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APPROXIMATION OF CONTINUOUS FUNCTIONS ON V.K. DZJADYK CURVES

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Abstract. In the given paper rational approximation is studied on closed curves of a complex plane for continuous functions in terms of the k-th modulus of continuity, $k \geq 1$. Here a rational function interpolates a continuous function at definite points.

1. **Introduction and main result**

Let Γ be an arbitrary restricted Jordan curve with two -component complements $\Omega = C\Gamma = \Omega_1 \bigcup \Omega_2, (0 \in \Omega_1, \infty \in \Omega_2)$. Let's consider the functions $w = \Phi_i(z)$, $(i = 1, 2)$, that conformally and univalently maps respectively Ω_i onto $\Omega_i^{'}$, $(\Omega_1^{'} = \{w : |w| < 1\}, \ \Omega_2^{'} = \{w : |w| > 1\})$, with norm $\Phi_1(0) = 0$, $\Phi_1^{'}(0) > 0$, $\Phi_2(\infty) = \infty$, $\lim\limits_{z\to\infty} \frac{1}{z}\Phi_2(z) > 0$. Let's extend each $\Phi_i(z), (i = 1, 2)$ continuously up to the bound $\Gamma = \partial\Omega_1 = \partial\Omega_2$ (generally speaking $\Phi_1(z) \neq \Phi_2(z)$ for $z \in \Gamma$). We preserve the notation $\Phi_i, (i = 1.2)$ for the extension.Let $z = \Psi_i(w)$ be the inverse mapping of $w = \Phi_i(z), (i = 1, 2)$.

For $A > 0$ and $B > 0$, we use the expression $A \preceq B$ (order inequality) if $A \leq CB$. The expression $A \asymp B$ means that $A \preceq B$ and $B \preceq A$ hold simultaneously.

By $C(\Gamma)$ we denote a class of functions continuous on Γ. For $\delta > 0$ and fixed $u_o \in (0, 1)$ we assume

$$U(z, \delta)) = \{\zeta : |\zeta - z| < \delta\}, d(\zeta, \Gamma) = \inf_{z\in\Gamma} |\zeta - z|,$$

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$$\Gamma_\delta = \bigcup_{z\in\Gamma} U(z,\delta) = \{\zeta : d(\zeta,\Gamma) < \delta\},$$

$$\Gamma_{1+u_0} = \{\zeta : \zeta \in \Omega_2; |\Phi_2(\zeta)| = 1 + u_0\},$$

$$\Gamma_{1-u_0} = \{\zeta : \zeta \in \Omega_1; |\Phi_1(\zeta)| = 1 - u_o\},$$

$$D^1_{u_o} = int\Gamma_{1+u_0}, D^2_{u_o} = ext\Gamma_{1-u_o},$$

$$D_{u_o} = D^1_{u_0} \bigcap D^2_{u_0},$$

$$\delta^*_n = \sup_{\zeta\in int\Gamma_{1+\frac{1}{n}} \cap\, ext\Gamma_{1-\frac{1}{n}}} d(\zeta,\Gamma), n = 1, 2, \cdots,$$

where under $int\,\Gamma$ we understand a finite domain whose boundary coincides with Γ, under $ext\,\Gamma$ we understand a finite domain whose domain coincides with Γ. Let R_n, $n = 0, 1, 2, ...$ be a set of all complex rational functions of power no higher than n. For $f \in C(\Gamma)$ we define

$$E_n(f,\Gamma) := \inf_{r_n\in R_n} \sup_{z\in\Gamma} |f(z) - r_n(z)| = \inf_{r_n\in R_n} \|f - r_n\|_\Gamma.$$

In connection with "simultaneous approximation and interpolation" the following claims are suggested in the paper $[14, p.310]$. Let $z_1, z_2, ..., z_p \in \Gamma$ be definite points and $f \in C(\Gamma)$. In this case for $\forall n \in N, n \geq p - 1$, there exists a rational function $r_n \in R_n$, for which

$$\|f - r_n\|_\Gamma \leq cE_n(f,\Gamma),$$

$$r_n(z_j) = f(z_j),\ \ (j = 1, 2, ..., p),$$

where $c > 0$ is independent n and f. The appropriate rational function is written in the following form

$$r_n(z) = r^*_n(z) + \sum_{j=1}^{p} \frac{q(z)}{q'(z_j)(z - z_j)}(f(z_j) - r^*_n(z_j)),$$

where

$$q(z) = \prod_{j=1}^{p}(z - z_j)$$

and $r^*_n \in R_n$ satisfies the following condition

$$\|f - r^*_n\|_\Gamma = E_n(f,\Gamma).$$

Definition 1.[11] *Let* Γ *be a rectifiable Jordan curve. By* $\theta(z,\delta), z\in\Gamma, 0<\delta<+\infty$ *we denote the length of a part of* Γ *getting into the* $U(z,\delta)=\zeta:|\zeta-z|<\delta$. *We attribute the curve* Γ *to the class* S *if it is fulfilled the condition*

$$\theta_\Gamma(\delta)\overset{df}{=}\sup_{z\in\Gamma}\theta(z,\delta)\asymp\delta.$$

Let's give the definition of a class of V.K. Dzjadyk curves B_k^* in a briefly and slightly modified form. (see [7, p.439-440]).

Definition 2. *We'll say that a rectifiable Jordan curve* Γ *belongs to the class* B_k^* *for some natural* k *if* $\Gamma\in S$ *and satisfies the following conditions*

(i) $|\tilde{z}-z|\asymp\rho_{1+\frac{1}{n}}(z)$,

where $\forall z\in\Gamma$, $\tilde{z}=\Psi_2((1+\frac{1}{n})\Phi_2(z))$, $\rho_{1+\frac{1}{n}}(z)=\inf\limits_{\zeta\in\Gamma_{1+\frac{1}{n}}}|\zeta-z|$,

(ii) $|\tilde{\zeta}-\zeta|^k\le|\tilde{\zeta}-z|^{k-1}|\tilde{z}-z|, \forall z,\zeta\in\Gamma.$

We'll study the functions for which the k-th modulus of continuity ($k\in N$) have been defined. There are some different definitions of such continuity modulus (see [6], [8], [13], [15]). The most convenient for our aim is the definition given by E.M. Dynkin [8].

Definition 3. *The k-th local modulus of continuity we'll call the quantity*

$$\omega_{f,k,z,\Gamma}(\delta)=E_{k-1}(f,\Gamma\cap U(z,\delta)),$$

where $f\in C(\Gamma), k\in N, z\in\Gamma, \delta>0.$

Definition 4. *The k-th global modulus of continuity we'll call the quantity*

$$\omega_{f,k,\Gamma}(\delta)=\sup_{z\in\Gamma}\omega_{f,k,z,\Gamma}(\delta).$$

In particular,

$$\omega_{f,k,\Gamma}(t\delta)\le c_1t^k\omega_{f,k,\Gamma}(\delta)\quad(t>1,\delta>0). \tag{1}$$

If $0<\delta<1$, there exists a constant c_2, that

$$\int_0^\delta\omega_{f,k,\Gamma}(t)\frac{dt}{t}\le c_2\omega_{f,k,\Gamma}(\delta). \tag{2}$$

The following theorems is the main result of the report.

Theorem 1. *Let* $\Gamma \in B_k^*,\ f \in C(\Gamma), k \in N$ *and* $z_1, z_2, ..., z_p \in \Gamma$ *be distinct points. Then for each* $n \in N, n \geq p+k$ *there exists a rational function* $r_n \in R_n$ *for which the following conditions are fulfilled*

$$|f(z) - r_n(z)| \leq c_1 \omega_{f,k,\Gamma}(\delta_n^*) \quad (z \in \Gamma), \tag{3}$$

$$r_n(z_j) = f(z_j) \quad (j = 1, 2, ..., p), \tag{4}$$

where the constant c_1 *is independent of* n.

The rational functions play an important role in many areas of applied mathematics and mechanics. It is actually the approximation of continuous functions by rational functions or some other functions, which can be found easily. The Theorem 1 studies rational approximations (in terms of the k-th modulus of continuity, $k \geq 1$) for continuous functions defined on closed curves Γ in the complex plane, which simultaneously interpolate at given points of Γ. The similar results for the analytic functions in different continua were obtained in the papers [5], [15], [1].

2. **Subsidiary facts**

By obtaining the main result we use an approximation of Cauchy kernel $(s-z)^{-1}$ by rational functions of the form

$$K_n(\zeta, z) = \sum_{j=-n}^{n} a_j(\zeta) z^j.$$

To construct rational functions a rational kernel suggested by V.K. Dzjadyk (see [7, ch.9] or [3, ch.3]) is used.

Lemma 1. *Let* Γ *be an arbitrary Jordan curve,* $0 < u_0 < 1$ *be an arbitrary fixed number,* $c = 2(1 + u_0)e^{2\pi}$. *Then for all natural* $n = 1.2, ...$ *and* $\zeta \in D_{u_0} \backslash \Gamma_{\delta^*(\frac{c}{n})}$ *there exists the function* $\Pi_n(\zeta, z) = \sum\limits_{j=-n}^{n} a_j(\zeta) z^j$ *with continuous with respect to* ζ *coefficients* $a_j, j = \overline{-n, n}$, *that for* $z \in \Gamma$ *and* $p = 0, 1$ *satisfies the inequalities*

$$\left|\frac{\partial^p}{\partial z^p}\left[\frac{1}{\zeta - z} - \Pi_n(\zeta, z)\right]\right| \preceq \delta^{*^2}\left(\frac{1}{n}\right)|\zeta - z|^{-p-3} \tag{5}$$

$$\left|\frac{\partial^p}{\partial z^p}\Pi_n(\zeta, z)\right| \preceq |\zeta - z|^{-p-1}. \tag{6}$$

The lemma is proved similarly to Corollary 4 of the paper [2]. It should be only noted that $\Pi_n(\zeta, z)$ is a polynomial kernel for $\zeta \in \left\{D^1_{u_o} \backslash \Gamma_{\delta^*\left(\frac{c}{n}\right)}\right\} \cap \Omega_2$. In the case $\zeta \in \left\{D^2_{u_o} \backslash \Gamma_{\delta^*\left(\frac{c}{n}\right)}\right\} \cap \Omega_1$ as it is shown in the paper [4] it gives us a rational function.

Lemma 2. *Let $\Gamma \in B^*_k, 0 < u_0 < 1$ be an arbitrary number. Then for any $n = 1, 2...$ and $\zeta \in D_{u_o}$ there exists a rational function $K_n(\zeta, z)$ with respect to the variable z with summable with respect to s coefficients for which for $z \in \Gamma$ and $p = 0, 1$ the inequalities*

$$\left|\frac{\partial^p}{\partial z^p}\left[\frac{1}{\zeta - z} - K_n(\zeta, z)\right]\right| \preceq \frac{1}{|\zeta - z|^{p+1}}\left[\frac{\delta^*_n}{|\zeta - z| + \delta^*_n}\right]^2, \tag{7}$$

$$\left|\frac{\partial^p}{\partial z^p} K_n(\zeta, z)\right| \preceq \left[|\zeta - z| + \delta^*_n\right]^{-p-1}. \tag{8}$$

are fulfilled.

Proof. Let n be sufficiently large. Assume $c = 2(1 + u_0)e^{2\pi}$. By compactness of $\overline{\Gamma_{\delta^*\left(\frac{c}{n}\right)}}$ we can distinguish a finite number of points $\zeta_1, \zeta_2, ..., \zeta_m \in \overline{\Gamma_{\delta^*\left(\frac{c}{n}\right)}}$, for which

$$\Gamma_{\delta^*\left(\frac{c}{n}\right)} \subset \bigcup_{k=1}^{m} U(\zeta_k, \delta^*_n).$$

Since $\Gamma \in B^*_k$, at each point $\zeta_k, k = \overline{1, m}$ we can construct the point $\zeta'_k \in D_{u_0} \backslash \Gamma_{\delta^*\left(\frac{c}{n}\right)}$ with the following condition

$$\left|\zeta_k - \zeta'_k\right| \preceq \delta^*_n. \tag{9}$$

We can easily see that

$$\left|\zeta'_k - z\right| \asymp |\zeta - z| + \delta^*_n, \;\; z \in \Gamma, \; \zeta \in U(\zeta_k, \delta^*_n). \tag{10}$$

By the identity

$$\frac{1}{\zeta - z} = \frac{1}{\zeta'_k - z} + \frac{\zeta'_k - \zeta}{\left(\zeta'_k - z\right)^2} + \left(\frac{\zeta'_k - \zeta}{\zeta'_k - z}\right)^2 \frac{1}{\zeta - z}$$

consider for $\zeta \in U(\zeta_k, \delta^*_n), k = \overline{1, m}$ the function

$$\lambda^{(k)}_n(\zeta, z) = \Pi_n\left(\zeta'_k, z\right) + \left(\zeta'_k - \zeta\right)\left(\Pi_{\left[\frac{n}{2}\right]}\left(\zeta'_k, z\right)\right)^2,$$

where $\Pi_n\left(\zeta_k^{'}, z\right))$ is the function from Lemma 1. We construct the required function $K_n(\zeta, z)$ as follows:

1) If $\zeta \in U(\zeta_1, \delta_n^*)$ we assume

$$K_n(\zeta, z) = \lambda_n^{(1)}(\zeta, z)$$

2) If $\zeta \in U(\zeta_k, \delta_n^*) \setminus \bigcup_{j=1}^{k-1} U(\zeta_j, \delta_n^*), k = \overline{2, m}$ we assume

$$K_n(\zeta, z) = \lambda_n^{(k)}(\zeta, z).$$

3) If $\zeta \in D_{u_0} \setminus \left\{ \bigcup_{j=1}^{m} U(\zeta_j, \delta_n^*) \right\}$, we take the appropriate function from Lemma 1. Now the affirmation of Lemma 2 follows from the above mentioned constructions, Lemma 1, estimates of the following easily verifiable relations

$$\frac{1}{\zeta - z} - \lambda_n^{(k)}(\zeta, z) = \left(\frac{\zeta_k^{'} - \zeta}{\zeta_k^{'} - z}\right)^2 \frac{1}{\zeta - z} + \left(\frac{1}{\zeta_k^{'} - z} - \Pi_n\left(\zeta_k^{'}, z\right)\right) +$$

$$+\left(\zeta_k^{'} - \zeta\right)\left[\frac{1}{\left(\zeta_k^{'} - z\right)^2} - \left(\Pi_{\left[\frac{n}{2}\right]}\left(\zeta_k^{'}, z\right)\right)^2\right],$$

$$\left|\frac{\partial^p}{\partial z^p}\left[\frac{1}{\zeta - z} - \lambda_n^{(k)}(\zeta, z)\right]\right| \preceq \frac{1}{|\zeta - z|^{p+1}}\left(\frac{\delta_n^*}{|\zeta - z| + \delta_n^*}\right)^2,$$

$$\left|\frac{\partial^p}{\partial z^p}\lambda_n^{(k)}(\zeta, z)\right| \preceq \left(|\zeta - z| + \delta_n^*\right)^{-p-1},$$

where $\zeta \in U(\zeta_k, \delta_n^*)$, $z \in \Gamma$, $p = 0, 1$.

Let's give a result in slightly modified form cited in the papers [8], [9], [12], [13, p.13-15].

Lemma 3. *Let $\Gamma \in B_k^*$ and $F \in C(\Gamma)$. Then we can continue the function $F(z)$ on the complex plane $\mathbf{C}$ so that the following relations be fulfilled (we keep denotation $F(z)$): (i) for $z \in \mathbf{C}\backslash\Gamma$*

$$\left|\frac{\partial F(z)}{\partial \bar{z}}\right| \leq c_1 \frac{\omega_{F,k,z,\Gamma}(c_2 d(z, \Gamma))}{d(z, \Gamma)},$$

where $c_1 = c_1(k, diam\Gamma)$.

(ii) if $\zeta \in \Gamma, z \in \mathbf{C}$, $|z - \zeta| < \delta$, $0 < \delta < \frac{1}{2} diam\Gamma$, then

$$|F(z) - R_{F,k,\zeta,\Gamma,\delta}(z)| \leq c_3 \omega_{F,k,\zeta,\Gamma}(c_4 \delta),$$

where $R_{F,k,\zeta,\Gamma,\delta}(z) \in R_{k-1}$ *is such a rational function that*

$$\|F - R_{F,k,\zeta,\Gamma,\delta}\|_{\Gamma\cap D(\zeta,\delta)} = \omega_{F,k,\zeta,\Gamma}(\delta),$$

and $c_3 = c_3(k)$.

(iii) If F *satisfies the Lipschitz condition on* Γ, *i.e.*

$$|F(z) - F(\zeta)| \le c\,|z-\zeta|\,, z,\zeta \in \Gamma,$$

then the continued function for $z,\zeta \in \mathbf{C}$ *satisfies the same condition. Here, instead of* c *there will be the constant* $c_4 = c_4(c, diam\Gamma, k)$.

3. **The proof of the main result (Theorem 1)**

Let's fix a point $z_0 \in \Gamma$ and assume for $\zeta \in \Gamma$

$$F(\zeta) = \int\limits_{\gamma(z_0,\zeta)} f(\xi)\,d\xi,$$

where $\gamma(z_0,\zeta) \subset \Gamma$ is an arc connecting the points z_0 and ζ.

We extend the function $F(\zeta)$ continuously on the complex plane $\mathbf{C}$. Let z and $\zeta \in \Gamma, |\zeta - z| \le \delta$, the arc $\gamma(z,\zeta) \subset int\Gamma$ connects these points, $mes\gamma(z,\zeta) \le c\,|z-\zeta|\,, c = c(\Gamma) \ge 1$. We'll have

$$F(\zeta) = F(z) + \int\limits_{\gamma(z,\zeta)} f(\xi)\,d\xi = \nu_\delta(\zeta,z) + + \int\limits_{\gamma(z,\zeta)} (f(\xi) - R_{f,k,z,\Gamma,c\delta}(\xi))\,d\xi,$$

$$\omega_{F,k+1,z,\Gamma}(\delta) \le \|F - \nu_\delta(.,z)\|_{\Gamma\cap D(z,\delta)} \preceq \delta\omega(\delta),$$

where $\omega(\delta) := \omega_{f,k,\Gamma}(\delta)$.

Using Lemma 3 for $\zeta \in G := \overline{int\Gamma_{1+\frac{1}{2}}} \cap \overline{ext\Gamma_{\frac{1}{2}}}$ we have

$$\left|\frac{\partial F(\zeta)}{\partial\bar{\zeta}}\right| \preceq \omega(d(\zeta,\Gamma)). \tag{11}$$

Besides, for $z \in \Gamma, \zeta \in \mathbf{C}$, $|z-\zeta)\,| \le \delta < diam\Gamma$ we have

$$|F(\zeta) - \nu_\delta(\zeta,z)| \preceq \delta\omega(\delta). \tag{12}$$

Indeed, for $\zeta \in \Gamma \cap D(z,\delta)$ the following inequality is valid

$$|\nu_\delta(\zeta,z) - R_{F,k+1,z,\Gamma,\delta}(\zeta)| \le |F(\zeta) - \nu_\delta(\zeta,z)| + |F(\zeta) - R_{F,k+1,z,\Gamma,\delta}(\zeta)| \preceq \delta\omega(\delta).$$

By Bernstein-Walsh lemma (see [14, p.77]) we have

$$\|\nu_\delta(.,z) - R_{F,k+1,z,\Gamma,\delta}\|_{D(z,\delta)} \preceq \delta\omega(\delta) \tag{13}$$

We introduce a rational kernel $Q_{\frac{n}{2}}(\zeta,z) := K_{\left[\frac{n}{2}\right]}(\zeta,z)$, where $K_n(\zeta,z)$ is a rational kernel from Lemma 2. By virtue of (3) and (4) $\zeta \in int\Gamma_{1+\frac{1}{2}} \cap ext\Gamma_{\frac{1}{2}}, z \in \Gamma$ we have

$$\left|\frac{1}{\zeta - z} - Q_{\frac{n}{2}}(\zeta,z)\right| \preceq \frac{1}{|\zeta - z|}\left(\frac{\delta_n^*}{|\zeta - z| + \delta_n^*}\right)^2, \tag{14}$$

$$\left|Q_{\frac{n}{2}}(\zeta,z)\right| \preceq \frac{1}{[|\zeta - z| + \delta_n^*]} \tag{15}$$

For $z \in \Gamma$ we give the approximate rational function by the formula

$$R_n(z) = -\frac{1}{\pi}\int\limits_G \frac{\partial F(\zeta)}{\partial\bar{\zeta}} Q^2_{\frac{n}{2}}(\zeta,z)\, dm(\zeta),$$

where $dm(\zeta)$ means integration with respect to the two-dimensional Lebesgue measure (area).

Let $z \in \Gamma$ and assume $U_n := U(z,\delta_n^*), \gamma_n := \partial U_n$. By lemma (iii) of the Lemma 3 $F \in ACL$ in $\mathbf{C}$ (absolutely continuous on all horizontal and verticals in $\mathbf{C}$). Then we apply the Green formula (see [10]) and have

$$\begin{aligned} f(z) - R_n(z) &= \tfrac{1}{\pi}\int\limits_{G\setminus U_n} \tfrac{\partial F(\zeta)}{\partial\bar{\zeta}}\left(Q^2_{\frac{n}{2}}(\zeta,z) - \tfrac{1}{(\zeta-z)^2}\right) dm(\zeta) \\ &+ \tfrac{1}{\pi}\int\limits_{U_n} \tfrac{\partial F(\zeta)}{\partial\bar{\zeta}} Q^2_{\frac{n}{2}}(\zeta,z)\, dm(\zeta) + \\ &+ f(z) - \tfrac{1}{2\pi i}\int\limits_{\gamma_n} \tfrac{F(\zeta)}{(\zeta-z)^2} d\zeta = U_1(z) + U_2(z) + U_3(z). \end{aligned} \tag{16}$$

Now, we estimate each $U_i(z), i = 1,2$. In relation (16) passing to polar coordinates and using (1), (2), (11), (14) and (15) the first two integrals are estimated in the following way:

$$|U_1(z)| = \left|\frac{1}{\pi}\int\limits_{G\setminus U_n} \frac{\partial F(\zeta)}{\partial\bar{\zeta}}\left(Q^2_{\frac{n}{2}}(\zeta,z) - \frac{1}{(\zeta - z)^2}\right) dm(\zeta)\right| \le$$

$$\le \frac{1}{\pi}\int\limits_{G\setminus U_n} \left|\frac{\partial F(\zeta)}{\partial\bar{\zeta}}\right| \left|Q^2_{\frac{n}{2}}(\zeta,z) - \frac{1}{(\zeta - z)^2}\right| dm(\zeta) \preceq \tag{17}$$

$$\preceq \int\limits_{\delta_n^*}^{c} \omega\left(r\right) \frac{\left(\delta_n^*\right)^3}{r^4} dr \preceq \omega\left(\delta_n^*\right) \delta_n^* \int\limits_{\delta_n^*}^{c} \frac{dr}{r^2} \preceq \omega\left(\delta_n^*\right),$$

$$\left|U_2\left(z\right)\right| = \left| \frac{1}{\pi} \int\limits_{U_n} \frac{\partial F\left(\zeta\right)}{\partial \bar{\zeta}} Q^2_{\frac{n}{2}}\left(\zeta, z\right) dm\left(\zeta\right) \right| \leq$$

$$\leq \frac{1}{\pi} \int\limits_{U_n} \left| \frac{\partial F\left(\zeta\right)}{\partial \bar{\zeta}} \right| \left| Q^2_{\frac{n}{2}}\left(\zeta, z\right) \right| dm\left(\zeta\right) \preceq \int\limits_{o}^{\delta_n^*} \frac{\omega\left(r\right)}{r} dr \preceq \omega\left(\delta_n^*\right). \tag{18}$$

Now let's estimate $U_3\left(z\right)$. We have

$$\left|U_3\left(z\right)\right| = \left| f\left(z\right) - \frac{1}{2\pi i} \int\limits_{\gamma_n} \frac{F\left(\zeta\right)}{\left(\zeta - z\right)^2} d\zeta \right| \leq \left| f\left(z\right) - \left(\nu_{\delta_n^*}\right)'_{\zeta}\left(z, z\right) \right| +$$

$$+ \frac{1}{2\pi} \left| \int\limits_{\gamma_n} \frac{F\left(\zeta\right) - \nu_{\delta_n^*}\left(\zeta, z\right)}{\left(\zeta - z\right)^2} d\zeta \right|. \tag{19}$$

The estimate

$$\left| f\left(z\right) - \left(\nu_{\delta_n^*}\right)'_{\zeta}\left(z, z\right) \right| = \left| f\left(z\right) - R_{f,k,z,\Gamma,c\delta_n^*}\left(z\right) \right| \leq \omega\left(c\delta_n^*\right) \preceq \omega\left(\delta_n^*\right) \tag{20}$$

is true. By inequalities (19), (20), and (12) we have

$$\left|U_3\left(z\right)\right| \preceq \omega\left(\delta_n^*\right) \tag{21}$$

Comparing estimates (16), (17), (18) and (21) we have

$$\left|f\left(z\right) - R_n\left(z\right)\right| \preceq \omega\left(\delta_n^*\right), z \in \Gamma. \tag{22}$$

Now, let's construct rational function for which conditions (3) and (4) are fulfilled. Let $n > 2p$. Let's construct the following functions

$$V_{\frac{n}{2+1}}\left(\zeta, z\right) = 1 - \left(\zeta - z\right) Q_{\frac{n}{2}}\left(\zeta, z\right), \zeta, z \in \Gamma,$$

$$u_n\left(z\right) = \sum_{j=1}^{p} \frac{q\left(z\right)}{q'\left(z_j\right)\left(z - z_j\right)} \left(f\left(z_j\right) - t_n\left(z_j\right)\right) V_{\frac{n}{2+1}}\left(z_j, z\right).$$

By (14) and (22) we have

$$\left|u_n\left(z\right)\right| \preceq \sum_{j}{}' \omega\left(\delta_n^*\right) \left(\frac{\delta_n^*}{\left|z - z_j\right| + \delta_n^*} \right)^2, z \in \Gamma,$$

where $\sum\limits_{j}^{\prime}$ means the sum in all j with $z_j \in \Gamma$.

Let's construct the required rational function in the following form

$$r_n(z) = t_n(z) + u_n(z). \tag{23}$$

Obviously the rational function of the form (23) satisfies conditions (3) and (4).

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