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THE STABILITY OF THE EQUILIBRIUM STATES FOR SOME MECHANICAL SYSTEMS

MIRCEA PUTA† AND DORIN WAINBERG

Abstract. In the first part of the paper some theoretical results (including the Lyapunov-Malkin theorem) are presented, followed in the second part by some of its applications in geometrical mechanics.

1. Theoretical aspects

To explain the stability concept, we need some basic notions and results from the theory of dynamical systems.

The lows of Dynamics are usually presented as equations of motion, which we will write as differential equations:

$$\dot{x} = f(x) \tag{1}$$

where

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

is a variable describing the state of the system. The function

$$f : \mathbb{R}^n \to \mathbb{R}^n$$

is smooth of x, and

$$\dot{x} = \frac{dx}{dt}.$$

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†Mircea Puta (1950-2007) died in a tragic car accident in July 26, 2007..

The set of all allowed x forms the state space of (1). When the time advances, the system's state is changed.

Definition 1. *An point $x_e \in \mathbb{R}^n$ is called equilibrium point for the system (1) if:*

$$f(x_e) = 0.$$

Remark 1. It is clear that the constant function

$$x(t) = x_e$$

is a solution for (1) and from the existence and uniqueness theorem it results that does not exists other solution containing x_e. So the unique trajectory starting at x_e is x_e itself, i.e. x_e does not changes in time.

Definition 2. *Let x_e be an equilibrium state for (1). We will say that x_e is nonlinear stable (or Lyapunov stable) if for any neighborhood U of x_e there exists a neighborhood V of x_e, $V \subset U$ such that any solution $x(t)$, initially in V (i.e. $x(0) \in V$), never leaves U.*

Definition 3. *If V in Definition 2 can be chosen such that*

$$\lim_{t \to \infty} x(t) = x_e,$$

then x_e is called asymptotically stable.

Definition 4. *An equilibrium state x_e that is not stable is called unstable.*

Let us consider the following system of differential equations of order one:

$$\begin{cases} \dot{x} = Ax + X(x, y) \\ \dot{y} = By + Y(x, y) \end{cases} \tag{2}$$

where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, A and B are constant matrices such that all eigenvalues of A are of nonzero real parts and all eigenvalues of B are of zero real parts, and the functions X, Y satisfy the following conditions:

i) $X(0, 0) = 0$,

ii) $dX(0, 0) = 0$,

iii) $Y(0, 0) = 0$,

iv) $dY(0, 0) = 0$.

We will take now the particular case of (2) in which the matrix B is O_n. The equations (2) become:

$$\begin{cases} \dot{x} = Ax + X(x, y) \\ \dot{y} = Y(x, y). \end{cases} \tag{3}$$

Theorem 1. *(Lyapunov-Malkin) Under the above conditions, if all eigenvalues of* A *have negative real parts and* $X(x, y)$ *and* $Y(x, y)$ *vanish when* $x = 0$*, then the equilibrium state*

$$x = 0, \ y = 0$$

of the system (2) is nonlinear stable with respect to (x, y) *and asymptotically stable with respect to* X*.*

For the proof of this basic result see Zenkov, Bloch and Marsden [5].

2. Two application of Lyapunov-Malkin theorem

In this section we study the stability of the equilibrium points for some concrete mechanical systems.

Example 1. (3-dimensional Toda lattice with two controls)

The dynamics of the generalized 3-dimensional Toda lattice with two controls is described by the following system:

$$\begin{cases} \dot{q}_1 = 2p_1^2 \\ \dot{q}_2 = 2p_2^2 - 2p_1^2 \\ \dot{q}_3 = -2p_2^2 \\ \dot{p}_1 = p_1 q_2 - p_1 q_1 + u_1 \\ \dot{p}_2 = p_2 q_3 - p_2 q_2 + u_2. \end{cases} \tag{4}$$

In what follows we shall employ the controls:

$$\begin{cases} u_1 = \alpha p_1 \\ u_2 = \beta p_2 \end{cases}$$

where $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta < 0$. Then the system (4) takes the following form:

$$\begin{cases} \dot{q}_1 = 2p_1^2 \\ \dot{q}_2 = 2p_2^2 - 2p_1^2 \\ \dot{q}_3 = -2p_2^2 \\ \dot{p}_1 = p_1 q_2 - p_1 q_1 + \alpha p_1 \\ \dot{p}_2 = p_2 q_3 - p_2 q_2 + \beta p_2. \end{cases} \tag{5}$$

Now we can consider:

$$x = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}, \quad y = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$X(x,y) = \begin{bmatrix} p_1 q_2 - p_1 q_1 \\ p_2 q_3 - p_2 q_2 \end{bmatrix}$$

$$Y(x,y) = \begin{bmatrix} 2p_1^2 \\ 2p_2^2 - 2p_1^2 \\ -2p_2^2 \end{bmatrix}.$$

In those conditions the system (5) can be written in the following equivalent form:

$$\begin{cases} \dot{x} = Ax + X(x,y) \\ \dot{y} = Y(x,y) \end{cases}$$

where:

$$A = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}.$$

We will verify the conditions in the Lyapunov-Malkin theorem. We have successively:

i) $$X(0,0) = \begin{bmatrix} p_1 q_2 - p_1 q_1 \\ p_2 q_3 - p_2 q_2 \end{bmatrix}_{(0,0,0,0,0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$Y(0,0) = \begin{bmatrix} 2p_1^2 \\ 2p_2^2 - 2p_1^2 \\ -2p_2^2 \end{bmatrix}_{(0,0,0,0,0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

ii) $X(0,y) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, for any $y \in \mathbb{R}^3$,

$Y(0,y) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, for any $y \in \mathbb{R}^3$.

iii)
$$\frac{DX}{Dx}\Big|_{(0,0)} = \begin{bmatrix} \frac{\partial(p_1q_2-p_1q_1)}{\partial p_1} & \frac{\partial(p_1q_2-p_1q_1)}{\partial p_2} \\ \frac{\partial(p_2q_3-p_2q_2)}{\partial p_1} & \frac{\partial(p_2q_3-p_2q_2)}{\partial p_2} \end{bmatrix}_{(0,0,0,0,0)} = \begin{bmatrix} q_2-q_1 & 0 \\ 0 & q_3-q_2 \end{bmatrix}_{(0,0,0,0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\frac{DX}{Dy}\Big|_{(0,0)} = \begin{bmatrix} \frac{\partial(p_1q_2-p_1q_1)}{\partial q_1} & \frac{\partial(p_1q_2-p_1q_1)}{\partial q_2} & \frac{\partial(p_1q_2-p_1q_1)}{\partial q_3} \\ \frac{\partial(p_2q_3-p_2q_2)}{\partial q_1} & \frac{\partial(p_2q_3-p_2q_2)}{\partial q_2} & \frac{\partial(p_2q_3-p_2q_2)}{\partial q_3} \end{bmatrix}_{(0,0,0,0,0)} = \begin{bmatrix} -p_1 & p_1 & 0 \\ 0 & -p_2 & p_2 \end{bmatrix}_{(0,0,0,0,0)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\frac{DY}{Dx}\Big|_{(0,0)} = \begin{bmatrix} \frac{\partial(2p_1^2)}{\partial p_1} & \frac{\partial(2p_1^2)}{\partial p_2} \\ \frac{\partial(2p_2^2-2p_1^2)}{\partial p_1} & \frac{\partial(2p_2^2-2p_1^2)}{\partial p_2} \\ \frac{\partial(-2p_2^2)}{\partial p_1} & \frac{\partial(-2p_2^2)}{\partial p_1} \end{bmatrix}_{(0,0,0,0,0)} = \begin{bmatrix} 4p_1 & 0 \\ -4p_1 & 4p_2 \\ 0 & -4p_2 \end{bmatrix}_{(0,0,0,0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$\frac{DY}{Dy}\Big|_{(0,0)} = \begin{bmatrix} \frac{\partial(2p_1^2)}{\partial q_1} & \frac{\partial(2p_1^2)}{\partial q_2} & \frac{\partial(2p_1^2)}{\partial q_3} \\ \frac{\partial(2p_2^2-2p_1^2)}{\partial q_1} & \frac{\partial(2p_2^2-2p_1^2)}{\partial q_2} & \frac{\partial(2p_2^2-2p_1^2)}{\partial q_3} \\ \frac{\partial(-2p_2^2)}{\partial q_1} & \frac{\partial(-2p_2^2)}{\partial q_1} & \frac{\partial(-2p_2^2)}{\partial q_3} \end{bmatrix}_{(0,0,0,0,0)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

iv) The characteristic polynomial of matrix A is

$$P_A(x) = \det \begin{bmatrix} \alpha - x & 0 \\ 0 & \beta - x \end{bmatrix} = (\alpha - x)(\beta - x).$$

It has negative roots α and β. Hence the eigenvalues of the matrix A are $x_1 = \alpha < 0,\ x_2 = \beta < 0$.

We can conclude with:

Remark 2. The equilibrium state $(0,0,0,0,0)$ for the system (4) is nonlinear stable.

Example 2. (Maxwell-Bloch equations with two controls)

The Maxwell-Bloch equations with two controls on axes Ox_1 and Ox_2 are writing in the following form:

$$\begin{cases} \dot{x}_1 = x_2 + u_1 \\ \dot{x}_2 = x_1 x_3 + u_2 \\ \dot{x}_3 = -x_1 x_2. \end{cases} \tag{6}$$

The controls u_1 and u_2 will be written as:

$$u_1(x_1, x_2, x_3) = \alpha x_1$$
$$u_2(x_1, x_2, x_3) = \beta x_2,$$

where $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta < 0$. Then our dynamics takes the following form:

$$\begin{cases} \dot{x}_1 = x_2 + \alpha x_1 \\ \dot{x}_2 = x_1 x_3 + \beta x_2 \\ \dot{x}_3 = -x_1 x_2. \end{cases} \tag{7}$$

If we take:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad y = x_3,$$
$$X(x,y) = \begin{bmatrix} 0 \\ x_1 x_3 \end{bmatrix},$$
$$Y(x,y) = -x_1 x_2$$

then the system (7) can be written in the following equivalent form:

$$\begin{cases} \dot{x} = Ax + X(x,y) \\ \dot{y} = Y(x,y) \end{cases} \tag{8}$$

where:

$$A = \begin{bmatrix} \alpha & 1 \\ 0 & \beta \end{bmatrix}.$$

Now we must verify the conditions from the Lyapunov-Malkin theorem for system (8). We have successively:

i) $X(0,0) = \begin{bmatrix} 0 \\ x_1x_3 \end{bmatrix}_{(0,0,0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

$Y(0,0) = [x_3]_{(0,0,0)} = 0.$

ii) $X(0,y) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad (\forall)\ y \in R^3,$

$Y(0,y) = 0, \quad (\forall)\ y \in R^3.$

iii) $\left.\frac{DX}{Dx}\right|_{(0,0)} = \begin{bmatrix} \frac{\partial 0}{\partial x_1} & \frac{\partial 0}{\partial x_2} \\ \frac{\partial (x_1x_3)}{\partial x_1} & \frac{\partial (x_1x_3)}{\partial x_2} \end{bmatrix}_{(0,0,0)}$

$= \begin{bmatrix} 0 & 0 \\ x_3 & 0 \end{bmatrix}_{(0,0,0)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$,

$\left.\frac{DX}{Dy}\right|_{(0,0)} = \begin{bmatrix} \frac{\partial 0}{\partial x_3} \\ \frac{\partial (x_1x_3)}{\partial x_3} \end{bmatrix}_{(0,0,0)}$

$= \begin{bmatrix} 0 \\ x_1 \end{bmatrix}_{(0,0,0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

$\left.\frac{DY}{Dx}\right|_{(0,0)} = \begin{bmatrix} -x_2 & -x_1 \end{bmatrix}_{(0,0,0)}$

$= \begin{bmatrix} 0 & 0 \end{bmatrix}_{(0,0,0)}$,

$\left.\frac{DY}{Dy}\right|_{(0,0)} = [0]_{(0,0,0)} = [0].$

iv) Again the characteristic polynomial of A is

$$P_A(x) = \det \begin{bmatrix} \alpha - x & 1 \\ 0 & \beta - x \end{bmatrix} = (\alpha - x)(\beta - x),$$

and it has negative roots. So we have the eigenvalues of the matrix A, $x_1 = \alpha < 0,\ x_2 = \beta < 0$.

We can conclude, by Lyapunov-Malkin theorem, that:

The equilibrium state $(0,0,0)$ for the system (6) is nonlinear stable.

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West University,
Faculty of Mathematics and Computer Science,
Timisoara, Romania

"1 Decembrie 1918" University,
Faculty of Sciences,
Alba Iulia, Romania
E-mail address: dwainberg@uab.ro, wainbergdorin@yahoo.com