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INTEGRAL PROPERTIES OF SOME FAMILIES OF MULTIVALENT FUNCTIONS WITH COMPLEX ORDER

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Abstract. In the present paper, we study integral properties of two families of p-valently analytic functions of complex order defined of the derivative operator of order m. The obtained results improve known results.

1. **Introduction**

Let $\mathcal{A}_p(n)$ denote the class of functions of the form

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \quad (a_k \geq 0; n, p \in \mathbb{N} := \{1, 2, 3, \cdots\}), \tag{1}$$

which are analytic and p-valent in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Upon differentiating both sides (1) m-times with respect to z, we have

$$f^{(m)}(z) = \frac{p!}{(p-m)!} z^{p-m} - \sum_{k=n+p}^{\infty} \frac{k!}{(k-m)!} a_k z^{k-m} \tag{2}$$

where $n, p \in \mathbb{N}; m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; p > m$.

Making use of the function $f^{(m)}(z)$ given by (2), Srivastava and Orhan [1]introduced the subclasses $\mathcal{R}^p_{n,m}(\lambda, b)$ and $\mathcal{L}^p_{n,m}(\lambda, b)$ of the p-valently analytic function class $\mathcal{A}_p(n)$, which consist of functions $f(z)$ satisfying the following inequality, respectively:

$$\left| \frac{1}{b} \left(\frac{z f^{(1+m)}(z) + \lambda z^2 f^{(2+m)}(z)}{\lambda z f^{(1+m)}(z) + (1-\lambda) f^{(m)}(z)} - (p-m) \right) \right| < 1 \tag{3}$$

and

$$\left| \frac{1}{b} \left(f^{(1+m)}(z) + \lambda z f^{(2+m)}(z) - (p-m) \right) \right| < p - m, \tag{4}$$

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where $z \in \mathbb{U}; p \in \mathbb{N}; m \in \mathbb{N}_0; p > m; 0 \leq \lambda \leq 1; b \in \mathbb{C} \setminus \{0\}$.

Also, in [1], Srivastava and Orhan proved the following characterization theorems of these subclasses.

Theorem A. *Let $f(z) \in \mathcal{A}_p(n)$ be given by (1). Then $f(z) \in \mathcal{R}^p_{n,m}(\lambda, b)$ if and only if*

$$\sum_{k=n+p}^{\infty} \frac{(k+|b|-p)k![\lambda(k-m-1)+1]}{(k-m)!} a_k \leq \frac{|b|p![\lambda(p-m-1)+1]}{(p-m)!}. \tag{5}$$

Theorem B. *Let $f(z) \in \mathcal{A}_p(n)$ be given by (1). Then $f(z) \in \mathcal{L}^p_{n,m}(\lambda, b)$ if and only if*

$$\sum_{k=n+p}^{\infty} \binom{k}{m}(k-m)[\lambda(k-m-1)+1]a_k \leq (p-m)\left[\frac{|b|-1}{m!} + \binom{p}{m}[\lambda(p-m-1)+1]\right]. \tag{6}$$

Also, let $\mathcal{I}_c : \mathcal{A}_p(n) \to \mathcal{A}_p(n)$ be an *integral operator* defined by $g = \mathcal{I}_c(f)$, where $c \in (-p, \infty), f \in \mathcal{A}_p(n)$ and

$$g(z) = \frac{c+p}{z^c} \int_0^z t^{c-1} f(t) dt. \tag{7}$$

We note that if $f \in \mathcal{A}_p(n)$ is a function of the form (1), then

$$g(z) = \mathcal{I}_c(f)(z) = z^p - \sum_{k=n+p}^{\infty} \frac{c+p}{c+k} a_k z^k. \tag{8}$$

The main object of the present work is to investigate the integral properties of p-valently functions belonging to the subclasses $\mathcal{R}^p_{n,m}(\lambda, b)$ and $\mathcal{L}^p_{n,m}(\lambda, b)$.

Our properties of the function classes $\mathcal{R}^p_{n,m}(\lambda, b)$ and $\mathcal{L}^p_{n,m}(\lambda, b)$ are motivated essentially by several earlier investigations including in [2].

2. Integral properties of the class $\mathcal{R}^p_{n,m}(\lambda, b)$

Theorem 1. *Let $p \in \mathbb{N}; m \in \mathbb{N}_0; p > m; b \in \mathbb{C} \setminus \{0\}$ and $c \in (-p, \infty)$. If $f \in \mathcal{R}^p_{n,m}(\lambda, b)$ and $g = \mathcal{I}_c(f)$, then $g \in \mathcal{R}^p_{n,m}(\lambda, \gamma)$ where*

$$|\gamma| = \frac{(c+p)|b|}{(c+p+n+|b|)} \tag{9}$$

and $|\gamma| < |b|$. The result is sharp.

Proof. From Theorem A and from (8), we have $g \in \mathcal{R}^p_{n,m}(\lambda,\gamma)$ if and only if

$$\sum_{k=n+p}^{\infty} \frac{(k+|\gamma|-p)k!(p-m)![\lambda(k-m-1)+1](c+p)}{(k-m)!|\gamma|p![\lambda(p-m-1)+1](c+k)} a_k \leq 1. \tag{10}$$

We note that for $k \geq n+p$ the inequalities

$$\frac{(k+|\gamma|-p)k!(p-m)![\lambda(k-m-1)+1](c+p)}{(k-m)!|\gamma|p![\lambda(p-m-1)+1](c+k)}$$

$$\leq \frac{(k+|b|-p)k!(p-m)![\lambda(k-m-1)+1]}{(k-m)!|b|p![\lambda(p-m-1)+1]} \tag{11}$$

imply (10), because $f \in \mathcal{R}^p_{n,m}(\lambda,b)$ and it is satisfies (5). This inequality is equivalent to

$$\frac{(k+|\gamma|-p)(c+p)}{|\gamma|(c+k)} \leq \frac{(k+|b|-p)}{|b|}$$

and we obtain

$$|\gamma| \geq \frac{(k-p)(c+p)|b|}{(k+|b|-p)(c+k)-|b|(c+p)} \ ; \ k \geq n+p \ ; \ \gamma = \gamma(p,k,c,b). \tag{12}$$

And now, we show that $|\gamma|$ is a decreasing function of k, $k \geq n+p$. Indeed, let

$$h(x) = \frac{(x-p)(c+p)|b|}{(x+|b|-p)(c+x)-|b|(c+p)} \quad ; x \in [n+p,\infty) \subset [n,\infty). \tag{13}$$

We have

$$h'(x) = \frac{-(x-p)^2(c+p)|b|}{((x+|b|-p)(c+x)-|b|(c+p))^2} < 0. \tag{14}$$

This implies

$$|\gamma(p,k,c,b)| \leq |\gamma| = |\gamma(p,n+p,c,b)| \ \ ; k \geq n+p. \tag{15}$$

The result is sharp, because

$$\mathcal{I}_c(f_b) = f_\gamma \tag{16}$$

where

$$f_b(z) = z^p - \frac{|b|p!(n+p-m)![\lambda(p-m-1)+1]}{(p-m)!(n+p)![n+|b|][\lambda(n+p-m-1)+1]} z^{n+p} \tag{17}$$

and

$$f_\gamma(z) = z^p - \frac{|\gamma|p!(n+p-m)![\lambda(p-m-1)+1]}{(p-m)!(n+p)![n+|\gamma|][\lambda(n+p-m-1)+1]} z^{n+p} \tag{18}$$

are extremal functions of $\mathcal{R}^p_{n,m}(\lambda, b)$ and $\mathcal{R}^p_{n,m}(\lambda, \gamma)$, respectively. Indeed we have

$$\mathcal{I}_c(f_b)(z) = z^p - \frac{|b|p!(n+p-m)![\lambda(p-m-1)+1](c+p)}{(p-m)!(n+p)![n+|b|][\lambda(n+p-m-1)+1](c+p+n)} z^{n+p} \tag{19}$$

Thus we deduce

$$\frac{|\gamma|}{n+|\gamma|} = \frac{|b|(c+p)}{[n+|b|](c+p+n)} \tag{20}$$

and this implies (16). From $\frac{|\gamma|}{n+|\gamma|} = \frac{|b|(c+p)}{[n+|b|](c+p+n)}$, we obtain $|\gamma| > 0$. Also, we have $|\gamma| < |b|$. Indeed,

$$|\gamma| - |b| = -\frac{[n+|b|]|b|}{|b|+(c+p+n)} < 0. \tag{21}$$

Remark 1. In Theorem 1, if we take $p = 1, m = 0, b = 1-\alpha$ and $\gamma = 1-\beta$, we obtain

$$\beta = 1 - \frac{(c+1)(1-\alpha)}{2-\alpha+c+n} \tag{22}$$

which was proved by Salagean [2].

3. Integral properties of the class $\mathcal{L}^p_{n,m}(\lambda, b)$

Theorem 2. *Let $p \in \mathbb{N}; m \in \mathbb{N}_0; p > m; b \in \mathbb{C} \setminus \{0\}$ and $c \in (-p, \infty)$. If $f \in \mathcal{L}^p_{n,m}(\lambda, b)$ and $g = \mathcal{I}_c(f)$, then $g \in \mathcal{L}^p_{n,m}(\lambda, \beta)$ where*

$$|\beta| = \frac{(c+p)|b| + n(1 - \frac{p!}{(p-m)!}[\lambda(p-m-1)+1])}{c+p+n} \tag{23}$$

and $|\beta| < |b|$. The result is sharp.

Proof. Using similar arguments as given by Theorem 1, we can get the result.

Remark 2. In (7), for $p = n = 1$, we obtain the integral operator of Bernardi [3],

$$I_c : \mathcal{A}_1(1) \to \mathcal{A}_1(1)$$

defined by $h = \mathcal{I}_c(f)$, where $c > -1$, $f \in \mathcal{A}_1(1)$,

$$h(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt.$$

Also, for $p = c = n = 1$, we obtain the integral operator of Libera [4],

$$I_1 : \mathcal{A}_1(1) \to \mathcal{A}_1(1)$$

defined by $h_1 = \mathcal{I}_1(f)$, where $f \in \mathcal{A}_1(1)$,

$$h_1(z) = \frac{2}{z} \int_0^z f(t)dt.$$

Corollary 1.*Let $b \in \mathbb{C} - \{0\}$, $c \in (-1, \infty)$. If $f \in \mathcal{L}^1_{1,0}(\lambda, b)$ and $h = I_c(f)$ is the Bernardi operator, then $h \in \mathcal{L}^1_{1,0}(\lambda, \beta)$, where*

$$|\beta| = \frac{c+1}{c+2} |b| .$$

Proof. In Theorem 2, we consider $p = n = 1$ and $m = 0$.

Corollary 2.*Let $b \in \mathbb{C} - \{0\}$. If $f \in \mathcal{L}^1_{1,0}(\lambda, b)$ and $h_1 = I_1(f)$ is the Libera operator, then $h \in \mathcal{L}^1_{1,0}(\lambda, \beta)$, where*

$$|\beta| = \frac{2}{3} |b| .$$

Proof. In Theorem 2, we consider $p = n = 1$, $m = 0$ and $c = 1$.

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