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ON THE STANCU TYPE LINEAR POSITIVE OPERATORS OF APPROXIMATION CONSTRUCTED BY USING THE BETA AND THE GAMMA FUNCTIONS

ELENA IULIA STOICA-LAZE

Abstract. The objective of this paper is to present some extensions of several classes of Stancu type linear positive operators of approximation by using the beta and the gamma functions.

Section 1 is devoted to consideration of the Stancu-Bernstein type operator S_m^α, introduced in 1968 by D.D. Stancu in the paper [28] starting from the Markov-Polya probability distribution.

In section 2 is considered the Stancu-Baskakov operator V_m^α, defined at (2.1) and (2.2). If $\alpha = 0$ then we obtain the Baskakov operator defined at (2.3).

Section 3 is devoted to the operator W_m^α of Stancu, Meyer-König and Zeller operator defined at (3.1) and (3.2) introduced γ in [22] by these authors starting from the Pascal probability distribution.

In section 4 is presented and discussed the Stancu beta operators of second-kind L_m, defined at (4.2), which was obtained by using Karl Pearson type VI, $b_{p,q}$, with positive parameters p and q.

1. The Stancu-Bernstein operator S_m^α

In the previous papers [32], [33], [34], professor D.D. Stancu has considered probabilistic methods for construction and investigation of some linear positive operators useful in approximation theory of functions.

First we mention that in 1968 he has introduced and investigated in [28] a new parameter-dependent linear polynomial operator S_m^α of Bernstein type associated

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to a function $f \in C[0,1]$, defined by the formula

$$(S_m^\alpha f)(x) = \sum_{k=0}^{m} p_{m,k}^\alpha(x) f\left(\frac{k}{m}\right), \tag{1.1}$$

where α is a non negative parameter and $w_{m,k}^\alpha$ is a polynomial which can be expressed by means of the factorial power $u^{(n,k)}$ of the non-negative order n and increment h, given by the formula

$$u^{(n,h)} = u(u-h)\dots(u-(n-1)h),$$

namely

$$p_{m,k}^\alpha(x) = \frac{1}{1^{(m,-\alpha)}}\binom{m}{k} x^{(k,-\alpha)}(1-x)^{(m-k,-\alpha)}. \tag{1.2}$$

If $\alpha = 0$ then this operator reduces to the classical operator B_m, introduced by Bernstein in 1912 in the paper [7] by starting from the binomial Bernoulli distribution.

By using the Markov-Pólya probability distribution (introduced by A. Markov [19] in 1917 and encountered in 1930 by G. Pólya [25] studying the contageous diseases. We mention that at this distribution we can arrive by using the following urn model. An urn contains a white balls and b black balls. One ball is drawn at random from this urn and then it is returned together with a constant number of c identical balls of the same color. This process is repeated m times. Denoting by Z_j the one-zero random variable, according as the j-th trial results in white or black, the probability that the total number of white balls $Z_1 + \dots + Z_m$ be equal with k $(0 \le k \le m)$ is given by

$$P(k; m, a, b, c) = \binom{m}{k} \frac{a(a+c)\dots(a+(k-1)c)b(b+c)\dots(b+(m-k-1)c)}{(a+b)(a+b+c)\dots(a+b+(m-1)c)}.$$

If we adopt the notations: $x = a/(a+b)$, $\alpha = c/(a+b)$, and we hold α a constant, allowing x to vary, we obtain the discrete probability distribution of Markov-Pólya. We can see that the probability to have

$$Y_m = \frac{1}{m}(Z_1 + \dots + Z_m) = \frac{k}{m}$$

is given just by the formula

$$p_{m,k}^\alpha(x) = \binom{m}{k} \frac{x^{(k,-\alpha)}(1-x)^{(m-k,-\alpha)}}{1^{(m,-\alpha)}}.$$

If we assume that $\alpha > 0$, then the operator S_m^α can be represented by means of the formula

$$(S_m^\alpha f)(x)\frac{1}{B\left(\dfrac{x}{\alpha},\dfrac{1-x}{\alpha}\right)}\int_0^1 t^{\frac{x}{\alpha}}(1-t)^{\frac{1-x}{\alpha}}(B_m f)(t)dt, \tag{1.3}$$

where B is the Euler (1707-1783) Beta function of the first kind, where

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt \quad (x,y>0).$$

This is a function of two variables x and y from $\mathbb{R}_+$.

Putting $x=t/(1+t)$ we obtain

$$B(x,y)=\int_0^\infty \frac{t^{x-1}dt}{(1+t)^{x+y}}.$$

The Beta function can be expressed by the Euler Gamma function $\Gamma(u)$, where $u>0$ and

$$\Gamma(u)=\int_0^\infty t^{u-1}e^{-u}dt.$$

Now let us make the remarks that

$$B(m,n)=(m-1)!(n-1)!/(m+n-1)!, \quad B\left(\frac{1}{2},\frac{1}{2}\right)=\pi, \quad \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi},$$

as can be seen in the books [9], [13].

Concerning the remainder of the approximation formula of the function f by the operator of D.D. Stancu (1.3), we should mention that it can be represented under the form

$$(R_m^\alpha f)(x)=(R_m^\alpha e_2)(x)(D_m^\alpha f)(x),$$

where

$$(R_m^\alpha e_2)(x)=\frac{1+\alpha m}{1+\alpha}\cdot\frac{x(1-x)}{m}$$

and

$$(D_m^\alpha f)(x)=\sum_{k=0}^{m-1} p_{m-1,k}^\alpha(x+\alpha,1-x+\alpha)\left[x,\frac{k}{m},\frac{k+1}{m};f\right],$$

the brackets representing the symbol for divided differences.

2. The Stancu-Baskakov operator V_m^α

If one uses the generalization given by D.D. Stancu in the paper [29] for the Fisher probability distribution

$$P(\chi = k) = q(k; n, x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}, \tag{2.1}$$

namely

$$P(\chi = k) = \binom{m+k-1}{k} \frac{B\left(\frac{1}{\alpha} + m, \frac{x}{\alpha} + k\right)}{B\left(\frac{1}{\alpha}, \frac{x}{\alpha}\right)},$$

where α and x are positive numbers and B is the Euler beta function, then we obtain the generalized Baskakov operator V_m^α considered by D.D. Stancu [29], defined by the formula

$$(V_m^\alpha f)(x) = \sum_{k=0}^{\infty} v_{m,k}^\alpha(x) f\left(\frac{k}{m}\right),$$

where $x \in [0, \infty)$ and

$$v_{m,k}^\alpha(x) = \binom{m+k-1}{k} \frac{(1+\alpha)(1+2\alpha)\dots(1+(m-1)\alpha)(x(x+\alpha))\dots(x+(k-1)\alpha)}{(1+x)(1+x+\alpha)\dots(1+x+(m+k-1)\alpha)}. \tag{2.2}$$

Since $v_{m,0}^\alpha(0) = 1$ and $v_{m,k}^\alpha(0) = 0$ if $k \geq 1$ one observes that we always have $(V_m^\alpha f)(0) = f(0)$.

We notice that the fundamental polynomials $v_{m,k}^\alpha$ can be expressed in a more compact form by means of the notion of factorial powers, namely

$$v_{m,k}^\alpha(x) = \binom{m+k-1}{k} \frac{1^{(m,-\alpha)} x^{(k,-\alpha)}}{(1+x)^{(m+k,-\alpha)}}.$$

The operator V_m^α includes as a special case the Baskakov operator Q_m, defined by the following formula

$$(Q_m f)(x) = \sum_{k=0}^{\infty} \binom{m+k-1}{k} \frac{x^k}{(1+x)^{m+k}} f\left(\frac{k}{m}\right), \tag{2.3}$$

which can be constructed if one uses the form considered by Fisher [10] for the negative binomial distribution.

We further note that if $\alpha > 0$ and $x > 0$ then one verifies directly that the fundamental polynomials $v_{m,k}^{\alpha}$ can also be represented in the following form

$$v_{m,\alpha}^{\alpha} = \binom{m+k-1}{k} \frac{B\left(\frac{1}{\alpha}+m, \frac{x}{\alpha}+k\right)}{B\left(\frac{1}{\alpha}, \frac{x}{\alpha}\right)} \tag{2.4}$$

in terms of the Euler beta function B.

3. The Stancu Meyer-König and Zeller operator W_m^{α}

If we use a generalization given by D.D. Stancu [29] for the Pascal probability distribution, then we can obtain the following general linear operator W_m^{α}, defined for a function f bounded on the interval $[0,1]$ namely

$$(W_m^{\alpha} f)(x) = \sum_{k=0}^{\infty} w_{m,k}^{\alpha}(x) f\left(\frac{k}{m+k}\right), \tag{3.1}$$

where for $0 \leq x < 1$ we have

$$w_{m,k}^{\alpha}(x) = \binom{m+k}{k} \frac{x^{(k,-\alpha)}(1-x)^{(m+1,-\alpha)}}{1^{(m+k+1,-\alpha)}}$$

$$= \binom{m+k}{k} \frac{x(x+\alpha)\dots(x+(k-1)\alpha)(1-x)(1-x+\alpha)\dots(1-x+m\alpha)}{(1+\alpha)(1+2\alpha)\dots(1+(m+k)\alpha)}.$$

One observes that if $x = 0$ then $W_m^{\alpha} f(0) = f(0)$, while if $x = 1$ it is convenient to take

$$(W_m^{\alpha} f) = \lim_{x \nearrow 1} (W_m^{\alpha} f)(x) = f(1).$$

It is obvious to see that W_m^{α} includes as a special case ($\alpha = 0$) the operator of Meyer-König and Zeller [22] defined by

$$(M_m f)(x) = \sum_{k=0}^{\infty} w_{m,k}(x) f\left(\frac{k}{m+k}\right), \tag{3.2}$$

where

$$w_{m,k}(x) = \binom{m+k}{k} x^k (1-x)^{m+1}$$

obtained by these authors by using the negative binomial probability distribution of Pascal.

In the paper [29] D.D. Stancu has given an integral representation of $W_m^\alpha f$ by using the Beta transformation of the operator M_m defined at (3.2), which is valid for $\alpha > 0$ and $0 < x < 1$ namely,

$$(W_m^\alpha f)(x) = \frac{1}{B\left(\frac{x}{\alpha}, \frac{1-x}{\alpha}\right)} \int_0^1 t^{\frac{x}{\alpha}-1}(1-t)^{\frac{1-x}{\alpha}}(M_m f)(t)dt. \tag{3.3}$$

Concerning this operator we want to mention that the paper [37] J. Swetits and B. Wood referring to a probabilistic method in connection with the Markov-Polya urn scheme, used by D.D. Stancu in [28] for constructing the operator S_m^α, have presented a variation of the Pascal urn scheme.

It permits to give a probabilistic interpretation of the operator M_m^α.

By using the notation

$$w_{m,k}^\alpha(u,v) = \binom{m+k}{k} \frac{u^{(k,-\alpha)}v^{(m+1,-\alpha)}}{(u+v)^{(m+1+k,-\alpha)}}$$

and the second order divided differences of the function f, D.D. Stancu has evaluated in the paper [33] the remainder term in the approximation formula of the function f:

$$f(x) = (M_m^\alpha f)(x) + (R_m^\alpha f)(x) \tag{3.4}$$

by means of $M_m^\alpha f$, namely

$$(R_m^\alpha f)(x) = -x(1-x)\sum_{k=0}^{\infty}(m+1+k)^{-1}w_{m-1,k}^\alpha(x+\alpha, 1-x+\alpha)\left[x, \frac{k}{m+k}, \frac{k+1}{m+1+k}; f\right],$$

where the brackets represent the symbol for divided differences.

4. **The Stancu beta operators of second kind**

By starting from the beta distribution of second kind $b_{p,q}$ (with positive parameters), which belongs to Karl Pearson's Type VI, one defines the beta second kind transformation $T_{p,q}$ of a function $g : [0,\infty) \to \mathbb{R}$ bounded and Lebesgue measurable in every interval $[a,b]$, where $0 < a < b < \infty$ such that $T_{p,q}|q| < \infty$. The moment of order r $(1 \le r < q)$ of the functional $T_{p,q}$ has the following value

$$\nu_r(p,q) = T_{p,q}e_r = \frac{p(p+1)\dots(p+r-1)}{(q-1)(q-2)\dots(q-r)}. \tag{4.1}$$

If one applies this transformation to the image of a function $f : [a, \infty) \to \mathbb{R}$ by the operator defined at (2.3) then we obtain the functional

$$F_m^f(p, q) = T_{p,q}(Q_m f),$$

given explicitly under the following form

$$F_m^f(p, q) = T_{p,q}(Q_m f)$$

$$= \sum_{k=0}^{m} \binom{m+k-1}{k} \frac{p(p+1)\dots(p+q-1)q(q+1)\dots(q+m-1)}{(p+q)(p+q+1)\dots(p+q+m+k-1)} f\left(\frac{k}{m}\right).$$

If we select $p = \dfrac{x}{\alpha}$ and $q = \dfrac{1}{\alpha}$, where $\alpha > 0$ then the preceding formulas leads us to the parameter dependent operator S_m^α introduced in 1970 in the paper of D.D. Stancu [29] (see also the paper [31] of the same author) as a generalization of the Baskakov operator [6]. By using the factorial powers, with the step $h = -\alpha$, it can be expressed under the following compact form

$$(L_m^\alpha f)(x) = \sum_{k=0}^{\infty} \binom{m+k-1}{k} \frac{x^{[k,-\alpha]} 1^{[m,-\alpha]}}{(1+x)^{[m+k,-\alpha]}} f\left(\frac{k}{m}\right).$$

It should be noticed that the operator $T_{\frac{x}{\alpha},\frac{1}{\alpha}} = T^\alpha$ was used in the paper [5] for obtaining the operator L_m^α by Adell and De la Cal.

Professor D.D. Stancu has introduced the new beta second kind approximation operator, defined by the formula

$$(L_m f)(x) = (T_{mx,m+1} f)(x) = \frac{1}{B(mx, m+1)} \int_0^\infty f(t) \frac{t^{mx-1} dt}{(1+t)^{mx+m+1}}. \tag{4.2}$$

Because

$$(L_m e_0)(x) = \int_0^\infty b_{mx,m+1}(t) dt = 1$$

it follows that the operator L_m reproduces the linear functions.

It is easily seen that this operator is of Feller's type but it is not an averaging operator.

If we use an inequality established by D.D. Stancu in [35], one can find an inequality given the order of approximation of the function f by means of $L_m f$, namely

$$|f(x) - (L_m f)(x)| \leq (1 + \sqrt{x(x+1)})\omega_1 \left(f; \frac{1}{\sqrt{m-1}}\right),$$

respectively

$$|f(x) - (L_m f)(x)| \leq (1 + x(x+1))\omega_2 \left(f; \frac{1}{\sqrt{m-1}}\right)$$

where $\omega_k(f, \delta)$ represents the modulus of continuity of order k $(k = 1, 2)$ of the function f.

According to the Bohman-Korovkin convergence criterion we can deduce that the sequence $(L_m f)$ converges uniformly in $[a, b]$ to the function f when m tends to infinity.

In the paper [29] D.D. Stancu has established an inequality of Lorentz type and an asymptotic formula of Voronovskaja type.

Ending this paper we mention that the operator of beta type of second kind of D.D. Stancu is distinct from other beta operators considered earlier by Mülbach [24], Lupaş [22], Upreti [38], Khan [14] and Adell [3].

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Babeş-Bolyai University
Doctoral Department of Mathematics
400370 Str. Kogălniceanu nr. 1
Cluj-Napoca, Romania
E-mail address: lazeln@yahoo.com