\eta_2(b-a).$$

On the other hand, for $t\in[a,b]$, we obtain

$$|A_1(x)(t)-A_2(x)(t)|\le$$
$$\le|\varphi_1(a)-\varphi_2(a)|+\int_a^t|f_1(s,x(s),x(g(s,x(s))))-f_2(s,x(s),x(g(s,x(s))))\,\mathrm{d}s|\le$$
$$\le\eta_1+\eta_2(b-a).$$

Consequently,

$$\|A_1(x)-A_2(x)\|_C\le\eta_1+\eta_2(b-a),\quad \forall x\in C_L([a-h,b],[a-h,b]).$$

Now, the proof follows from Lemma 4.1. □

5. Data dependence on parameter: Lipschitz-continuity

In this section we will use the following abstract result:

Lemma 5.1. *Let (X,d) be a complete metric space, $J \subset \mathbb{R}$ and $A : X \times J \to X$ an operator. We suppose that:*

(i) $\exists\, \alpha \in]0,1[$ *such that*

$$d\big(A(x_1,\lambda), A(x_2,\lambda)\big) \le \alpha d(x_1,x_2), \quad \forall\, x_1,x_2 \in X,\ \lambda \in J;$$

(ii) $\exists\, l > 0$ *such that*

$$d\big(A(x,\lambda_1), A(x,\lambda_2)\big) \le l|\lambda_1 - \lambda_2|, \quad \forall\, x \in X,\ \lambda_1,\lambda_2 \in J.$$

Then

(a) $\forall\, \lambda \in J$, *the operator* $A(\cdot,\lambda) : X \to X$ *has a unique fixed point,* $x^*(\lambda) \in X$;

(b) $d(x^*(\lambda_1), x^*(\lambda_2)) \le \dfrac{l}{1-\alpha}|\lambda_1 - \lambda_2|, \quad \forall \lambda_1,\lambda_2 \in J.$

Proof. Evidently, from the condition (i) the operator $A(\cdot,\lambda)$ is a contraction. Therefore, the fixed point equation $A(x,\lambda) = x$ has a unique solution $x^*(\lambda) \in X$, corresponding to an arbitrary value $\lambda \in J$. Moreover, for $\lambda_1,\lambda_2 \in J$ we have

$$\begin{aligned} & d(x^*(\lambda_1), x^*(\lambda_2)) = d(A(x^*(\lambda_1),\lambda_1), A(x^*(\lambda_2),\lambda_2)) \le \\ & \le d(A(x^*(\lambda_1),\lambda_1), A(x^*(\lambda_1),\lambda_2)) + d(A(x^*(\lambda_1),\lambda_2), A(x^*(\lambda_2),\lambda_2)) \le \\ & \le l|\lambda_1 - \lambda_2| + \alpha \cdot d(x^*(\lambda_1), x^*(\lambda_2)), \end{aligned}$$

and consequently

$$d(x^*(\lambda_1), x^*(\lambda_2)) \le \frac{l}{1-\alpha}|\lambda_1 - \lambda_2|.$$

Accordingly, we have the proof. □

Now we consider the problem

$$\begin{cases} x'(t) = f(t, x(t), x(g(x,t)), \lambda), & t \in [a,b],\ \lambda \in J, \\ x(t) = \varphi(t,\lambda), & t \in [a-h,a],\ \lambda \in J,\ h > 0, \end{cases} \tag{9}$$

and, for $L>0$, the corresponding operator A, given as follows:

$$A : C_L([a-h,b],[a-h,b]) \times J \to C_L([a-h,b],[a-h,b]) \times J,$$

$$A(x,\lambda) := \begin{cases} \varphi(t,\lambda), & t\in[a-h,a],\ \lambda\in J; \\ \varphi(a,\lambda) + \displaystyle\int_a^t f(s,x(s),x(g(s,x(s))),\lambda)\mathrm{d}s, & t\in[a,b], \quad \lambda\in J. \end{cases} \tag{10}$$

Based upon Lemma 5.1 we have the next result:

Theorem 5.1. *We suppose that*

(i) $f\in C([a,b]\times[a-h,b]^2\times J,\mathbb{R})$;

(ii) $g\in C([a,b]\times[a-h,b],[a-h,b])$;

(iii) $\varphi\in C_L([a-h,a],[a-h,b])\times J$, *and* $\exists l_\varphi>0$ *such that*

$$|\varphi(t,\lambda_1)-\varphi(t,\lambda_2)|\le l_\varphi;$$

(iv) *there exists* $L_f>0$ *such that*

$$|f(t,u_1,u_2,\lambda)-f(t,v_1,v_2,\lambda)|\le L_f(|u_1-v_1|+|u_2-v_2|),$$

$\forall t\in[a,b],\ u_i,v_i\in[a-h,b],\ \lambda\in J,\ i=1,2$;

(v) *there exists* $L_f>0$ *such that*

$$|f(t,u,v,\lambda_1)-f(t,u,v,\lambda_2)|\le l_f|\lambda_1-\lambda_2|,$$

$\forall t\in[a,b],\ u,v\in[a-h,b],\ \lambda_i\in J,\ i=1,2$;

(vi) *there exists* $L_g>0$ *such that*

$$|g(t,u)-g(t,v)|\le L_g|u-v|,$$

$\forall t\in[a,b],\ u,v\in[a-h,b]$;

(vii) $L_f(b-a)[2+LL_g]<1$.

Then

(a) $\forall\,\lambda\in J$, *the operator* $A(\cdot,\lambda):X\to X$ *defined by* (10) *has a unique fixed point,* $x^*(\lambda)\in X$;

(b) $\|x^*(\lambda_1), x^*(\lambda_2)\|_C \leq \dfrac{l_\varphi + l_f(b-a)}{1 - L_f(b-a)[2 + LL_g]}|\lambda_1 - \lambda_2|, \quad \forall \lambda_1, \lambda_2 \in J.$

Proof. From the proof of Theorem 3.1, for all $t \in [a, b]$ we have

$$|A(x_1, \lambda)(t) - A(x_2, \lambda)(t)| =$$

$$= \left| \int_a^t f(s, x_1(s), x_1(g(s, x_1(s))), \lambda) \mathrm{d}s - \int_a^t f(s, x_2(s), x_2(g(s, x_2(s))), \lambda) \mathrm{d}s \right| \leq$$

$$\leq L_f(b-a)[2 + LL_g]\|x_1 - x_2\|_C,$$

and taking $\alpha := L_f(b-a)[2 + LL_g]$, due to the condition (vi), the first assumption of Lemma 5.1 is satisfied.

Furthermore, for $t \in [a-h, a]$, we have:

$$|A(x, \lambda_1) - A(x, \lambda_2)| = |\varphi(t, \lambda_1) - \varphi(t, \lambda_2)| \leq l_\varphi |\lambda_1 - \lambda_2|.$$

On the other hand, if $t \in [a, b]$, we obtain:

$$|A(x, \lambda_1) - A(x, \lambda_2)| \leq |\varphi(a, \lambda_1) - \varphi(a, \lambda_2)| +$$

$$+ \int_a^t |f(s, x(s), x(g(s, x(s))), \lambda_1) - f(s, x(s), x(g(s, x(s))), \lambda_2)| \mathrm{d}s \leq$$

$$\leq l_\varphi |\lambda_1 - \lambda_2| + l_f(b-a)|\lambda_1 - \lambda_2| = [l_\varphi + l_f(b-a)]|\lambda_1 - \lambda_2|.$$

One can be observe that $l := l_\varphi + l_f(b-a)$ has the same property as the one from Lemma 5.1.

Consequently, the proof is complete. □

6. **Data dependence: differentiability**

Henceforward we will need the following result, which is very useful for proving solutions of operatorial equations to be differentiable with respect to parameters.

Theorem 6.1 (Fibre contraction principle (I. A. Rus [14])). *Let (X, d) and (Y, ρ) be two metric spaces and*

$$A : X \times Y \to X \times Y, \ (B : X \to X, \, C : X \times Y \to Y),$$

$$A(x, y) = (B(x), C(x, y))$$

a triangular operator.

We suppose that

(i) (Y, ρ) *is a complete metric space;*

(ii) *the operator* B *is a Picard operator;*

(iii) *there exists* $L_C \in [0, 1[$ *such that* $C(x, \cdot) : Y \to Y$ *is an* L_C*-contraction, for all* $x \in X$;

(iv) *if* $(x^*, y^*) \in F_A$, *then* $C(\cdot, y^*)$ *is continuous in* x^*.

Then the operator A *is a Picard operator.*

For some applications of the fibre contraction principle see I. A. Rus [15], E. Egri and I. A. Rus [6], E. Egri [5].

Consider the following problem with parameter:

$$x'(t; \lambda) = f(t, x(t; \lambda), x(g(t, x(t; \lambda)); \lambda); \lambda), \quad t \in [a, b], \tag{11}$$

$$x(t; \lambda) = \varphi(t; \lambda), \; t \in [a - h, a], \tag{12}$$

with $\lambda \in J \subset \mathbb{R}$ a compact subset.

We have:

Theorem 6.2. *Suppose that we have satisfied the conditions below:*

(P_1) $h > 0$, $J \subset \mathbb{R}$, *a compact interval;*

(P_2) $\varphi(t, \cdot) \in C^1(J, \mathbb{R})$, *for all* $t \in [a - h, a]$;
$\varphi(\cdot, \lambda) \in C^1_L([a - h, a], [a - h, a])$, *and*
$\varphi'(a, \lambda) = f(a, \varphi(a; \lambda), \varphi(g(a, \varphi(a; \lambda)); \lambda); \lambda)$;

(P_3) $f \in C^1([a, b] \times [a - h, b]^2 \times J, \mathbb{R})$;
$g \in C([a, b] \times [a - h, b], [a - h, b])$;

(P_4) *there exists* $L_f > 0$ *such that*

$$\left| \frac{\partial f(t, u_1, u_2; \lambda)}{\partial u_i} \right| \le L_f,$$

for all $t \in [a, b]$, $u_i \in [a - h, b]$, $i = 1, 2$, $\lambda \in J$;

(P_5) $m_f, M_f \in \mathbb{R}$ *are such that*

(1) $m_f \le f(t, u_1, u_2; \lambda) \le M_f$, $\forall\, t \in [a, b]$, $u_1, u_2 \in [a - h, b]$, $\lambda \in J$;

(2) $a \leq h + \varphi(a;\lambda) + \min\{0, m_f(b-a)\}$;

(3) $b \geq \varphi(a;\lambda) + \max\{0, M_f(b-a)\}$;

(P_6) $\max\{|m_f|, |M_f|\} \leq L$;

(P_7) $L_f(b-a)(2 + LL_g) < 1$.

Then

(1) *the problem* (11) + (12) *has in* $C_L([a-h,b],[a-h,b])$ *a unique solution,* $x^*(\cdot,\lambda)$;

(2) $x^*(t,\cdot) \in C^1(J,\mathbb{R})$, $\forall t \in [a-h,b]$.

Proof. Since we are in the conditions of Theorem 3.1, we certainly have that the problem (11)+(12) has in $C_L([a-h,b],[a-h,b])$ a unique solution, $x^*(\cdot,\lambda)$. Therefore, statement (1) from the theorem is satisfied.

To justify the affirmation (2), first observe that the problem (11) + (12) is equivalent with the following fixed point equation

$$x(t;\lambda) = \begin{cases} \varphi(t;\lambda), & t \in [a-h,a],\ \lambda \in J, \\ \varphi(a;\lambda) + \displaystyle\int_a^t f(t, x(t;\lambda), x(g(t,x(t;\lambda));\lambda);\lambda)\,\mathrm{d}s, & t \in [a,b],\ \lambda \in J. \end{cases} \tag{13}$$

We try to fit in the fibre contraction principle. For this purpose we consider the operator

$$B : C_L([a-h,b]\times J,[a-h,b]) \to C_L([a-h,b]\times J,[a-h,b]),$$

where

$$B(x)(t;\lambda) := \text{ the right hand side of (13).}$$

From the proof of the existence and uniqueness theorem 3.1 this operator is well-defined. We denote by X its domain (codomain). Observe that the set X endowed with the Chebyshev metric

$$d_C(x,y) = \max_{t,\lambda} |x(t;\lambda) - y(t;\lambda)|, \text{for all } x, y \in X,$$

is a Banach space.

From the contraction principle, in the conditions of the theorem, the operator B is a Picard operator with $F_B = \{x^*\}$.

We consider the subset $X_1 \subset X$,

$$X_1 := \left\{ x \in X \,\middle|\, \frac{\partial x}{\partial t} \in C([a-h,b] \times J, \mathbb{R}) \right\}.$$

Remark that $x^* \in X_1$, $B(X_1) \subset X_1$ and $B : X_1 \to X_1$ is Picard operator. Thus, B fulfills (ii) from the fibre contraction theorem.

Let $Y := C([a-h,b] \times J, \mathbb{R})$. We want to prove that there exists $\frac{\partial x^*}{\partial \lambda}$ and, moreover, $\frac{\partial x^*}{\partial \lambda} \in Y$. If we suppose that there exists $\frac{\partial x^*}{\partial \lambda}$, then from (13) we have

$$\frac{\partial x^*(t;\lambda)}{\partial \lambda} = \frac{\partial \varphi(a;\lambda)}{\partial \lambda} + \int_a^t \frac{\partial f(s, x^*(s;\lambda), x^*(g(s, x^*(s;\lambda));\lambda);\lambda)}{\partial u_1} \cdot \frac{\partial x^*(s;\lambda)}{\partial \lambda} \, \mathrm{d}s +$$
$$+ \int_a^t \frac{\partial f(s, x^*(s;\lambda), x^*(g(s, x^*(s;\lambda));\lambda);\lambda)}{\partial u_2} \cdot \frac{\partial x^*(g(s, x^*(s;\lambda));\lambda)}{\partial t} \cdot \frac{\partial g(s, x^*(s;\lambda))}{\partial v} \cdot$$
$$\cdot \frac{\partial x^*(s;\lambda)}{\partial \lambda} \, \mathrm{d}s + \int_a^t \frac{\partial f(s, x^*(s;\lambda), x^*(g(s, x^*(s;\lambda));\lambda);\lambda)}{\partial u_2} \cdot \frac{\partial x^*(g(s, x^*(s;\lambda));\lambda)}{\partial \lambda} \, \mathrm{d}s +$$
$$+ \int_a^t \frac{\partial f(s, x^*(s;\lambda), x^*(x^*(s;\lambda);\lambda);\lambda)}{\partial \lambda} \, \mathrm{d}s, \qquad t \in [a,b],\ \lambda \in J.$$

The obtained relation suggests us to consider the operator

$$C : X_1 \times Y \to Y$$

$$(x, y) \mapsto C(x, y)$$

defined by

$$C(x,y)(t;\lambda) := \frac{\partial \varphi(a;\lambda)}{\partial \lambda} + \int_a^t \frac{\partial f(s, x(s;\lambda), x(g(s, x(s;\lambda));\lambda);\lambda)}{\partial u_1} \cdot y(s;\lambda) \, \mathrm{d}s +$$
$$+ \int_a^t \frac{\partial f(s, x(s;\lambda), x(g(s, x(s;\lambda));\lambda);\lambda)}{\partial u_2} \cdot \frac{\partial x(g(s, x(s;\lambda));\lambda)}{\partial t} \cdot \frac{\partial g(s, x(s;\lambda))}{\partial v} \cdot$$
$$\cdot y(s;\lambda) \mathrm{d}s + \int_a^t \frac{\partial f(s, x(s;\lambda), x(g(s, x(s;\lambda));\lambda);\lambda)}{\partial u_2} \cdot \frac{\partial x(g(s, x(s;\lambda));\lambda)}{\partial \lambda} \mathrm{d}s +$$
$$+ \int_a^t \frac{\partial f(s, x(s;\lambda), x(g(x(s;\lambda));\lambda);\lambda)}{\partial \lambda} \, \mathrm{d}s, \qquad t \in [a,b],\ \lambda \in J.$$

and

$$C(x,y)(t,\lambda) := \frac{\partial \varphi(t;\lambda)}{\partial \lambda}, \text{ for } t \in [a_1, a],\ \lambda \in J.$$

In this way we can consider the triangular operator

$$A : X_1 \times Y \to X_1 \times Y$$

$$(x,y) \mapsto (B(x), C(x,y)).$$

As we have seen earlier, $B : X_1 \to X_1$ is a Picard operator. Realize that $C(x,\cdot) : Y \to Y$ is a contraction on $[a,b]$. Indeed, we have

$$|C(x,y_1)(t;\lambda) - C(x,y_2)(t;\lambda)| =$$

$$\Big|\int_a^t \frac{\partial f(s,x(s;\lambda),x(g(s,x(s;\lambda));\lambda);\lambda)}{\partial u_1} \cdot \big[y_1(s;\lambda) - y_2(s;\lambda)\big]\mathrm{d}s+$$

$$+\int_a^t \frac{\partial f(s,x(s;\lambda),x(g(s,x(s;\lambda));\lambda);\lambda)}{\partial u_2} \cdot \frac{\partial x(g(s,x(s;\lambda));\lambda)}{\partial t} \cdot \frac{\partial g(s,x(s;\lambda))}{\partial v} \cdot$$

$$\cdot \big[y_1(s;\lambda) - y_2(s;\lambda)\big]\mathrm{d}s\Big| \le$$

$$\le L_f \int_a^t \left|1 + \frac{\partial x(g(s,x(s;\lambda));\lambda)}{\partial t} \cdot \frac{\partial g(s,x(s;\lambda))}{\partial v}\right| \cdot |y_1(s;\lambda) - y_2(s;\lambda)|\,\mathrm{d}s \le$$

$$\le L_f(1 + LL_g)\int_a^t |y_1(s;\lambda) - y_2(s;\lambda)|\,\mathrm{d}s \le$$

$$\le L_f(1 + LL_g)(b-a)||y_1 - y_2||_C \le$$

$$\le L_f(b-a)(2 + LL_g)||y_1 - y_2||_C.$$

In this way we got

$$||C(x,y_1) - C(x,y_2)||_C \le L_C \cdot ||y_1 - y_2||_C, \quad \text{with} \quad L_C := L_f(b-a)(2 + LL_g).$$

So, we are in the condition of the fibre contraction principle, and consequently, A is a Picard operator, i.e. the sequences defined by

$$x_{n+1} := B(x_n),$$

$$y_{n+1} := C(x_n, y_n), \quad n \in \mathbb{N}$$

converges uniformly (with respect to $t \in [a-h,b],\ \lambda \in J$) to the unique fixed point of the operator A, $(x^*, y^*) \in F_A$, for all $x_0, y_0 \in C([a-h,b] \times J, [a-h,b])$.

Taking $x_0 = 0$, $y_0 = \dfrac{\partial x_0}{\partial \lambda} = 0$, we get $y_1 = \dfrac{\partial x_1}{\partial \lambda}$. By induction it can be proved that $y_n = \dfrac{\partial x_n}{\partial \lambda}$, $\forall\, n \in \mathbb{N}$. So,

$$x_n \overset{unif.}{\longrightarrow} x^* \text{ as } n \to \infty,$$

$$\frac{\partial x_n}{\partial \lambda} \overset{unif.}{\longrightarrow} y^* \text{ as } n \to \infty.$$

From these, using a theorem of Weierstrass we have that x^* is differentiable and $\dfrac{\partial x^*}{\partial \lambda} = y^* \in Y$. □

7. Example

To check our results consider the following Cauchy problem

$$x'(t) = \frac{1}{5}\left[x\left(\frac{1}{6}[t + x(t)]\right) - \frac{1}{6}x(t) - \frac{1}{24}t\right] + \frac{1}{2}, \qquad t \in [0,2], \tag{14}$$

$$x(t) = t, \qquad t \in [-1,0]. \tag{15}$$

We look for the solution $x \in C([-1,2],[-1,2]) \cap C^1([0,2],[-1,2])$ of the problem (14)+(15). For this purpose we apply Theorem 2.1. First observe that we have $a = 0$, $b = 2$, $h = 1$, $\varphi(t) = t$, and

$$g(t,u) = \frac{1}{6}(t+u), \text{ for all } t \in [0,2],\ u \in [-1,2],$$

$$f(t,u_1,u_2) = \frac{1}{5}\left[u_2 - \frac{1}{6}u_1 - \frac{1}{24}t\right] + \frac{1}{2}, \text{ for all } t \in [0,2],\ u_1,u_2 \in [-1,2],$$

with

$$m_f = \frac{13}{60},\ M_f = \frac{14}{15},\ L_f = \frac{1}{5},\ L_g = \frac{1}{6}.$$

Since all the conditions of Theorem 2.1 are fulfilled, the problem (14)+(15) has in $C([-1,2],[-1,2])$ at least a solution. Moreover, considering $\frac{1}{2} \le L < 3$, due to Theorem3.1, this solution is unique on the set $C_L([-1,2],[-1,2])$, and it is the limit of the sequence $(x_n)_{n\ge 0}$ of successive approximation, given by the recursive relation

$$x_{n+1} = \begin{cases} t, & t \in [-1,0], \\ \displaystyle\int_0^t \left\{\frac{1}{5}\left[x_n\left(\frac{1}{6}[s + x_n(s)]\right) - \frac{1}{6}x_n(s) - \frac{1}{24}s\right] + \frac{1}{2}\right\} \mathrm{d}s, & t \in [0,2], \end{cases}$$

Doing some calculations with Maple, it seems that the sequence of successive approximation is convergent to the function $x(t) = \frac{t}{2}$. Indeed, this is the unique solution of the problem, since it satisfies the functional differential equation (14).

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Babeş-Bolyai University,
Department of Computer Science,
530241 Miercurea-Ciuc, Str. Topliţa, nr.20,
jud. Harghita, Romania
E-mail address: egriedit@yahoo.com

Babeş-Bolyai University,
Department of Applied Mathematics,
Str. M. Kogălniceanu Nr. 1,
400084 Cluj-Napoca, Romania
E-mail address: iarus@math.ubbcluj.ro