STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIV**, Number 2, June 2009

VARIATIONAL ANALYSIS OF A ELASTIC-VISCOPLASTIC CONTACT PROBLEM WITH FRICTION AND ADHESION

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Abstract. The aim of this paper is to study the process of frictional contact with adhesion between a body and an obstacle. The material's behavior is assumed to be elastic-viscoplastic, the process is quasistatic, the contact is modeled by the Signorini condition and the friction is described by a non local Coulomb law coupled with adhesion. The adhesion process is modelled by a bonding field on the contact surface. We derive a variational formulation of the problem, then, under a smallness assumption on the coefficient of friction, we prove an existence and uniqueness result of a weak solution for the model. The proof is based on arguments of time-dependent variational inequalities, differential equations and Banach fixed point theorem.

1. Introduction

The adhesive contact between deformable bodies, when a glue is added to prevent relative motion of the surfaces, has received recently increased attention in the mathematical literature. Basic modelling can be found in [10], [12], [14] and [6]. Analysis of models for adhesive contact can be found in [2]-[4], [13] and in the recent monographs [17],[18]. An application of the theory of adhesive contact in the medical field of prosthetic limbs was considered in [15], [16]; there, the importance of the bonding between the bone-implant and the tissue was outlined, since debonding may lead to decrease in the persons ability to use the artificial limb or joint.

Received by the editors: 15.05.2008.

2000 *Mathematics Subject Classification.* 74H10, 74H10, 74M15, 74F25, 49J40.

Key words and phrases. Elastic-viscoplastic material, non local Coulomb friction law, adhesion, variational inequality, weak solution, fixed point.

Contact problems for elastic and elastic-viscoelastic bodies with adhesion and friction appear in many applications of solids mechanics such as the fiber-matrix interface of composite materials. A consistent model coupling unilateral contact, adhesion and friction is proposed by Raous, Cangémi and Cocu in [14]. Adhesive problems have been the subject of some recent publications (see for instance [14], [9], [1], [3], [6]). The novelty in all the above papers is the introduction of a surface internal variable, the bonding field, denoted in this paper by β; it describes the pointwise fractional density of active bonds on the contact surface, and sometimes referred to as the intensity of adhesion. Following [10], [11], the bonding field satisfies the restrictions $0 \leq \beta \leq 1$; when $\beta = 1$ at a point of the contact surface, the adhesion is complete and all the bonds are active; when $\beta = 0$ all the bonds are inactive, severed, and there is no adhesion; when $0 < \beta < 1$ the adhesion is partial and only a fraction β of the bonds is active. We refer the reader to the extensive bibliography on the subject in [12], [14], [15]. Such models contain a new internal variable β which represents the adhesion intensity over the contact surface, it takes values between 0 and 1, and describes the fractional density of active bonds on the contact surface.

Elastic quasistatic contact problems with *Signorini* conditions and local *Coulomb* friction law were recently studied by Cocu and Rocca in [5]. Other elastic-viscoplastic contact models with *Signorini* conditions and non local *Coulomb* friction law were variationally analyzed in [7], [8] There exists at least one solution to such problems if the friction coefficient is sufficiently small.

The aims of this paper is to extend the result when non local Coulomb friction law coupled with adhesion are taken into account at the interface and the material behavior is assumed to be elastic-viscoplastic.

The paper is structured as follows. In Section 2 we present the elastic-viscoplastic contact model with fiction and adhesion and provide comments on the contact boundary conditions. In Section 3 we list the assumptions on the data, derive the variational formulation. In Sections 4, we present our main existence and uniqueness results, Theorems 4.1, which state the unique weak solvability of the Signorini adhesive contact problem with non local Coulomb friction law conditions.

2. Problem statement

We consider an elastic-viscoplastic body, which occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$, with a smooth boundary $\partial\Omega = \Gamma$ divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that $meas(\Gamma_1) > 0$. Let $[0, T]$ be the time interval of interest, where $T > 0$. The body is clamped on $\Gamma_1 \times (0, T)$ and therefore the displacement field vanishes there, it is also submitted to the action of volume forces of density f_0 in $\Omega \times (0, T)$ and surface tractions of density f_2 on $\Gamma_2 \times (0, T)$. On $\Gamma_3 \times (0, T)$, the body is in adhesive contact with friction with an obstacle the so-called foundation. The friction is modelled by a non local Coulomb law. We denote by ν the outward normal unit vector on Γ.

With these assumptions, the classical formulation of the elastic-viscoplastic contact problem with friction and adhesion is the following.

Problem $\mathcal{P}$. *Find a displacement field $u : \Omega \times [0, T] \to R^d$, a stress field $\sigma : \Omega \times [0, T] \to S^d$, and a bonding field $\beta : \Omega \times [0, T] \to R$ such that*

$$\dot{\sigma} = \mathcal{E}\varepsilon(\dot{u}) + \mathcal{G}(\sigma, \varepsilon(u)) \qquad \text{in } \Omega \times (0, T), \tag{2.1}$$

$$\text{Div}\sigma + f_0 = 0 \qquad \text{in } \Omega \times (0, T), \tag{2.2}$$

$$u = 0 \qquad \text{on } \Gamma_1 \times (0, T), \tag{2.3}$$

$$\sigma\nu = f_2 \qquad \text{on } \Gamma_2 \times (0, T), \tag{2.4}$$

$$u_\nu \leq 0,\ \sigma_\nu - \gamma_\nu\beta^2 R_\nu(u_\nu) \leq 0,\ u_\nu(\sigma_\nu - \gamma_\nu\beta^2 R_\nu(u_\nu)) = 0 \text{ on } \Gamma_3 \times (0, T), \tag{2.5}$$

$$\left\{\begin{array}{l} |\sigma_\tau + \gamma_\tau\beta^2 R_\tau(u_\tau)| \leq \mu p(|R(\sigma_\nu) - \gamma_\nu\beta^2 R_\nu(u_\nu)|), \\ |\sigma_\tau + \gamma_\tau\beta^2 R_\tau(u_\tau)| < \mu p(|R(\sigma_\nu) - \gamma_\nu\beta^2 R_\nu(u_\nu)|) \Rightarrow u_\tau = 0, \\ |\sigma_\tau + \gamma_\tau\beta^2 R_\tau(u_\tau) = \mu p(|R(\sigma_\nu) - \gamma_\nu\beta^2 R_\nu(u_\nu)|) \Rightarrow \exists\lambda \geq 0, \\ \text{such that } \sigma_\tau + \gamma_\tau\beta^2 R_\tau(u_\tau) = -\lambda u_\tau. \end{array}\right. \quad \text{on } \Gamma_3 \times (0, T), \tag{2.6}$$

$$\dot{\beta} = -(\beta(\gamma_\nu R_\nu(u_\nu)^2 + \gamma_\tau\|R_\tau(u_\tau)\|^2) - \epsilon_a)_+ \qquad \text{on } \Gamma_3 \times (0, T), \tag{2.7}$$

$$u(0) = u_0,\ \sigma(0) = \sigma_0 \text{ in } \Omega,\ \beta(0) = \beta_0 \text{ on } \Gamma_3 \tag{2.8}$$

We now provide some comments on equations and conditions (2.1)-(2.8). The material is assumed to be elastic-viscoplastic with a constitutive law of the form (2.1), where $\mathcal{E}$ and $\mathcal{G}$ are constitutive functions which will be described below. We denote by $\varepsilon(u)$ the linearized strain tensor. The equilibrium equation is given by (2.2), where "*Div*" denotes the divergence operator for tensor valued functions. Equations (2.3) and (2.4) represent the displacement and traction boundary conditions.

Conditions (2.5) represent he Signorini contact condition with adhesion where u_ν is the normal displacement $\sigma_\nu$represents the normal stress, γ_ν denote a given adhesion coefficient and R_ν is the truncation operator define by

$$R_\nu(s) = \begin{cases} L & \text{if } s < -L, \\ -s & \text{if } -L \leq s \leq 0, \\ 0 & \text{if } s > 0, \end{cases}$$

where $L > 0$ is the characteristic length of the bond, beyond which it does not offer any additional traction. The introduction of operator R_ν, together with the operator R_τ defined below , is motivated by the mathematical arguments but it is not restrictive for physical point of view, since no restriction on the size of the parameter L is made in what follows. Thus, by choosing L very large, we can assume that $R_\nu(u_\nu) = u_\nu$ and, therefore, from (2.5) we recover the contact conditions

$$u_\nu \leq 0,\ \sigma_\nu - \gamma_\nu \beta^2 u_\nu \leq 0,\ u_\nu(\sigma_\nu - \gamma_\nu \beta^2 u_\nu) = 0 \text{ on } \Gamma_3 \times (0,T),$$

It follows from (2.5) that there is no penetration between the body and the foundation, since $u_\nu \leq 0$ during the process.

Conditions (2.6) are a non local Coulomb friction law conditions coupled with adhesion, where u_τ, σ_τ denote tangential components of vector u and tensor σ respectively. R_τ is the truncation operator given by

$$R_\tau(v) = \begin{cases} v & \text{if } \|v\| \leq L, \\ L\frac{v}{\|v\|} & \text{if } \|v\| > L. \end{cases}$$

This condition shows that the magnitude of the shear on the contact surface depends on the bonding field and on the tangential displacement, but as long as it does not exceed the bond length L.

R will represent a normal regularization operator that is , linear and continues operator $R: H^{-\frac{1}{2}}(\Gamma) \longrightarrow L^2(\Gamma)$. We shall need it to regularize the normal trace of the stress witch is too rough on Γ. p is a non-negative function, the so-called friction bound, $\mu \geq 0$ is the coefficient of friction. The friction law was used with $p(r) = r_+$. A new version of *Coulomb* law consists to take

$$p(r) = r(1 - \alpha r)_+,$$

where α is a small positive coefficient related to the hardness and the wear of the contact surface and $r_+ = \max\{0, r\}$.

Also, note that when the bonding field vanishes, then the contact conditions (2.5) and (2.6) become the classic Signorini contact with a non local Coulomb friction law conditions were used in ([8]), that is

$$u_\nu \leq 0, \ \sigma_\nu \leq 0, \ u_\nu \sigma_\nu = 0 \text{ on } \Gamma_3 \times (0, T),$$

$$\begin{cases} |\sigma_\tau| \leq \mu p(|R(\sigma_\nu)|), \\ |\sigma_\tau| < \mu p(|R(\sigma_\nu)|) \Rightarrow u_\tau = 0, & \text{on } \Gamma_3 \times (0, T), \\ |\sigma_\tau| = \mu p(|R(\sigma_\nu)|) \Rightarrow \exists \lambda \geq 0, \text{ such that } \sigma_\tau = -\lambda u_\tau. \end{cases}$$

The evolution of the bonding field is governed by the differential equation (2.7) with given positive parameters γ_ν, γ_τ and ϵ_a, where $r_+ = \max\{0, r\}$. Here and below in this paper, a dot above a function represents the derivative with respect to the time variable. We note that the adhesive process is irreversible and, indeed, once debonding occurs bonding cannot be reestablished, since $\dot{\beta} \leq 0$. Finally, (2.8) is the initial condition in which β_0 is a given bonding field.

3. **Variational formulation and preliminaries**

In this section, we list the assumptions on the data and derive a variational formulation for the contact problem. To this end we need to introduce some notation and preliminary material.

Here and below $\mathbb{S}^d$ represents the space of second order symmetric tensors on $\mathbb{R}^d$. We recall that the inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by

$$u \cdot v = u_i v_i \ , \qquad \|v\| = (v \cdot v)^{\frac{1}{2}} \qquad \forall\, u, v \in \mathbb{R}^d,$$

$$\sigma \cdot \tau = \sigma_{ij} \tau_{ij} \ , \qquad \|\tau\| = (\tau \cdot \tau)^{\frac{1}{2}} \qquad \forall\, \sigma, \tau \in \mathbb{S}^d.$$

Here and everywhere in this paper, i, j, k, l run from 1 to d, summation over repeated indices is applied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g. $u_{i,j} = \frac{\partial u_i}{\partial x_j}$.

Everywhere below, we use the classical notation for L^p and Sobolev spaces associated to Ω and Γ. Moreover, we use the notation $L^2(\Omega)^d$, $H^1(\Omega)^d$ and $\mathcal{H}$ and $\mathcal{H}_1$ for the following spaces :

$$L^2(\Omega)^d = \{\ v = (v_i) \mid v_i \in L^2(\Omega)\ \}, \quad H^1(\Omega)^d = \{\ v = (v_i) \mid v_i \in H^1(\Omega)\ \},$$

$$\mathcal{H} = \{\ \tau = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega)\ \}, \quad \mathcal{H}_1 = \{\ \tau \in \mathcal{H} \mid \tau_{ij,j} \in L^2(\Omega)\ \}.$$

The spaces $L^2(\Omega)^d$, $H^1(\Omega)^d$, $\mathcal{H}$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$(u, v)_{L^2(\Omega)^d} = \int_\Omega u \cdot v \, dx, \quad (u, v)_{H^1(\Omega)^d} = \int_\Omega u \cdot v \, dx + \int_\Omega \nabla u \cdot \nabla v \, dx,$$

$$(\sigma, \tau)_{\mathcal{H}} = \int_\Omega \sigma \cdot \tau \, dx, \quad (\sigma, \tau)_{\mathcal{H}_1} = \int_\Omega \sigma \cdot \tau \, dx + \int_\Omega \operatorname{Div} \sigma \cdot \operatorname{Div} \tau \, dx,$$

and the associated norms $\|\cdot\|_{L^2(\Omega)^d}$, $\|\cdot\|_{H^1(\Omega)^d}$, $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}_1}$, respectively. Here and below we use the notation

$$\varepsilon(v) = (\varepsilon_{ij}(v)), \quad \varepsilon_{ij}(v) = \frac{1}{2}(v_{i,j} + v_{j,i}) \quad \forall\, v \in H^1(\Omega)^d,$$

$$\operatorname{Div} \tau = (\tau_{ij,j}) \quad \forall\, \tau \in \mathcal{H}_1.$$

For every element $v \in H^1(\Omega)^d$ we also write v for the trace of v on Γ and we denote by v_ν and v_τ the normal and tangential components of v on Γ given by $v_\nu = v \cdot \nu$, $v_\tau = v - v_\nu \nu$.

Let now consider the closed subspace of $H^1(\Omega)^d$ defined by

$$V = \{\ v \in H^1(\Omega)^d \mid v = 0 \text{ on } \Gamma_1\ \}.$$

Since $meas\,(\Gamma_1) > 0$, the following Korn's inequality holds:

$$\|\varepsilon(v)\|_{\mathcal{H}} \geq c_K \, \|v\|_{H^1(\Omega)^d} \quad \forall\, v \in V, \tag{3.1}$$

where $c_K > 0$ is a constant which depends only on Ω and Γ_1. Over the space V we consider the inner product given by

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}} \tag{3.2}$$

and let $\|\cdot\|_V$ be the associated norm. It follows from Korn's inequality (3.1) that $\|\cdot\|_{H^1(\Omega)^d}$ and $\|\cdot\|_V$ are equivalent norms on V and, therefore, $(V, \|\cdot\|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, (3.1) and (3.2), there exists a constant c_0 depending only on the domain Ω, Γ_1 and Γ_3 such that

$$\|v\|_{L^2(\Gamma_3)^d} \leq c_0 \|v\|_V \qquad \forall v \in V. \tag{3.3}$$

For every real Hilbert space X we use the classical notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, $1 \leq p \leq \infty$, $k \geq 1$ and we also introduce the set

$$\mathcal{Q} = \{\, \theta \in L^\infty(0,T;L^2(\Gamma_3)) \mid 0 \leq \theta(t) \leq 1 \ \ \forall\, t \in [0,T],\ \text{a.e. on } \Gamma_3 \,\}.$$

Finally, if X_1 and X_2 are two Hilbert spaces endowed with the inner products $(\cdot,\cdot)_{X_1}$ and $(\cdot,\cdot)_{X_2}$ and the associated norms $\|\cdot\|_{X_1}$ and $\|\cdot\|_{X_2}$, respectively, we denote by $X_1 \times X_2$ the product space together with the canonical inner product $(\cdot,\cdot)_{X_1 \times X_2}$ and the associated norm $\|\cdot\|_{X_1 \times X_2}$.

In the study of the problem $\mathcal{P}$, we consider the following assumptions on the problem data.

$$\begin{cases} \mathcal{E}\colon \Omega\times\mathcal{S}_d \longrightarrow \mathcal{S}_d \text{ is a symmetric and positive definite tensor :} \\ (a)\ \ \mathcal{E}_{ijkl}\in L^\infty(\Omega) \ \text{ for every } i,j,k,l=\overline{1,d}; \\ (b)\ \ \mathcal{E}\sigma\cdot\tau=\sigma\cdot\mathcal{E}\tau \text{ for every } \sigma,\,\tau\in\mathcal{S}_d; \\ (c)\ \ \text{there exists } \alpha>0 \text{ such that } \mathcal{E}\sigma\cdot\sigma\geq\alpha\left|\sigma\right|^2 \ \ \forall\sigma\in\mathcal{S}_d, \ \text{a.e. in } \Omega \end{cases} \tag{3.4}$$

$$\begin{cases} \mathcal{G}:\Omega\times\mathcal{S}_d\times\mathcal{S}_d\longrightarrow\mathcal{S}_d \text{ and} \\ (a)\ \ \text{there exists } L_{\mathcal{G}}>0 \text{ such that :} \\ \qquad \left|\mathcal{G}\left(\cdot,\sigma_1,\varepsilon_1\right)-\mathcal{G}\left(\cdot,\sigma_2,\varepsilon_2\right)\right|\leq L_{\mathcal{G}}(\left|\sigma_1-\sigma_2\right|+\left|\varepsilon_1-\varepsilon_2\right|) \\ \qquad \text{for every } \sigma_1,\sigma_2,\varepsilon_1,\varepsilon_2\in\mathcal{S}_d \text{ a.e. in } \Omega; \\ (b)\ \ \mathcal{G}\left(\cdot,\sigma,\varepsilon\right) \ \text{ is a measurable function with respect to the Lebesgue} \\ \qquad \text{measure on } \Omega \text{ for every } \varepsilon,\sigma\in\mathcal{S}_d; \\ (c)\ \ \mathcal{G}\left(\cdot,0,0\right)\in\mathcal{H}. \end{cases} \tag{3.5}$$

$$\begin{cases} \text{The friction function p}:\Gamma_3\times\mathbb{R}_+\longrightarrow\mathbb{R}_+ \text{ verifies} \\ (a)\ \ \text{there exists } M>0 \text{ such that :} \\ \qquad \left|p\left(x,r_1\right)-p\left(x,r_2\right)\right|\leq M\left|r_{1-}r_2\right| \\ \qquad \text{for every } r_1,r_2\in\mathbb{R}_+, \quad \text{a.e. } x\in\Gamma_3; \\ (b)\ \ x\mapsto p\left(x,r\right) \ \text{ is measurable on } \Gamma_3, \ \text{for every } r\in\mathbb{R}_+; \\ (c)\ \ p\left(x,0\right)=0, \quad \text{a.e. } x\in\Gamma_3. \end{cases} \tag{3.6}$$

We also suppose that the body forces and surface tractions have the regularity

$$f_0\in W^{1,\infty}(0,T;L^2(\Omega)^d), \quad f_2\in W^{1,\infty}(0,T;L^2(\Gamma_2)^d), \tag{3.7}$$

and we define the function $f:[0,T]\to V$ by

$$(f(t),v)_V=\int_\Omega f_0(t)\cdot v\,dx+\int_{\Gamma_2} f_2(t)\cdot v\,da, \tag{3.8}$$

for all $u, v \in V$ and $t \in [0,T]$, and we note that the condition (3.7) implies that

$$f \in W^{1,\infty}(0,T;V). \tag{3.9}$$

For the Signorini problem we use the convex subset of admissible displacements given by

$$U_{ad} = \{v \in H_1 \mid \; v = 0 \text{ on } \Gamma_1\,, \; v_\nu \leq 0 \text{ on } \Gamma_3\} \tag{3.10}$$

The adhesion coefficients γ_ν, γ_τ and the limit bound ϵ_a satisfy the conditions

$$\gamma_\nu,\; \gamma_\tau \in L^\infty(\Gamma_3), \quad \epsilon_a \in L^2(\Gamma_3), \quad \gamma_\nu,\, \gamma_\tau,\, \epsilon_a \geq 0 \quad \text{a.e. on } \Gamma_3 \tag{3.11}$$

while the friction coefficient μ is such that

$$\mu \in L^\infty(\Gamma_3), \quad \mu(x) \geq 0 \quad \text{a.e. on } \Gamma_3. \tag{3.12}$$

Finally, we assume that the initial data verifies

$$\beta_0 \in L^2(\Gamma_3), \quad 0 \leq \beta_0 \leq 1 \;\text{ a.e. on } \Gamma_3. \tag{3.13}$$

We define the adhesion functional $j_{ad} : L^\infty(\Gamma_3) \times V \times V \to \mathbb{R}$ by

$$j_{ad}(\beta, u, v) = \int_{\Gamma_3} \big(-\gamma_\nu \beta^2 R_\nu(u_\nu) v_\nu + \gamma_\tau \beta^2 R_\tau(u_\tau) \cdot v_\tau \big)\, da, \tag{3.14}$$

and the friction functional $j_{fr} : L^\infty(\Gamma_3) \times \mathcal{H}_1 \times V \times V \to \mathbb{R}$ by

$$j_{fr}(\beta, \sigma, u, v) = \int_{\Gamma_3} \mu p(|R(\sigma_\nu) - \gamma_\nu \beta^2 R_\nu(u_\nu)|) \cdot |v_\tau|\, da, \tag{3.15}$$

The initial conditions u_0, σ_0 and β_0 satisfy

$$u_0 \in U_{ad}, \quad \sigma_0 \in \mathcal{H}_1, \quad \beta_0 \in L^2(\Gamma_3) \cap \mathcal{Q}, \tag{3.16}$$

and

$$(\sigma_0, \varepsilon(v) - \varepsilon(u_0))_{\mathcal{H}} + j_{ad}(\beta_0, \sigma_0, v - u_0) + j_{fr}(\beta_0, \sigma_0, \xi_0, v) - j_{fr}(\beta_0, \sigma_0, \xi_0, u_0) \geq$$
$$\geq (f_0, v - u_0)_V + (f_2, v - u_0)_{L^2(\Gamma_2)^d} \quad \forall\, v \in U_{ad}. \tag{3.17}$$

Let us remark that assumption (3.16) and (3.17) involve regularity conditions of the initial data u_0, σ_0 and β_0 and a compatibility condition between u_0, σ_0, β_0, f_0 and f_2.

By a standard procedure based on Green's formula combined with (2.2)-(2.4) and (3.8), we can derive the following variational formulation of problem $\mathcal{P}$, in terms of displacement, stress and bonding fields.

Proof.[**Problem** $\mathcal{P}^V$] *Find a displacement field* $u : [0,T] \to V$*, a stress field* $\sigma : [0,T] \to H_1$ *and a bonding field* $\beta : [0,T] \to L^2(\Gamma_3)$ *such that*

$$\dot{\sigma} = \mathcal{E}\varepsilon(\dot{u}) + \mathcal{G}(\sigma, \varepsilon(u)) \qquad \text{in } \ \Omega \times (0,T), \tag{3.18}$$

$$\begin{gathered} u(t) \in U_{ad}, \quad (\sigma(t), \varepsilon(v) - \varepsilon(u(t)))_{\mathcal{H}} + j_{ad}(\beta(t), u(t), v - u(t)) + \\ j_{fr}(\beta(t), \sigma(t), u(t), v) - j_{fr}(\beta(t), \sigma(t), u(t), u(t)) \geq (f(t), v - u(t))_V \\ \forall\, v \in U_{ad}, \ t \in [0,T], \end{gathered} \tag{3.19}$$

$$\dot{\beta}(t) = -\Big(\beta(t)\big(\gamma_\nu R_\nu(u_\nu(t))^2 + \gamma_\tau \|R_\tau(u_\tau(t))\|^2\big) - \epsilon_a\Big)_+ \text{ a.e. on } t \in (0,T),$$

$$u(0) = u_0, \qquad \sigma(0) = \sigma_0, \qquad \beta(0) = \beta_0. \tag{3.20}$$

□

In the rest of this section, we derive some inequalities involving the functionals j_{ad}, and j_{fr} which will be used in the following sections. Below in this section β, β_1, β_2 denote elements of $L^2(\Gamma_3)$ such that $0 \leq \beta,\ \beta_1,\ \beta_2 \leq 1$ a.e. on Γ_3, u_1, u_2, v_1, v_2, u and v represent elements of V; σ, σ_1, σ_2 denote elements of $\mathcal{H}_1$ and c is a generic positive constants which may depend on Ω, Γ_1, Γ_3, p, γ_ν, γ_τ and L, whose value may change from place to place. For the sake of simplicity, we suppress in what follows the explicit dependence on various functions on $\boldsymbol{x} \in \Omega \cup \Gamma_3$.

First, we remark that the j_{ad} is linear with respect to the last argument and therefore

$$j_{ad}(\beta, u, -v) = -j_{ad}(\beta, u, v). \tag{3.21}$$

Next, using (3.14) and the inequalities $|R_\nu(u_{1\nu})| \leq L$, $\|R_\tau(u_\tau)\| \leq L$, $|\beta_1| \leq 1$, $|\beta_2| \leq 1$, for the previous inequality, we deduce that

$$j_{ad}(\beta_1, u_1, u_2 - u_1) + j_{ad}(\beta_2, u_2, u_1 - u_2) \leq c \int_{\Gamma_3} |\beta_1 - \beta_2|\, \|u_1 - u_2\|\, da,$$

then, we combine this inequality with (3.3), to obtain

$$j_{ad}(\beta_1, u_1, u_2 - u_1) + j_{ad}(\beta_2, u_2, u_1 - u_2) \leq c\,\|\beta_1 - \beta_2\|_{L^2(\Gamma_3)}\|u_1 - u_2\|_V. \qquad (3.22)$$

Next, we choose $\beta_1 = \beta_2 = \beta$ in (3.22) to find

$$j_{ad}(\beta, u_1, u_2 - u_1) + j_{ad}(\beta, u_2, u_1 - u_2) \leq 0. \qquad (3.23)$$

Similar manipulations, based on the Lipschitz continuity of operators R_ν, R_τ show that

$$|j_{ad}(\beta, u_1, v) - j_{ad}(\beta, u_2, v)| \leq c\,\|u_1 - u_2\|_V\|v\|_V. \qquad (3.24)$$

Also, we take $u_1 = v$ and $u_2 = 0$ in (3.23), then we use the equalities $R_\nu(0) = 0$, $R_\tau(0) = 0$ and (3.22) to obtain

$$j_{ad}(\beta, v, v) \geq 0. \qquad (3.25)$$

Next, we use (3.15), (3.6)(a), keeping in mind (3.3), propriety of R and the inequalities $|R_\nu(u_{1\nu})| \leq L,\ \|R_\tau(u_\tau)\| \leq L,\ |\beta_1| \leq 1,\ |\beta_2| \leq 1$ we obtain

$$j_{fr}(\beta_1, \sigma_1, u_1, u_2) - j_{fr}(\beta_1, \sigma_1, u_1, u_1) + j_{fr}(\beta_2, \sigma_2, u_2, u_1) - j_{fr}(\beta_2, \sigma_2, u_2, u_2) \leq$$
$$\leq c_0^2 M\ \|\mu\|_{L^\infty(\Gamma_3)}(\|\beta_2 - \beta_1\|_{L^2(\Gamma_3)} + \|\sigma_{2-}\sigma_1||_{\mathcal{H}_1})||u_2 - u_1\|_V. \qquad (3.26)$$

Now, by using (3.6)(a) and (3.12), it follows that the integral in (3.15) is well defined. Moreover, we have

$$j_{fr}(\beta, \sigma, u, v) \leq c_0^2 M\|\mu\|_{L^\infty(\Gamma_3)}\|(\|\sigma||_{\mathcal{H}_1} + \|\beta\|_{L^2(\Gamma_3)})||u\|_V\|v\|_V. \qquad (3.27)$$

The inequalities (3.22)-(3.27) combined with equalities (3.21) will be used in various places in the rest of the paper.

4. **Existence and uniqueness result**

Our main result which states the unique solvability of Problem $\mathcal{P}^V$, is the following.

Theorem 4.1. *Assume that assumptions* (3.4)-(3.7) *and* (3.11)-(3.13) *hold. Then, there exists* $\mu_0 > 0$ *depending only on* Ω, Γ_1, Γ_3, $\mathcal{E}$ *and* p *such that, if* $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$, *then Problem* $\mathcal{P}^V$ *has a unique solution* (u, σ, β). *Moreover, the solution satisfies*

$$u \in W^{1,\infty}(0, T; V), \tag{4.1}$$

$$\sigma \in W^{1,\infty}(0, T; \mathcal{H}_1), \tag{4.2}$$

$$\beta \in W^{1,\infty}(0, T; L^2(\Gamma_3)) \cap \mathcal{Q}. \tag{4.3}$$

A triple of functions $(u,\ \sigma,\ \beta)$ which satisfies (2.1), and (3.19)-(3.20) is called a weak solution of the frictional adhesive contact Problem $\mathcal{P}$. We conclude by Theorem 4.1. that, under the assumptions (3.4)-(3.7) and (3.11)-(3.13), if $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$, then there exists a unique weak solution of Problem $\mathcal{P}$ which verifies (4.1)-(4.3), that we present in what follows.

The proof of the Theorem 4.1 will be carried out in several steps. It based on fixed-point arguments. To this end, we assume in the following that (3.4)-(3.7)and (3.11)-(3.13) hold; below, c is a generic positive constants which may depend on Ω, Γ_1, Γ_3, $\mathcal{E}$ *and* p, γ_ν, γ_τ and L, whose value may change from place to place. For the sake of simplicity, we suppress in what follows the explicit dependence on various functions on $\boldsymbol{x} \in \Omega \cup \Gamma_3$.

For each $\eta = (\eta_1, \eta_2) \in L^\infty(0, T; \mathcal{H} \times L^2(\Gamma_3))$ we introduce the function $z_\eta = (z_\eta^1, z_\eta^2) \in W^{1,\infty}(0, T; \mathcal{H} \times L^2(\Gamma_3))$ defined by

$$z_\eta(t) = \int_0^t \eta(s)ds + z_0 \qquad \forall t \in [0, T], \tag{4.4}$$

where

$$z_0 = (\sigma_0 - \mathcal{E}\varepsilon(u_0), \beta_0). \tag{4.5}$$

In the first step, we consider the following variational problem.

Proof.[Problem $\mathcal{P}^\eta$] *Find a displacement field* $u_\eta : [0, T] \to V$, *a stress field* $\sigma_\eta : [0, T] \to \mathcal{H}_1$ *such that*

$$\sigma_\eta(t) = \mathcal{E}\varepsilon(u_\eta(t)) + z_\eta^1(t) \tag{4.6}$$

$$u_\eta(t) \in U_{ad}, \quad (\sigma_\eta(t), \varepsilon(v) - \varepsilon(u_\eta(t)))_{\mathcal{H}} + j_{ad}(z_\eta^2(t), u_\eta(t), v - u_\eta(t)) +$$
$$+ j_{fr}(z_\eta^2(t), \sigma_\eta, u_\eta(t), v) - j_{fr}(z_\eta^2(t), \sigma_\eta, u_\eta(t), u_\eta(t)) \geq \tag{4.7}$$
$$\geq (f(t), v - u_\eta(t))_V \quad \forall v \in U_{ad}.$$

□

We have the following result.

Lemma 4.2. *There exists* $\mu_0 > 0$ *which depends on* Ω, Γ_1, Γ_3, $\mathcal{E}$ *and* p *such that, if* $\|\mu\|_{L^\infty(\Gamma_3)} < \mu_0$*, then, Problem* $\mathcal{P}^\eta$ *has a unique solution having the regularity* $u_\eta \in W^{1,\infty}(0, T, V)$*,* $\sigma_\eta(t) \in W^{1,\infty}(0, T; \mathcal{H}_1)$*. Moreover,*

$$u_\eta(0) = u_0, \qquad \sigma_\eta(0) = \sigma_0 \tag{4.8}$$

Proof. *Using Riez's representation theorem we may define the operator* $A_\eta(t) : V \to V$ *and the element* $f_\eta(t) \in V$ *by*

$$(A_\eta(t) u_\eta(t), v)_V = (\mathcal{E}\varepsilon(u_\eta(t)), \varepsilon(v))_{\mathcal{H}} + j_{ad}(z_\eta^2(t), u_\eta(t), v)$$
$$\forall\ t \in [0, T], \quad \forall\ w,\ v \in U_{ad}, \tag{4.9}$$

$$(f_\eta(t), v)_V = (f(t), v)_V - (z_\eta^1(t), \varepsilon(v))_{\mathcal{H}}$$
$$\forall\ t \in [0, T], \quad \forall v \in U_{ad}, \tag{4.10}$$

Let $t \in [0, T]$*. We use the assumption (3.4), the equalities (3.21) and the inequalities (3.23) and (3.24) to prove that* $A_\eta(t)$ *is a strongly monotone Lipschitz continuous operator on V. Moreover,by (3.10) we have that* U_{ad} *is a closed convex non-empty set of* V*. Using (3.15), we can easily check that* $j_{fr}(z_\eta^2(t), \sigma_\eta, u_\eta(t), .)$ *is a continuous seminorm on* V *and moreover, it satisfies (3.26) and(3.27). Then by an existence and uniqueness result on elliptic quasivariational inequalities, drabla it follows that there exists a unique solution* $u_\eta(t)$ *such that*

$$u_\eta(t) \in U_{ad}.$$

$$(A_\eta(t)u_\eta(t), v)_V + j_{fr}(, z_\eta^2(t), \sigma_\eta, u_\eta(t), v) - j_{fr}(z_\eta^2(t), \sigma_\eta, u_\eta(t), u_\eta(t)) \geq$$
$$\geq (f_\eta(t), v - u_\eta(t))_V \qquad \forall\, v \in U_{ad}.$$

Taking $\sigma_\eta\,(t))$, defined by (4.6) and using (4.4), we deduce that $\sigma_\eta \in W^{1,\infty}(0,T;\mathcal{H})$ and (4.7). Let us remark, that for $v = u_{g\eta}\,(t) \mp \varphi \;\; \forall \varphi \in D\,(\Omega)^d$, it comes from (4.6) and Green's formula

$$\mathrm{Div}\sigma_\eta(t) + f_0(t) = 0. \tag{4.11}$$

Keeping in mind that $f_0 \in W^{1,\infty}(0,T;L^2\,(\Omega)^d)$ it follows that $\sigma_\eta \in W^{1,\infty}(0,T;\mathcal{H}_1)$. Therefore, the existence and uniqueness of $(u_\eta(t), \sigma_\eta\,(t)) \in V \times \mathcal{H}_1$ solution of problem $\mathcal{P}^\eta$ is established under smallness assumption. The initial conditions (4.8) follows from (3.17), (4.4) and (4.5) and the uniqueness of the problem for $t = 0$.

Let now $t_1,\ t_2 \in [0,T]$, Using (3.4), (3.1) and (4.4)we obtain

$$\|u_\eta(t_1) - u_\eta(t_2)\|_V \leq c(\|f(t_1) - f(t_2)\|_V + ||z_\eta(t_1) - z_\eta(t_2)\|_{\mathcal{H}\times L^2(\Gamma_3)} +$$
$$+ \|\sigma_\eta(t_1) - \sigma_\eta(t_2)\|), \tag{4.12}$$

and from (4.6), (4.11) and (4.12), it result that

$$\|\sigma_\eta(t_1) - \sigma_\eta(t_2)\|_V \leq c(\|f(t_1) - f(t_2)\|_V + ||z_\eta^1(t_1) - z_\eta^1(t_2)\|_{\mathcal{H}}). \tag{4.13}$$

Recall that $f \in W^{1,\infty}(0,T;V)$, $z_\eta = (z_\eta^1, z_\eta^2) \in W^{1,\infty}(0,T;\mathcal{H} \times L^2(\Gamma_3))$, it follows from (4.12) and (4.13) that $u_\eta \in W^{1,\infty}(0,T;V)$ and $\sigma_\eta \in W^{1,\infty}(0,T;\mathcal{H}_1)$. □

We denote by $\beta_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3))$ the function defined by

$$\beta_\eta = z_\eta^2, \tag{4.14}$$

and consider the mapping $F:\ [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$ defined by

$$F(t,\beta_\eta) = -(\beta_\eta(t)(\gamma_\nu R_\nu((u_\eta)_\gamma(t))^2 + \gamma_\tau\|R_\tau((u_\eta)_\tau(t))\|^2) - \epsilon_a)_+, \tag{4.15}$$

for all $t \in [0,T]$ and $\beta_\eta \in W^{1,\infty}(0,T;L^2(\Gamma_3))$

Using the assumptions (3.4), (3.5), (4.4), and (4.5), we may consider the operator

$$\Lambda\eta : L^\infty(0,T;\mathcal{H} \times L^2(\Gamma_3)) \longrightarrow L^\infty(0,T;\mathcal{H} \times L^2(\Gamma_3))$$

define by

$$\Lambda\eta = (\mathcal{G}(\sigma_\eta, \varepsilon(u_\eta)), F(t, \beta_\eta)) \quad \forall \eta \in L^\infty(0,T;\mathcal{H} \times L^2(\Gamma_3)) \tag{4.16}$$

where (σ_η, u_η) is the solution of the variational problem $\mathcal{P}^\eta$.

In the last step, we will prove the following result.

Lemma 4.3. *There exists a unique element* $\eta^* = (\eta_1^*, \eta_2^*)$ *such that* $\Lambda\eta^* = \eta^*$ *and* $\eta^* \in L^\infty(0,T;\mathcal{H} \times L^2(\Gamma_3))$

Proof. Let $\eta_1 = (\eta_1^1, \eta_1^2)$ and $\eta_2 = (\eta_2^1, \eta_2^2) \in L^\infty(0,T;\mathcal{H} \times L^2(\Gamma_3))$ and let $t \in [0,T]$. We use similar arguments to those used in the proof of (4.10) to deduce that

$$\|u_{\eta_1} - u_{\eta_2}\|_V \leq c(||z_{\eta_1} - z_{\eta_2}\|_{\mathcal{H}\times L^2(\Gamma_3)} + \|\sigma_{\eta_1} - \sigma_{\eta_2}\|_{\mathcal{H}}), \tag{4.17}$$

and from (3.4), (3.5) and (4.6), we obtain that

$$\|\sigma_{\eta_1} - \sigma_{\eta_2}\|_{\mathcal{H}} \leq c(\|u_{\eta_1} - u_{\eta_2}\|_V + ||z_{\eta_1} - z_{\eta_2}\|_{\mathcal{H}\times L^2(\Gamma_3)}), \tag{4.18}$$

from (4.17) and (4.18), it results that

$$\|u_{\eta_1} - u_{\eta_2}\|_V \leq c||z_{\eta_1} - z_{\eta_2}\|_{\mathcal{H}\times L^2(\Gamma_3)}. \tag{4.19}$$

On the other hand, it follows from (4.15) that

$$||F_{\eta_2}(t, \beta_{\eta_2}) - F_{\eta_1}(t, \beta_{\eta_1})||_{L^2(\Gamma_3)} \leq$$
$$\leq c\,\|\beta_{\eta_1}(t) R_\nu(u_{\eta_1\nu}(t))^2 - \beta_{\eta_2}(t) R_\nu(u_{\eta_2\nu}(t))^2\|_{L^2(\Gamma_3)} +$$
$$+ \|\beta_{\eta_1}(t)\|R_\tau(u_{\eta_1\tau}(t))\|^2 - \beta_{\eta_2}(t)\|R_\tau(u_{\eta_2\tau}(t))\|^2\|_{L^2(\Gamma_3)}.$$

Using the definition of R_ν and R_τ and writing $\beta_{\eta_1} = \beta_{\eta_1} - \beta_{\eta_2} + \beta_{\eta_2}$, we get

$$||F(t, \beta_{\eta_2}) - F(t, \beta_{\eta_1})||_{L^2(\Gamma_3)} \leq c\,\|\beta_{\eta_1}(t) - \beta_{\eta_2}(t)\|_{L^2(\Gamma_3)} + c\|u_{\eta_1}(t) - u_{\eta_2}(t)\|_{L^2(\Gamma_3)}.$$

We now use (4.17), (4.18), (4.14) and (4.5) to deduce

$$\|\Lambda\eta_2(t) - \Lambda\eta_1(t)\|_{\mathcal{H}\times L^2(\Gamma_3)} \leq c\|z_{\eta_2}(t) - z_{\eta_1}(t)\|_{\mathcal{H}\times L^2(\Gamma_3)}.$$

From (3.5), (4.4), (4.16) and the last inequalities, it result that

$$\|\Lambda\eta_2(t) - \Lambda\eta_1(t)\|_{\mathcal{H}\times L^2(\Gamma_3)} \leq c\int_0^t \|\eta_2(t) - \eta_1(t)\|_{\mathcal{H}\times L^2(\Gamma_3)} ds. \tag{4.20}$$

Denoting now by Λ^p the power of the operator Λ, (4.20) implies by recurrence that

$$\|\Lambda\eta_2(t) - \Lambda\eta_1(t)\|_{\mathcal{H}\times L^2(\Gamma_3)} \leq c \int_0^t \int_0^s \qquad \int_0^q \|\eta_2(t) - \eta_1(t)\|_{\mathcal{H}\times L^2(\Gamma_3)} dr \quad ds,$$

for all $t \in [0,T]$ and $p \in N$. Hence, it follows that

$$\|\Lambda^p\eta_2 - \Lambda^p\eta_1\|_{L^\infty(0,T,\mathcal{H}\times L^2(\Gamma_3))} \leq \frac{c^nT^n}{n!}\|\eta_2 - \eta_1\|_{L^\infty(0,T,\mathcal{H}\times L^2(\Gamma_3))}, \quad \forall p \in N. \quad (4.21)$$

and since $\lim_{p\to\infty} \dfrac{c^pT^p}{p!} = 0$, inequality (4.21) shows that for p sufficiently large $\Lambda^p : L^\infty(0,T;\mathcal{H} \times L^2(\Gamma_3)) \to L^\infty(0,T;\mathcal{H} \times L^2(\Gamma_3))$ is a contraction. Then, we conclude by using the Banach fixed point theorem that Λ has a unique fixed point $\eta^* \in L^\infty(0,T;\mathcal{H} \times L^2(\Gamma_3))$ such that $\Lambda\eta^* = \eta^*$. Hence, from (4.16) it results for all $t \in [0,T]$,

$$\eta^*(t) = (\eta^{*1}(t), \eta^{*2}(t)) = (\mathcal{G}(\sigma_{\eta^*}(t), \varepsilon(u_{\eta^*}(t))), (F(t, \beta_{\eta^*}(t)))) \qquad (4.22)$$

□

Now, we have all the ingredients to provide the proof of Theorem 4.1.

Proof.[Proof of Theorem 4.1.] *Existence.* Let $\eta^* \in L^\infty(0,T;\mathcal{H}\times L^2(\Gamma_3))$ be the fixed point of Λ and let $(u_{\eta*}, \sigma_{\eta*}) \in W^{1,\infty}(0,T;\mathcal{H}_1 \times V)$ be the solution of Problem $\mathcal{P}^{\eta^*}$. Let also $\beta_{\eta^*} \in W^{1,\infty}(0,T;L^2(\Gamma_3))$ be the solution of Problem $\mathcal{P}^\eta$ for $\eta = \eta^*$. We shall prove that $(u_{\eta^*}, \sigma_{\eta^*}, \beta_{\eta^*})$ is a unique solution of Problem $\mathcal{P}^V$.

The regularity expressed in (4.1) follow from Lemma 4.1, Lemma 4.3 and the fixed point of operators Λ.

The initial conditions (3.20) follow from (4.5), (4.14) and (4.8) for $\eta = \eta^*$. Moreover, the equalities (3.18) and (3.20) follow from (4.4), (4.6), Lemma 4.4, (4.12) and (4.16) for $\eta = \eta^*$ since

$$\begin{aligned}
\dot{\sigma}_{\eta^*}(t) &= \mathcal{E}\varepsilon(u_{\eta^*}(t)) + \dot{z}^1_{\eta^*}(t) && a.e.\ t \in (0,T) \\
z^1_{\eta^*}(t) &= \eta^{*1}(t) = \mathcal{G}(\sigma_{\eta^*}(t), \varepsilon(u_{\eta^*}(t))) && a.e.\ t \in (0,T) \\
\dot{\beta}_{\eta^*}(t) &= \dot{z}^2_{\eta^*}(t) = \eta^{*2}(t) = F(t, \beta_{\eta^*}(t)) && a.e.\ t \in (0,T)
\end{aligned}$$

Uniqueness. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of operators Λ defined by (4.16). Indeed, let $(u, \sigma, \beta) \in W^{1,\infty}(0, T; V \times \mathcal{H}_1 \times L^2(\Gamma_3))$ be another solution of Problem $\mathcal{P}^V$.

We denote by $\eta \in L^\infty(0, T; \mathcal{H} \times L^2(\Gamma_3))$ the function defined by

$$\eta(t) = (\mathcal{G}(\sigma, \varepsilon(u)), F(t, \beta)), \qquad \forall t \in [0, T], \tag{4.23}$$

and let $z_\eta \in W^{1,\infty}(0, T; \mathcal{H} \times L^2(\Gamma_3))$ be the function given by (4.4) and (4.5). It results that (u, σ) is a solution to Problem $\mathcal{P}_\eta$ and since by Lemma 4.1, this problem has a unique solution denoted (u_η, σ_η), we obtain

$$u = u_\eta \text{ and } \sigma = \sigma_\eta. \tag{4.24}$$

Then, we replace $(u, \sigma) = (u_\eta, \sigma_\eta) = (u_{\eta*}, \sigma_{\eta*})$ in (3.20) and use the initial condition (3.20) to see that β is a solution to Problem $\mathcal{P}_\eta$. Since by Lemma 4.2, this last problem has a unique solution denoted β_η, we find

$$\beta = \beta_\eta. \tag{4.25}$$

We use now (4.16) and (4.25) to obtain that $\eta = (\mathcal{G}(\sigma_\eta, \varepsilon(u_\eta)), F_\eta(t, \beta_\eta))$, i.e. η is a fixed point of the operator Λ. It follows now from Lemma 4.3 that

$$\eta = \eta^*. \tag{4.26}$$

The uniqueness part of the theorem is now a consequence of (4.24), (4.25) and (4.26). □

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