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A FRICTIONLESS ELASTIC-VISCOPLASTIC CONTACT PROBLEM WITH NORMAL COMPLIANCE, ADHESION AND DAMAGE

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Abstract. We study a quasistatic frictionless contact problem with normal compliance, adhesion and damage for elastic-viscoplastic material.The adhesion of the contact surfaces is modeled with a surface variable, the bonding field, whose evolution is described by a first order differential equation. The mechanical damage of the material, caused by excessive stess or strains, is described by a damage function whose evolution is modeled by an inclusion of parabolic type. We provide a variational formulation of the problem and prove the existence and uniqueness of a weak solution. The proofs are based on time-dependent variational equalities, classical results on elliptic and parabolic variational inequalities, differential equations and fixed point arguments.

1. **Introduction**

We consider a mathematical model for a quasistatic process of frictionless contact between an elastic-viscoplastic body and an obstacle, within the framework of small deformation theory. The contact is modeled with normal compliance. The effect of damage due to the mechanical stress or strain is included in the model. Such situation is common in many engeneering applications where the forces acting on the system very periodically leading to the appearence and growth of microcracks which may deteriorate the mechanism of the system. Because of the safety issue

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of mechanical equipments, considerable efforts were been devoted to modeling and numerically simulating damage.

Early models for mechanical damage derived from the termodyamical considerations appeared in [9, 10], where numerical simulations were included. Mathematical analysis of one-dimensional problems can be found in [11]. In all these papers the damage of the material is described with a damage function α, restricted to have values between zero and one. When $\alpha = 1$ there is no damage in the material, when $\alpha = 0$, the material is completely damaged, when $0 < \alpha < 1$ there is partial damage and the system has a reduced load carrying capacity. Quasistatic contact problems with damage have been investigated in [13, 14, 17]. In this paper, the inclusion used for the evolution of the damage field is

$$\dot{\alpha} - k \triangle \alpha + \partial\varphi_K (\alpha) \ni \Phi (\boldsymbol{\sigma}, \boldsymbol{\varepsilon} (\mathbf{u}) , \alpha) ,$$

where K denotes the set of admissible damage functions defined by

$$K = \left\{\xi \in H^1 (\Omega) \,/\, 0 \leq \xi \leq 1 \text{ a.e. in } \Omega\right\} ,$$

k is a positive coefficient, $\partial\varphi_K$ represents the subdifferential of the indicator function of the set K and Φ is a given constitutive function which describes the sources of the damage in the system. In the present paper we consider a rate type elastic-viscoplastic material with constitutive relation

$$\dot{\boldsymbol{\sigma}} = \mathcal{E}\boldsymbol{\varepsilon} \left(\dot{\mathbf{u}}\right) + \mathcal{G} (\boldsymbol{\sigma}, \boldsymbol{\varepsilon} (\mathbf{u}) , \alpha) ,$$

where $\mathcal{E}$ is a fourth order tensor, $\mathcal{G}$ is a nonlinear constitutive function and α is the damage field and the adhesion between the body and the obstacle is taken into account during the conact. The adhesive contact between bodies, when a glue is added to keep surfaces from relative motion, is receiving increased attention in the mathematical literature. Analysis of models for adhesive contact can be found in [2, 3, 4, 6, 12, 15, 20]. The novelty in all the above papers is the introduction of a surface internal variable, the *bonding field*, denoted in the paper by β; it describes the pointwise fractional density of active bonds on the contact surface, and sometimes referred to as the *intensity of adhesion.* Following [7, 8], the bonding field satisfies the restrictions $0 \leq \beta \leq 1$; when $\beta = 1$ at a point of the contact surface, the adhesion is complete and all the bonds are active, when $\beta = 0$ all the bonds are inactive, severed, and there is no adhesion; when $0 < \beta < 1$ the adhesion is partial and only a fraction β of the bonds is active. We refer the reader to the extensive bibliography on the subject in [16,18,19].

The paper is structured as follows. In section 2 we present the notation and some preliminaries. In section 3 we present the mechanical problem, we list the assumptions and in section 4 we give and prove our main existence and uniqueness result, Theorem 4.1. The proof is based on monotone operator theory, classical results on parabolic inequalities and Banach fixed point arguments.

2. **Notation and preliminaries**

In this short section, we present the notation we shall use and some preliminary material. For more details, we refer the reader to [5]. We denote by S_d the space of second order symmetric tensors on $\mathbb{R}^d$,($d = 2, 3$), while (.) and $|.|$ represent the inner product and the Euclidean norm on S_d and $\mathbb{R}^d$, respectively. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a regular boundary Γ and let ν denote the unit outer normal on Γ. we shall use the notation

$$H = L^2(\Omega)^d = \{\mathbf{u} = (u_i) \ / \ u_i \in L^2(\Omega)\},$$

$$\mathcal{H} = \{\boldsymbol{\sigma} = (\sigma_{ij}) \ / \ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\},$$

$$H_1 = \{\mathbf{u} = (u_i) \in H \ / \ \varepsilon(\mathbf{u}) \in \mathcal{H}\},$$

$$\mathcal{H}_1 = \{\boldsymbol{\sigma} \in \mathcal{H} \ / \ Div\ \boldsymbol{\sigma} \in H\},$$

where $\varepsilon : H_1 \to \mathcal{H}$ and $\mathrm{Div} : \mathcal{H}_1 \to H$ are the deformation and divergence operators, respectively, defined by

$$\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \qquad \boldsymbol{\varepsilon}_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}),\ Div\ \boldsymbol{\sigma} = (\sigma_{ij,j}).$$

Here and below, the indices i and j run between 1 to d, the summation convention over repeated indices is used and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable. The spaces $H, \mathcal{H}, H_1$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$(\mathbf{u},\mathbf{v})_H = \int_\Omega u_i v_i dx \qquad \forall \mathbf{u},\mathbf{v} \in H,$$

$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij}\, \tau_{ij} dx \qquad \forall \boldsymbol{\sigma},\boldsymbol{\tau} \in \mathcal{H},$$

$$(\mathbf{u},\mathbf{v})_{H_1} = (\mathbf{u},\mathbf{v})_H + (\boldsymbol{\varepsilon}(\mathbf{u}),\boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} \qquad \forall \mathbf{u},\mathbf{v} \in H_1,$$

$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} + (Div\ \boldsymbol{\sigma}, Div\ \boldsymbol{\tau})_H \qquad \forall \boldsymbol{\sigma},\boldsymbol{\tau} \in \mathcal{H}_1.$$

The associated norms on the spaces $H, \mathcal{H}, H_1$ and $\mathcal{H}_1$ are denoted by $|\ \ |_H$, $|\ \ |_{\mathcal{H}}$, $|\ \ |_{H_1}$ and $|\ \ |_{\mathcal{H}_1}$, respectively. Let $H_\Gamma = H^{1/2}(\Gamma)^d$ and let $\gamma : H_1 \to H_\Gamma$ be the trace map. For every element $\mathbf{v} \in H_1$ we also use the notation $\mathbf{v}$ to denote the trace $\gamma\mathbf{v}$ of $\mathbf{v}$ on Γ and we denote by $\mathfrak{v}_\nu$ and $\mathbf{v}_\tau$ the *normal* and *tangential* components of $\mathbf{v}$ on the boundary Γ given by

$$v_\nu = \mathbf{v}.\boldsymbol{\nu}, \qquad \mathbf{v}_\tau = \mathbf{v} - v_\nu \boldsymbol{\nu}. \tag{2.1}$$

Similarly, for a regular (say C^1) tensor field $\boldsymbol{\sigma} : \Omega \to S_d$, we define its *normal* and *tangential* components by

$$\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}).\boldsymbol{\nu}, \qquad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}, \tag{2.2}$$

and we recall that the following Green's formula holds

$$(\boldsymbol{\sigma},\boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + (\mathrm{Div}\ \boldsymbol{\sigma},\mathbf{v})_H = \int_\Gamma \boldsymbol{\sigma}\boldsymbol{\nu}.\mathbf{v}\ da \qquad \forall \mathbf{v} \in H_1. \tag{2.3}$$

Finally, for any real Hilbert space X, we use the classical notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, where $1 \le p \le +\infty$, and $k \ge 1$. We denote by $C(0,T;X)$ and $C^1(0,T;X)$ the space of continuous and continuously differentiable functions from $[0,T]$ to X, respectively, with the norms

$$|f|_{C(0,T;X)} = \max_{t\in[0,T]} |f(t)|_X,$$

$$|f|_{C^1(0,T;X)} = \max_{t\in[0,T]} |f(t)|_X + \max_{t\in[0,T]} \left|\dot{f}(t)\right|_X,$$

respectively. Moreover, we use the dot above to indicate the derivative with respect to the time variable and, for a real number r, we use r_+ to present its positive part, that is $r_+ = \max\{0, r\}$. Finally, for the convenience of the reader, we recall the following version of the classical theorem of Cauchy-Lipschitz (see, e.g., [21, p. 60]).

Theorem 1. *Assume that* $(X, |.|_X)$ *is a real Banach space and* $T > 0$. Let $F(t,.) : X \to X$ *be an operator defined a.e. on* $(0,T)$ *satisfying the following conditions:* 1- $\exists\, L_F > 0$ *such that* $|F(t,x) - F(t,y)|_X \le L_F\, |x-y|_X \quad \forall x, y \in X,$ *a.e.* $t \in (0,T)$. 2- $\exists\, p \ge 1$ *such that* $t \longmapsto F(t,x) \in L^p(0,T;X) \quad \forall x \in X$. *Then for any* $x_0 \in X$, *there exists a unique function* $x \in W^{1,p}(0,T;X)$ *such that*

$$\dot{x}(t) = F(t, x(t)) \qquad a.e.\ t \in (0,T),$$

$$x(0) = x_0.$$

Theorem 2.1 will be used in section 4 to prove the unique solvability of the intermediate problem involving the bonding field.

Moreover, if X_1 and X_2 are real Hilbert spaces, then $X_1 \times X_2$ denotes the product Hilbert space endowed with the canonical inner product $(.,.)_{X_1 \times X_2}$.

3. **Problem statement**

A viscoplastic body occupies the domain $\Omega \subset \mathbb{R}^d$ with the boundary Γ divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that $meas\,(\Gamma_1) > 0$. The time interval of interest is $[0,T]$ where $T > 0$. The body is clamped on Γ_1 and so the displacement field vanishes there. A volume force of density $\mathbf{f}_0$ acts in $\Omega \times (0,T)$ and surface tractions of density $\mathbf{f}_2$ act on $\Gamma_2 \times (0,T)$. We assume that the body is in adhesive frictionless contact with an obstacle, the so called foundation, over the potential contact surface Γ_3. Moreover, the process is quasistatic, i.e. the inertial terms are neglected in the equation of motion. We use an elasto-viscoplastic constitutive law with damage to model the material's behavior and an ordinary differential equation to describe the evolution of the bonding field. The mechanical formulation of the frictionless problem with normal compliance is as follows.

Problem P. *Find a displacement field* $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$*, a stress field* $\boldsymbol{\sigma} : \Omega \times [0,T] \to S_d$*, a damage field* $\alpha : \Omega \times [0,T] \to \mathbb{R}$ *and a bonding field* $\beta : \Gamma_3 \times [0,T] \to [0,1]$ *such that*

$$\dot{\boldsymbol{\sigma}} = \mathcal{E}\boldsymbol{\varepsilon}\left(\dot{\mathbf{u}}\right) + \mathcal{G}\left(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}\left(\mathbf{u}\right), \alpha\right), \tag{3.1}$$

$$\dot{\alpha} - k \triangle \alpha + \partial\varphi_K\left(\alpha\right) \ni \Phi\left(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}\left(\mathbf{u}\right), \alpha\right), \tag{3.2}$$

$$Div\ \boldsymbol{\sigma} + \mathbf{f}_0 = 0 \text{ in } \Omega \times (0,T), \tag{3.3}$$

$$\mathbf{u} = 0 \text{ on } \Gamma_1 \times (0,T), \tag{3.4}$$

$$\boldsymbol{\sigma\nu} = \mathbf{f}_2 \text{ on } \Gamma_2 \times (0,T), \tag{3.5}$$

$$-\sigma_\nu = p_\nu\left(u_\nu\right) - \gamma_\nu\ \beta^2\left(-R\left(u_\nu\right)\right)_+ \text{ on } \Gamma_3 \times (0,T), \tag{3.6}$$

$$\boldsymbol{\sigma}_\tau = 0 \text{ on } \Gamma_3 \times (0,T), \tag{3.7}$$

$$\frac{\partial\alpha}{\partial\nu} = 0 \text{ on } \Gamma \times (0,T), \tag{3.8}$$

$$\dot{\beta} = -\left[\gamma_\nu\beta\left[\left(-R\left(u_\nu\right)\right)_+\right]^2 - \epsilon_a\right]_+ \text{ on } \Gamma_3 \times (0,T), \tag{3.9}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0,\ \alpha(0) = \alpha_0 \text{ in } \Omega, \tag{3.10}$$

$$\beta(0) = \beta_0 \text{ on } \Gamma_3. \tag{3.11}$$

The relation (3.1) represents the viscoplastic constitutive law with damage, the evolution of the damage field is governed by the inclusion given by the relation (3.2), k is a constant, $\partial\varphi_K$ denotes the subdifferential of the indicator function φ_K of K which represents the set of admissible damage functions satisfying $0 \leq \alpha \leq 1$ and Φ is a given constitutive function which describes damage sources in the system. (3.3) represents the equilibrium equation, (3.4) and (3.5) are the displacement and traction boundary conditions, respectively. (3.6) represents the normal compliance contact condition with adhesion in which γ_ν and ϵ_a are given adhesion coefficients and R is the truncation operator defined by

$$R\left(s\right) = \begin{cases} -L & \text{if } s \leq -L, \\ s & |s| < L, \\ L & \text{if } s \geq L. \end{cases} \tag{3.12}$$

Here $L > 0$ is the caracteristic length of the bond, beyonding which it does not offer any additional traction. The introduction of R is motivated by the mathematical arguments but it is not restrictive for physical point of view, since no restriction on the size of the parameter L is made in what follows. Also, p_ν is a given positive function which will be decribed below. In this condition the interpenetrability between the body and the foundation is allowed, that is u_ν may be positive on Γ_3. The contribution of the adhesive to normal traction is represented by the term $\gamma_\nu \beta \left(-R\left(u_\nu\right)\right)_+$, the adhesive traction is tensile, and is proportional, with proportionality coefficient γ_ν, to the square of the intensity of adhesion, and to the normal displacement, but as in various papers see e.g. [2, 3] and the references threin. Condition (3.7) represents the frictionless contact condition and shows that the tangential stress vanishes on the contact surface during the process. (3.8) represents a homogeneous Newmann boundary condition where $\frac{\partial \alpha}{\partial \nu}$ represents the normal derivative of α. Next, equation (3.9) represents the ordinary differential equation which describes the evolution of the bonding field and it was already used in [2], see also [19] for more details. Here, γ_ν and ϵ_a are given adhesion coefficients which may depend on $\mathbf{x} \in \Gamma_3$ and R is the truncation operator given by (3.12). Notice that in this model once debonding occurs bonding connot be reestablished since, as it follows from (3.9), $\dot{\beta} \leq 0$. In (3.10), we consider the initial conditions where $\mathbf{u}_0$ is the initial displacement, $\boldsymbol{\sigma}_0$ is the initial stress and α_0 is the initial damage. Finally, (3.11) is the initial condition, in which β_0 denotes the initial bonding field. Let Z denote the bonding fields set

$$Z = \left\{\beta \in L^2\left(\Gamma_3\right) \ / \ 0 \leq \beta \leq 1 \text{ a.e. on } \Gamma_3\right\},$$

and for displacement field we need the closed subspace of H_1 defined by

$$V = \{\mathbf{v} \in H_1 | \mathbf{v} = 0 \text{ on } \Gamma_1\}.$$

Since meas $(\Gamma_1) > 0$, Korn's inequality holds and there exists a constant $C_K > 0$, that depends only on Ω and Γ_1 such that

$$\left|\boldsymbol{\varepsilon}\left(\mathbf{v}\right)\right|_{\mathcal{H}} \geq C_K \left|\mathbf{v}\right|_{H_1} \quad \forall \mathbf{v} \in V.$$

On V we consider the inner product and the associated norm given by

$$(\mathbf{u},\mathbf{v}) = (\boldsymbol{\varepsilon}(\mathbf{u}),\boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}}, \qquad |\mathbf{v}|_V = |\boldsymbol{\varepsilon}(\mathbf{v})|_{\mathcal{H}} \qquad \forall \mathbf{u},\mathbf{v}\in V.$$

It follows from Korn's inequality that $|.|_{H_1}$ and $|.|_V$ are equivalent norms on V and therefore $(V,|.|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a constant C_0, depending only on Ω, Γ_1 and Γ_3 such that

$$|\mathbf{v}|_{L^2(\Gamma_3)^d} \le C_0\,|\mathbf{v}|_V \qquad \forall \mathbf{v}\in V. \tag{3.13}$$

In the study of the mechanical problem (3.1)-(3.11), we make the following assumptions. The operator $\mathcal{E}:\Omega\times S_d\to S_d$ satisfies

$$\begin{cases} \text{(a) } \mathcal{E}=(e_{ijkh})\ /\ e_{ijkh}\in L^\infty(\Omega), \\ \text{(b) } \mathcal{E}\,\boldsymbol{\sigma}\,.\,\boldsymbol{\tau}=\boldsymbol{\sigma}\,.\,\mathcal{A}\,.\,\boldsymbol{\tau} \quad \forall\boldsymbol{\sigma},\,\boldsymbol{\tau}\in S_d, \text{ a.e. in } \Omega, \\ \text{(c) } \mathcal{E}\,\boldsymbol{\sigma}\,.\,\boldsymbol{\sigma}\ge m_{\mathcal{E}}\ |\boldsymbol{\sigma}|^2\ \ \forall\boldsymbol{\sigma}\in S_d, \text{ for some } m_{\mathcal{E}}>0. \end{cases} \tag{3.14}$$

The operator $\mathcal{G}:\Omega\times S_d\times S_d\times\mathbb{R}\to S_d$ satisfies

$$\begin{cases} \text{(a) There exists a constant } L_{\mathcal{G}}>0 \text{ such that} \\ |\mathcal{G}(\mathbf{x},\boldsymbol{\sigma}_1,\ \boldsymbol{\varepsilon}_1,\alpha_1)-\mathcal{G}(\mathbf{x},\boldsymbol{\sigma}_2,\boldsymbol{\varepsilon}_2,\alpha_2)|\le L_{\mathcal{G}}\ \ (|\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2|+|\boldsymbol{\varepsilon}_1-\boldsymbol{\varepsilon}_2|+|\alpha_1-\alpha_2|) \\ \forall\boldsymbol{\sigma}_1,\ \boldsymbol{\sigma}_2,\ \boldsymbol{\varepsilon}_1,\ \boldsymbol{\varepsilon}_2\in S_d,\ \alpha_1,\ \alpha_2\in\mathbb{R}, \text{ a.e. } \mathbf{x}\in\Omega; \\ \text{(b) } \mathbf{x}\longmapsto\mathcal{G}(\mathbf{x},\boldsymbol{\sigma},\boldsymbol{\varepsilon},\alpha) \text{ is a Lebesgue measurable function on } \Omega \\ \forall\boldsymbol{\sigma},\boldsymbol{\varepsilon}\in S_d,\ \forall\alpha\in\mathbb{R}; \\ \text{(c) } \mathbf{x}\longmapsto\mathcal{G}(\mathbf{x},\mathbf{0},\mathbf{0},0)\in\mathcal{H}. \end{cases} \tag{3.15}$$

The damage function $\Phi:\Omega\times S_d\times S_d\times\mathbb{R}\to\mathbb{R}$ satisfies

$$\begin{cases} \text{(a) There exists a constant } L > 0 \text{ such that} \\ |\Phi(\mathbf{x}, \boldsymbol{\sigma}_1,\ \boldsymbol{\varepsilon}_1, \alpha_1) - \Phi(\mathbf{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2, \alpha_2)| \leq L\left(|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2| + |\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2| + |\alpha_1 - \alpha_2|\right) \\ \forall \boldsymbol{\sigma}_1,\ \boldsymbol{\sigma}_2,\ \boldsymbol{\varepsilon}_1,\ \boldsymbol{\varepsilon}_2 \in S_d,\ \alpha_1,\ \alpha_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Omega; \\ \text{(b) } \mathbf{x} \longmapsto \Phi(\mathbf{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \alpha) \text{ is a Lebesgue measurable function on } \Omega \\ \forall \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in S_d,\ \forall \alpha \in \mathbb{R}; \\ \text{(c) } \mathbf{x} \longmapsto \Phi(\mathbf{x}, \mathbf{0}, \mathbf{0}, 0) \in \mathcal{H}. \end{cases} \tag{3.16}$$

The normal compliance function $p_\nu : \Gamma_3 \times \mathbb{R}^d \to \mathbb{R}_+$ satisfies

$$\begin{cases} \text{(a) There exists } L_\nu > 0 \text{ such that} \\ |p_\nu(\mathbf{x}, \mathbf{r}_1) - p_\nu(\mathbf{x}, \mathbf{r}_2)| \leq L_\nu |\mathbf{r}_1 - \mathbf{r}_2| \quad \forall \mathbf{r}_1, \mathbf{r}_2 \in \mathbb{R}^d, \text{ a.e. } \mathbf{x} \in \Gamma_3. \\ \text{(b) } (p_\nu(\mathbf{x}, \mathbf{r}_1) - p_\nu(\mathbf{x}, \mathbf{r}_2)).(\mathbf{r}_1 - \mathbf{r}_2) \geq 0 \quad \forall \mathbf{r}_1, \mathbf{r}_2 \in \mathbb{R}^d, \text{ a.e. } \mathbf{x} \in \Gamma_3. \\ \text{(c) } \mathbf{r} \mapsto p_\nu(., \mathbf{r}) \text{ is Lebesgue measurable on } \Gamma_3,\ \forall \mathbf{r} \in \mathbb{R}^d. \\ \text{(d) The mapping } p_\nu(., \mathbf{r}) = 0 \quad \text{for all } \mathbf{r} \leq 0. \end{cases} \tag{3.17}$$

The adhesion coefficients satisfy

$$\gamma_\nu \in L^\infty(\Gamma_3),\ \gamma_\nu \geq 0,\ \epsilon_a \in L^\infty(\Gamma_3),\ \epsilon_a \geq 0. \tag{3.18}$$

We also suppose that the body forces and surface traction have the regularity

$$\mathbf{f}_0 \in C(0, T; H), \qquad \mathbf{f}_2 \in C(0, T; L^2(\Gamma_2)^d). \tag{3.19}$$

Finally we assume that the initial data satisfy the following conditions

$$\mathbf{u}_0 \in V,\ \boldsymbol{\sigma}_0 \in \mathcal{H}_1, \tag{3.20}$$

$$\alpha_0 \in K, \tag{3.21}$$

$$\beta_0 \in Z. \tag{3.22}$$

We define the bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ by

$$a(\xi, \varphi) = k \int_\Omega \nabla\xi \,.\, \nabla\varphi \, dx. \tag{3.23}$$

Next, we denote $\mathbf{f} : [0, T] \to V$ the function defined by

$$(\mathbf{f}(t), \mathbf{v})_V = \int_{\Omega} \mathbf{f}_0(t) .\mathbf{v}\ dx + \int_{\Gamma_2} \mathbf{f}_2(t) .\mathbf{v}\ da \qquad \forall \mathbf{v} \in V, \text{ a.e. } t \in (0, T). \tag{3.24}$$

The adhesion functional $j_{ad} : L^{\infty}(\Gamma_3) \times V \times V \to \mathbb{R}$ defined by

$$j_{ad}(\beta, \mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} -\gamma_\nu\ \beta^2 (-R(u_\nu))_+ v_\nu\ da. \tag{3.25}$$

In addition to the functional (3.25), we need the normal compliance functional j_{nc} : $V \times V \to \mathbb{R}$ given by

$$j_{nc}(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} p_\nu(u_\nu) v_\nu\ da. \tag{3.26}$$

Keeping in mind (3.17)-(3.18), we observe that the integrals in (3.25) and (3.26) are well defined and we note that conditions (3.19) imply

$$\mathbf{f} \in C(0, T; V). \tag{3.27}$$

Finally we assume the following condition of compatibility

$$(\boldsymbol{\sigma}_0, \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + j_{ad}(\beta_0, \mathbf{u}_0, \mathbf{v}) + j_{nc}(\mathbf{u}_0, \mathbf{v}) = (\mathbf{f}(0), \mathbf{v})_V \qquad \forall \mathbf{v} \in V. \tag{3.28}$$

Using standard arguments based on green's formula (2.3) we can derive the following variational formulation of the frictionless problem with normal compliance (3.1)-(3.11) as follows.

Problem PV. Find a displacement field $\mathbf{u} : [0, T] \to V$, *a stress field* $\boldsymbol{\sigma}$: $[0, T] \to \mathcal{H}$ *a damage field* $\alpha : [0, T] \to H^1(\Omega)$ *and a bonding field* $\beta : [0, T] \to L^2(\Gamma_3)$ *such that*

$$\dot{\boldsymbol{\sigma}}(t) = \mathcal{E}\boldsymbol{\varepsilon}\left(\dot{\mathbf{u}}(t)\right) + \mathcal{G}\left(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{u}(t)), \alpha(t)\right), \qquad \text{a.e. } t \in (0, T), \tag{3.29}$$

$$\begin{aligned} &\alpha(t) \in K \text{ for all } t \in [0, T], \ (\dot{\alpha}(t), \xi - \alpha(t))_{L^2(\Omega)} + a(\alpha(t), \xi - \alpha(t)) \\ &\qquad \geq (\Phi(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{u}(t)), \alpha(t)), \xi - \alpha(t))_{L^2(\Omega)}\ \forall \xi \in K, \end{aligned} \tag{3.30}$$

$$\begin{aligned} &(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + j_{ad}(\beta(t), \mathbf{u}(t), \mathbf{v}) + j_{nc}(\mathbf{u}(t), \mathbf{v}) \\ &\qquad = (\mathbf{f}(t), \mathbf{v})_V \qquad \forall \mathbf{v} \in V,\ \forall t \in [0, T], \end{aligned} \tag{3.31}$$

$$\dot{\beta}(t) = -\left[\gamma_\nu\ \beta(t)\ \ \left[(-R(u_\nu(t)))_+\right]^2 - \epsilon_a\right]_+ \qquad \text{a.e. } t \in (0, T), \tag{3.32}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \boldsymbol{\sigma}(0) = \boldsymbol{\sigma}_0, \alpha(0) = \alpha_0, \beta(0) = \beta_0. \tag{3.33}$$

We notice that the variational problem PV is formulated in terms of displacement, stress field, damage field and bonding field. The existence of the unique solution of problem PV is stated and proved in the next section. To this end, we consider the following remark whose estimates will be used in different places of the paper.

Remark 1. *From* (3.32) *we obtain that* $\beta(\mathbf{x}, t) \leq \beta_0(\mathbf{x})$, *since* $\beta_0(\mathbf{x}) \in Z$ *then* $\beta(\mathbf{x}, t) \leq 1$ *for all* $t \geq 0$, *a.e. on* Γ_3. *If* $\beta(\mathbf{x}, t_0) = 0$ *for all* $t = t_0$ *it follows from* (3.32) *that* $\dot{\beta}(\mathbf{x}, t) = 0$ *for all* $t \geq t_0$, *therefore,* $\beta(\mathbf{x}, t) = 0$ *for all* $t \geq t_0$. *We conclude that* $0 \leq \beta(\mathbf{x}, t) \leq 1$ $\forall t \in [0, T]$, *a.e.* $\mathbf{x} \in \Gamma_3$.

In the sequel we consider that C is a generic positive constant which depends on $\Omega, \Gamma_1, \Gamma_3, \gamma_\nu, L$ and may change from place to place. First, we remark that j_{ad} and j_{nc} are linear with respect to the last argument and therefore

$$j_{ad}(\beta, \mathbf{u}, -\mathbf{v}) = -j_{ad}(\beta, \mathbf{u}, \mathbf{v}), \qquad j_{nc}(\mathbf{u}, -\mathbf{v}) = -j_{nc}(\mathbf{u}, \mathbf{v}). \tag{3.34}$$

Next, using (3.25) as well as the properties of the operator R , (3.12), we find

$$j_{ad}(\beta_1, \mathbf{u}_1, \mathbf{v}) - j_{ad}(\beta_2, \mathbf{u}_2, \mathbf{v}) = \int_{\Gamma_3} \gamma_\nu \; \beta_1^2[(-R(u_{2\nu}))_+ - (-R(u_{1\nu}))_+] \; v_\nu \; da$$

$$+ \int_{\Gamma_3} \gamma_\nu \; (\beta_2^2 - \beta_1^2)(-R(u_{2\nu}))_+ \; v_\nu \; da \leq C \int_{\Gamma_3} |\beta_1 - \beta_2| \, |\mathbf{v}| \; da,$$

and from (3.13) we obtain

$$j_{ad}(\beta_1, \mathbf{u}_1, \mathbf{v}) - j_{ad}(\beta_2, \mathbf{u}_2, \mathbf{v}) \leq c \; |\beta_1 - \beta_2|_{\; L^2(\Gamma_3)} \; |\mathbf{v}|_V \; . \tag{3.35}$$

Now, we use (3.26) to see that

$$|j_{nc}(\mathbf{u}_1, \mathbf{v}) - j_{nc}(\mathbf{u}_2, \mathbf{v})| \leq \int_{\Gamma_3} |p_\nu(u_{1\nu}) - p_\nu(u_{2\nu})| \; |v_\nu| \; da,$$

and therefore (3.17) (a) and (3.13) imply

$$|j_{nc}(\mathbf{u}_1, \mathbf{v}) - j_{nc}(\mathbf{u}_2, \mathbf{v})| \leq C \, | \, \mathbf{u}_1 - \mathbf{u}_2|_V \; |\mathbf{v}|_V \, . \tag{3.36}$$

We use again (3.26) to see that

$$j_{nc}(\mathbf{u}_1, \mathbf{u}_2 - \mathbf{u}_1) + j_{nc}(\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) = \int_{\Gamma_3} (p_\nu(\mathbf{u}_{1\nu}) - p_\nu(\mathbf{u}_{2\nu}))\ (u_{2\nu} - u_{1\nu})\ da,$$

and therefore (3.17) (b) implies

$$j_{nc}(\mathbf{u}_1, \mathbf{u}_2 - \mathbf{u}_1) + j_{nc}(\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2) \leq 0. \tag{3.37}$$

The inequalities (3.35)-(3.37) combined with equalities (3.34) will be used in various places in the rest of the paper.

4. **Well posedness of the problem**

The main result in this section is the following existence and uniqueness result.

Theorem 2. *Assume that* (3.14)-(3.22) *and* (3.28) *hold. Then, problem PV has a unique solution* $\{\mathbf{u}, \boldsymbol{\sigma}, \beta, \alpha\}$ which satisfies

$$\mathbf{u} \in C(0, T; V),$$

$$\boldsymbol{\sigma} \in C(0, T; \mathcal{H}_1),$$

$$\beta \in W^{1,\infty}\left(0, T; L^2(\Gamma_3)\right),$$

$$\alpha \in W^{1,2}\left(0, T; L^2(\Omega)\right) \cap L^2\left(0, T; H^1(\Omega)\right). \tag{4.1}$$

A quadruplet $(\mathbf{u}, \boldsymbol{\sigma}, \beta, \alpha)$ which satisfies (3.29)-(3.33) is called a weak solution to the compliance contact problem P. We conclude that, under the stated assumptions, problem (3.1)-(3.11) has a unique weak solution satisfying (4.1). We turn now to the proof of Theorem 4.1 which is carried out in several steps. To this end, we assume in the following that (3.14)-(3.22) and (3.28) hold. Below, C denotes a generic positive constant which may depend on $\Omega, \Gamma_1, \Gamma_3, \mathcal{E}, \gamma_\nu, L$ and T but does not depend on t nor of the rest of input data, and whose value may change from place to place. Moreover, for the sake of simplicity, we supress, in what follows, the explicit dependence of various functions on $\mathbf{x} \in \Omega \cup \Gamma$. The proof of Theorem 4.1 will be carried out in several steps. In the first step we solve the differential equation in (3.32) for the adhesion field, where $\mathbf{u}$ is given, and study the continuous dependence of the adhesion solution with respect to $\mathbf{u}$.

Lemma 3. *For every* $\mathbf{u} \in C(0,T;V)$, there exists a unique solution

$$\beta_u \in W^{1,\infty}\left(0,T;L^2(\Gamma_3)\right)$$

satisfying

$$\dot{\beta}_u(t) = -\left[\gamma_\nu\ \beta_u(t)\ \ \left[(-R(u_\nu(t)))_+\right]^2 - \epsilon_a\right]_+ \qquad \text{a.e. } t \in (0,T),$$

$$\beta_u(0) = \beta_0.$$

Moreover, $\beta_u(t) \in Z$ *for* $t \in [0,T]$, *a.e. on* Γ_3, *and there exists a constant* $C > 0$, *such that, for all* $\mathbf{u}_1,\ \mathbf{u}_2 \in C(0,T;V)$,

$$\left|\beta_{u_1}(t) - \beta_{u_2}(t)\right|^2_{L^2(\Gamma_3)} \le C \int_0^t |\mathbf{u}_1(s) - \mathbf{u}_2(s)|^2_V\ ds \qquad \forall t \in [0,T].$$

Proof. Consider the mapping $F : [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$ defined by

$$F(t,\beta) = -\left[\gamma_\nu \beta(t)\left[(-R(\mathbf{u}_\nu))_+\right]^2 - \epsilon_a\right]_+,$$

$\forall t \in [0,T]$ and $\beta \in L^2(\Gamma_3)$. It follows from the properties of the truncation operator R that F is Lipschitz continuous with respect to the second argument, uniformly in time. Moreover, for any $\beta \in L^2(\Gamma_3)$, the mapping $t \longmapsto F(t,\beta)$ belongs to $L^\infty\left(0,T,L^2(\Gamma_3)\right)$. Thus, the existence and the uniqueness of the solution β_u follows from the classical theorem of Cauchy-Lipschitz given in Theorem 2.1. Notice also that the argument used in Remark 3.1 shows that $0 \le \beta_u(t) \le 1$ for all $t \in [0,T]$, a.e. on Γ_3. Therefore, from the definition of the set Z, we find that $\beta_u(t) \in Z$ for all $t \in [0,T]$, which concludes the proof of the Lemma. Now let $\mathbf{u}_1,\ \mathbf{u}_2 \in C(0,T;V)$ and let $t \in [0,T]$. We have, for $i = 1,2$,

$$\beta_{u_i}(t) = \beta_0 - \int_0^t \left[\gamma_\nu\ \beta_{u_i}(t)\ \ \left[(-R(u_{i\nu}(t)))_+\right]^2 - \epsilon_a\right]_+ ds,$$

and then

$$\left|\beta_{u_1}(t) - \beta_{u_2}(t)\right|_{L^2(\Gamma_3)}$$

$$\leq C\int_0^t \left|\beta_{u_1}(s)\ \left[(-R(u_{1\nu}(s)))_+\right]^2-\beta_{u_2}(s)\ \left[(-R(u_{2\nu}(s)))_+\right]^2\right|_{L^2(\Gamma_3)}\ ds.$$

Using the definition of the truncation operator R given by (3.12) and considering $\beta_{u_1}=\beta_{u_1}-\beta_{u_2}+\beta_{u_2}$ we find

$$\left|\beta_{u_1}(t)-\beta_{u_2}(t)\right|_{L^2(\Gamma_3)}$$

$$\leq C\left(\int_0^t \left|\beta_{u_1}(s)-\beta_{u_2}(s)\right|_{L^2(\Gamma_3)}\ ds+\int_0^t \left|\mathbf{u}_1(s)-\mathbf{u}_2(s)\right|_{L^2(\Gamma_3)^d}\ ds\right).$$

Applying Gronwall's inequality, it follows that

$$\left|\beta_{u_1}(t)-\beta_{u_2}(t)\right|^2_{L^2(\Gamma_3)}\leq C\int_0^t \left|\mathbf{u}_1(s)-\mathbf{u}_2(s)\right|^2_{L^2(\Gamma_3)^d}\ ds,$$

and using (3.13) we obtain the second part of Lemma 4.2. □

Now we consider the following viscoplastic problem and we prove an existence and uniqueness result for (3.29), (3.31) and (3.33) with the corresponding initial condition.

Problem QV. Find a displacement field $\mathbf{u}:[0,T]\to V$, a damage field $\alpha:[0,T]\to H^1(\Omega)$ and a stress field $\boldsymbol{\sigma}:[0,T]\to\mathcal{H}$ satisfying (3.29) and

$$(\boldsymbol{\sigma}(t),\boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}}+j_{ad}(\beta_{\mathbf{u}}(t),\mathbf{u}(t),\mathbf{v})+j_{nc}(\mathbf{u}(t),\mathbf{v})$$

$$=(\mathbf{f}(t),\mathbf{v})_V\qquad\forall\mathbf{v}\in V,\ \forall t\in[0,T],\tag{4.2}$$

$$\mathbf{u}(0)=\mathbf{u}_0,\boldsymbol{\sigma}(0)=\boldsymbol{\sigma}_0,\alpha(0)=\alpha_0.\tag{4.3}$$

Let $(\boldsymbol{\eta},\omega)\in C(0,T;\mathcal{H}\times L^2(\Omega))$ and let $\mathfrak{z}_\eta(t)=\int_0^t\boldsymbol{\eta}(s)\ ds+\boldsymbol{\sigma}_0-\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_0)$, then

$$\mathfrak{z}_\eta\in C^1(0,T;\mathcal{H}),$$

and consider the following variational problem.

Problem QV_η. Find a displacement field $\mathbf{u}_\eta:[0,T]\to V$ and a stress field $\boldsymbol{\sigma}_\eta:[0,T]\to\mathcal{H}$ such that

$$\boldsymbol{\sigma}_\eta(t)=\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t))+\mathfrak{z}_\eta(t),\ \forall t\in[0,T],\tag{4.4}$$

$$(\boldsymbol{\sigma}_\eta(t),\boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}}+j_{ad}(\beta_{u_\eta}(t),\mathbf{u}_\eta(t),\mathbf{v})+j_{nc}(\mathbf{u}_\eta(t),\mathbf{v})$$

$$= (\mathbf{f}(t), \mathbf{v})_V \qquad \forall \mathbf{v} \in V,\ \forall t \in [0, T], \tag{4.5}$$

$$\mathbf{u}_\eta(0) = \mathbf{u}_0, \boldsymbol{\sigma}_\eta(0) = \boldsymbol{\sigma}_0. \tag{4.6}$$

To solve problem QV_η we consider $\boldsymbol{\theta} \in C(0, T; V)$ and we construct the following intermediate problem.

Problem $QV_{\eta\theta}$. *Find a displacement field* $\mathbf{u}_{\eta\theta} : [0, T] \to V$ *and a stress field* $\boldsymbol{\sigma}_{\eta\theta} : [0, T] \to \mathcal{H}$ *such that*

$$\boldsymbol{\sigma}_{\eta\theta}(t) = \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_{\eta\theta}(t)) + \mathfrak{z}_\eta(t), \tag{4.7}$$

$$(\boldsymbol{\sigma}_{\eta\theta}(t), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + (\boldsymbol{\theta}(t), \mathbf{v})_V = (\mathbf{f}(t), \mathbf{v})_V \qquad \forall \mathbf{v} \in V,\ \forall t \in [0, T], \tag{4.8}$$

$$\mathbf{u}_{\eta\theta}(0) = \mathbf{u}_0, \boldsymbol{\sigma}_{\eta\theta}(0) = \boldsymbol{\sigma}_0. \tag{4.9}$$

Lemma 4. *There exists a unique solution* $(\mathbf{u}_{\eta\theta}, \boldsymbol{\sigma}_{\eta\theta})$ *of the problem* $QV_{\eta\theta}$ *which satisfies* $\mathbf{u}_{\eta\theta} \in C(0, T; V)$, $\boldsymbol{\sigma}_{\eta\theta} \in C(0, T; \mathcal{H}_1)$.

Proof. We define the operator $A : V \to V$ by

$$(A\,\mathbf{u}, \mathbf{v})_V = (\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}},\ \forall \mathbf{u}, \mathbf{v} \in V. \tag{4.10}$$

Using (3.14), it follows that A is a strongly monotone Lipschitz operator, thus A is invertible and $A^{-1} : V \to V$ is also a strongly monotone Lipschitz operator. It follows that there exists a unique function $\mathbf{u}_{\eta\theta}$ which satisfies

$$\mathbf{u}_{\eta\theta} \in C(0, T; V), \tag{4.11}$$

$$A\,\mathbf{u}_{\eta\theta}(t) = \mathbf{h}_{\eta\theta}(t), \tag{4.12}$$

where $\mathbf{h}_{\eta\theta} \in C(0, T; V)$ is such that

$$(\mathbf{h}_{\eta\theta}(t), \mathbf{v})_V = (\mathbf{f}(t), \mathbf{v})_V - (\mathfrak{z}_\eta(t), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} - (\boldsymbol{\theta}(t), \mathbf{v})_V\ \ \forall \mathbf{v} \in V,\ \forall t \in [0, T]. \tag{4.13}$$

It follows from, (4.12) that $\mathbf{u}_{\eta\theta} \in C(0, T; V)$. Consider $\boldsymbol{\sigma}_{\eta\theta}$ defined in (4.7), since, $\mathfrak{z}_\eta \in C^1(0, T; \mathcal{H})$, $\mathbf{u}_{\eta\theta} \in C(0, T; V)$ we deduce that $\boldsymbol{\sigma}_{\eta\theta} \in C(0, T; \mathcal{H})$. Since $\mathrm{Div}\boldsymbol{\sigma}_{\eta\theta} = -\mathbf{f}_0 \in C(0, T; H)$, we further have $\boldsymbol{\sigma}_{\eta\theta} \in C(0, T; \mathcal{H}_1)$. This concludes the existence part of Lemma 4.3. The uniqueness of the solution follows from the unique solvability of the time-dependent equation (4.12). Finally $(\mathbf{u}_{\eta\theta}, \boldsymbol{\sigma}_{\eta\theta})$ is the unique solution of problem $QV_{\eta\theta}$ obtained in Lemma 4.3, which concludes the proof. □

Let $\Lambda\boldsymbol{\theta}(t)$ denote the element of V defined by

$$(\Lambda\boldsymbol{\theta}(t), \mathbf{v})_V = j_{ad}(\beta_{u_{\eta\theta}}(t), \mathbf{u}_{\eta\theta}(t), \mathbf{v}) + j_{nc}(\mathbf{u}_{\eta\theta}(t), \mathbf{v}) \quad \forall \mathbf{v} \in V,\ \forall t \in [0,T]. \tag{4.14}$$

We have the following result.

Lemma 5. For each $\boldsymbol{\theta} \in C(0,T;V)$ the function $\Lambda\boldsymbol{\theta} : [0,T] \to V$ belongs to $C(0,T;V)$. Moreover, there exists a unique element $\boldsymbol{\theta}^* \in C(0,T;V)$ such that $\Lambda\boldsymbol{\theta}^* = \boldsymbol{\theta}^*$.

Proof. Let $\boldsymbol{\theta} \in C(0,T;V)$ and let $t_1,\ t_2 \in [0,T]$. Using (3.35), (3.36) and (4.14) we obtain

$$|\Lambda\boldsymbol{\theta}(t_1) - \Lambda\boldsymbol{\theta}(t_2)|_V \le C\left(\left|\beta_{u_{\eta\theta}}(t_1) - \beta_{u_{\eta\theta}}(t_2)\right|_{L^2(\Gamma_3)} + |\mathbf{u}_{\eta\theta}(t_1) - \mathbf{u}_{\eta\theta}(t_2)|_V\right). \tag{4.15}$$

By Lemma 4.3, $\mathbf{u}_{\eta\theta} \in C(0,T;V)$ and, by Lemma 4.2, $\beta_{u_{\eta\theta}} \in W^{1,\infty}\left(0,T;L^2(\Gamma_3)\right)$, then we deduce from inequality (4.15) that $\Lambda\boldsymbol{\theta} \in C(0,T;V)$. Let now $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2 \in C(0,T;V)$ and denote $\mathbf{u}_{\eta\theta_i} = \mathbf{u}_i$ and $\beta_{u_{\eta\theta_i}} = \beta_{u_i}$ for $i = 1,2$. Using again the relations (3.35), (3.36) and (4.14) we find

$$|\Lambda\boldsymbol{\theta}_1(t) - \Lambda\boldsymbol{\theta}_2(t)|_V^2 \le C\left(\left|\beta_{u_1}(t) - \beta_{u_2}(t)\right|_{L^2(\Gamma_3)}^2 + |\mathbf{u}_1(t) - \mathbf{u}_2(t)|_V^2\right). \tag{4.16}$$

Then by Lemma 4.2, we have

$$\left|\beta_{u_1}(t) - \beta_{u_2}(t)\right|_{L^2(\Gamma_3)}^2 \le C\int_0^t |\mathbf{u}_1(s) - \mathbf{u}_2(s)|_{L^2(\Gamma_3)}^2 \, ds,$$

and by (3.13) we get

$$\left|\beta_{u_1}(t) - \beta_{u_2}(t)\right|_{L^2(\Gamma_3)}^2 \le C\int_0^t |\mathbf{u}_1(s) - \mathbf{u}_2(s)|_V^2 \, ds.$$

Use the previous inequality in (4.16) to obtain

$$|\Lambda\boldsymbol{\theta}_1(t) - \Lambda\boldsymbol{\theta}_2(t)|_V^2 \le C\left(|\mathbf{u}_1(t) - \mathbf{u}_2(t)|_V^2 + \int_0^t |\mathbf{u}_1(s) - \mathbf{u}_2(s)|_V^2 \, ds\right). \tag{4.17}$$

Moreover, from (4.8) it follows that

$$(\mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_1) - \mathcal{E}\boldsymbol{\varepsilon}(\mathbf{u}_2), \boldsymbol{\varepsilon}(\mathbf{u}_1) - \boldsymbol{\varepsilon}(\mathbf{u}_2))_{\mathcal{H}} + (\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2, \mathbf{u}_1 - \mathbf{u}_2)_V = 0 \text{ on } (0,T). \tag{4.18}$$

Hence

$$|\mathbf{u}_1(t) - \mathbf{u}_2(t)|_V \le C \; |\boldsymbol{\theta}_1(t) - \boldsymbol{\theta}_2(t)|_V \;\; \forall t \in [0,T]. \tag{4.19}$$

Now from the inequalities (4.17) and (4.19) we have

$$|\Lambda\boldsymbol{\theta}_1(t) - \Lambda\boldsymbol{\theta}_2(t)|_V^2 \le C \left(|\boldsymbol{\theta}_1(t) - \boldsymbol{\theta}_2(t)|_V^2 + \int_0^t |\boldsymbol{\theta}_1(s) - \boldsymbol{\theta}_2(s)|_V^2 \; ds \right) \;\; \forall t \in [0,T].$$

Applying Gronwall's inequality we obtain

$$|\Lambda\boldsymbol{\theta}_1(t) - \Lambda\boldsymbol{\theta}_2(t)|_V^2 \le C \int_0^t |\boldsymbol{\theta}_1(s) - \boldsymbol{\theta}_2(s)|_V^2 \; ds \; \forall t \in [0,T].$$

Reiterating this inequality n times yields

$$|\Lambda^n\boldsymbol{\theta}_1 - \Lambda^n\boldsymbol{\theta}_2|_{C(0,T;V)}^2 \le \frac{(CT)^n}{n!} \; |\boldsymbol{\theta}_1 - \boldsymbol{\theta}_2|_{C(0,T;V)}^2 ,$$

which implies that for n sufficiently large a power Λ^n of Λ is a contraction in the Hilbert space $C(0,T;V)$. Then, there exists a unique $\boldsymbol{\theta}^* \in C(0,T;V)$ such that $\Lambda^n\boldsymbol{\theta}^* = \boldsymbol{\theta}^*$ and $\boldsymbol{\theta}^*$ is also the unique fixed point of Λ. □

Lemma 6. *There exists a unique solution of problem QV_η satisfying $\mathbf{u}_\eta \in C(0,T;V)$, $\boldsymbol{\sigma}_\eta \in C(0,T;\mathcal{H}_1)$.*

Proof. Let $\boldsymbol{\theta}^* \in C(0,T;V)$ be the fixed point of Λ, Lemma 4.3 implies that $(\mathbf{u}_{\eta\theta^*}, \boldsymbol{\sigma}_{\eta\theta^*}) \in C(0,T;V) \times C(0,T;\mathcal{H}_1)$ is the unique solution of $QV_{\eta\theta}$ for $\boldsymbol{\theta} = \boldsymbol{\theta}^*$. since $\Lambda\boldsymbol{\theta}^* = \boldsymbol{\theta}^*$ and from the relations (4.14), (4.7), (4.8) and (4.9), we obtain that $(\mathbf{u}_\eta, \boldsymbol{\sigma}_\eta) = (\mathbf{u}_{\eta\theta^*}, \boldsymbol{\sigma}_{\eta\theta^*})$ is the unique solution of QV_η. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operator Λ given in (4.14). □

Now for $(\boldsymbol{\eta},\omega) \in C(0,T;\mathcal{H} \times L^2(\Omega))$, we suppose that the assumptions of Theorem 4.1 hold and we consider the following intermediate problem for the damage field.

Probem PV_ω. Find a a damage field $\alpha_\omega : [0,T] \to H^1(\Omega)$ such that $\alpha_\omega(t) \in$ K, for all $t \in [0,T]$ and

$$(\dot{\alpha_\omega}(t), \xi - \alpha_\omega(t))_{L^2(\Omega)} + a(\alpha_\omega(t), \xi - \alpha_\omega(t))$$

$$\geq \left(\omega\left(t\right),\xi-\alpha_{\omega}\left(t\right)\right)_{L^{2}(\Omega)}\ \forall\xi\in K,\ \text{a.e. } t\in\left(0,T\right) \tag{4.20}$$

$$\alpha_{\omega}\left(0\right)=\alpha_{0} \tag{4.21}$$

Lemma 7. *Problem PV_{ω} has a unique soltion α_{ω} such that*

$$\alpha_{\omega}\in W^{1,2}\left(0,T;L^{2}\left(\Omega\right)\right)\cap L^{2}\left(0,T;H^{1}\left(\Omega\right)\right). \tag{4.22}$$

Proof. We use (3.21), (3.23) and a classical existence and uniqueness result on parabolic inequalities (see for instance [1 p. 124]). □

As a consequence of the problems QV_{η} and PV_{ω}, we may define the operator $\mathcal{L}:C(0,T;\mathcal{H}\times L^{2}\left(\Omega\right))\rightarrow C(0,T;\mathcal{H}\times L^{2}\left(\Omega\right))$ by

$$\mathcal{L}(\boldsymbol{\eta},\omega)=(\mathcal{G}(\boldsymbol{\sigma}_{\eta},\varepsilon(\mathbf{u}_{\eta}),\alpha_{\omega}),\Phi(\boldsymbol{\sigma}_{\eta},\varepsilon(\mathbf{u}_{\eta}),\alpha_{\omega})), \tag{4.23}$$

for all $(\boldsymbol{\eta},\omega)\in C(0,T;\mathcal{H}\times L^{2}\left(\Omega\right))$. Then we have.

Lemma 8. *The operator $\mathcal{L}$ has a unique fixed point*

$$(\boldsymbol{\eta}^{*},\omega^{*})\in C(0,T;\mathcal{H}\times L^{2}\left(\Omega\right)).$$

Proof. Let $(\boldsymbol{\eta}_{1},\omega_{1})$, $(\boldsymbol{\eta}_{2},\omega_{2})\in C(0,T;\mathcal{H}\times L^{2}\left(\Omega\right))$, let $t\in[0,T]$ and use the notation $\mathbf{u}_{\eta i}=\mathbf{u}_{i}$, $\boldsymbol{\sigma}_{\eta i}=\boldsymbol{\sigma}_{i}$, $\mathfrak{z}_{\eta i}=\mathfrak{z}_{i}$ and $\alpha_{\omega_{i}}=\alpha_{i}$ for $i=1,2$. Taking into account the relations (3.15), (3.16) and (4.23), we deduce that

$$|\mathcal{L}(\boldsymbol{\eta}_{1},\omega_{1})-\mathcal{L}(\boldsymbol{\eta}_{2},\omega_{2})|_{\mathcal{H}\times L^{2}(\Omega)}$$

$$\leq C\left(\left|\mathbf{u}_{1}\left(t\right)-\mathbf{u}_{2}\left(t\right)\right|_{V}+\left|\alpha_{1}\left(t\right)-\alpha_{2}\left(t\right)\right|_{L^{2}(\Omega)}+\left|\boldsymbol{\sigma}_{1}\left(t\right)-\boldsymbol{\sigma}_{2}\left(t\right)\right|_{\mathcal{H}}\right). \tag{4.24}$$

Using (4.5) we obtain

$$\left(\mathcal{E}\varepsilon\left(\mathbf{u}_{1}\right)-\mathcal{E}\varepsilon\left(\mathbf{u}_{2}\right),\varepsilon\left(\mathbf{u}_{1}\right)-\varepsilon\left(\mathbf{u}_{2}\right)\right)_{\mathcal{H}}=j_{ad}(\beta_{u_{2}},\mathbf{u}_{2},\mathbf{u}_{1}-\mathbf{u}_{2})-j_{ad}(\beta_{u_{1}},\mathbf{u}_{1},\mathbf{u}_{1}-\mathbf{u}_{2})$$

$$+j_{nc}(\mathbf{u}_{2},\mathbf{u}_{1}-\mathbf{u}_{2})-j_{nc}(\mathbf{u}_{1},\mathbf{u}_{1}-\mathbf{u}_{2})+\left(\mathfrak{z}_{2}-\mathfrak{z}_{1},\varepsilon\left(\mathbf{u}_{1}\right)-\varepsilon\left(\mathbf{u}_{2}\right)\right)_{\mathcal{H}}\ \text{a.e. } t\in\left(0,T\right). \tag{4.25}$$

Keeping in mind (3.35), (3.37) and (3.14) we find

$$\left|\mathbf{u}_{1}\left(t\right)-\mathbf{u}_{2}\left(t\right)\right|_{V}\leq C\left(\left|\beta_{u_{1}}\left(t\right)-\beta_{u_{2}}\left(t\right)\right|_{L^{2}(\Gamma_{3})}+\left|\mathfrak{z}_{1}\left(t\right)-\mathfrak{z}_{2}\left(t\right)\right|_{\mathcal{H}}\right), \tag{4.26}$$

and

$$|\mathbf{u}_1(t)-\mathbf{u}_2(t)|_V^2 \le C\left(\left|\beta_{u_1}(t)-\beta_{u_2}(t)\right|^2_{L^2(\Gamma_3)}+|\mathfrak{z}_1(t)-\mathfrak{z}_2(t)|^2_{\mathcal{H}}\right).$$

By Lemma 4.2, we obtain

$$|\mathbf{u}_1(t)-\mathbf{u}_2(t)|_V^2 \le C\left(|\mathfrak{z}_1(t)-\mathfrak{z}_2(t)|^2_{\mathcal{H}}+\int_0^t |\mathbf{u}_1(s)-\mathbf{u}_2(s)|_V^2\ ds\right)$$

$$\le C(\int_0^t |\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)|^2_{\mathcal{H}}\ ds+\int_0^t |\mathbf{u}_1(s)-\mathbf{u}_2(s)|_V^2\ ds). \tag{4.27}$$

Applying Gronwall's inequality yields

$$|\mathbf{u}_1(t)-\mathbf{u}_2(t)|_V^2 \le C\int_0^t |\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)|^2_{\mathcal{H}}\ ds, \tag{4.28}$$

which implies

$$|\mathbf{u}_1(t)-\mathbf{u}_2(t)|_V \le C\int_0^t |\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)|_{\mathcal{H}}\ ds. \tag{4.29}$$

Moreover, by (4.4) we find

$$|\boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t)|_{\mathcal{H}} \le C\left(|\mathbf{u}_1(t)-\mathbf{u}_2(t)|_V+|\mathfrak{z}_1(t)-\mathfrak{z}_2(t)|_{\mathcal{H}}\right).$$

Substituting (4.29) in the previous inequality we obtain

$$|\boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t)|_{\mathcal{H}} \le C\int_0^t |\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)|_{\mathcal{H}}\ ds. \tag{4.30}$$

From (4.20) we deduce that

$$\left(\dot{\alpha_1},\alpha_2-\alpha_1\right)_{L^2(\Omega)}+a(\alpha_1,\alpha_2-\alpha_1)$$

$$\ge(\omega_1,\alpha_2-\alpha_1)_{L^2(\Omega)}\ \text{ a.e. } t\in(0,T),$$

and

$$\left(\dot{\alpha_2},\alpha_1-\alpha_2\right)_{L^2(\Omega)}+a(\alpha_2,\alpha_1-\alpha_2)$$

$$\ge(\omega_2,\alpha_1-\alpha_2)_{L^2(\Omega)}\ \text{ a.e. } t\in(0,T).$$

Adding the previous inequalities we obtain

$$\left(\dot{\alpha_1}-\dot{\alpha_2},\alpha_1-\alpha_2\right)_{L^2(\Omega)}+a(\alpha_1-\alpha_2,\alpha_1-\alpha_2)$$

$$\le|\omega_1-\omega_2|_{L^2(\Omega)}|\alpha_1-\alpha_2|_{L^2(\Omega)}\ \text{ a.e. } t\in(0,T).$$

Integrating the previous inequality on $[0,t]$, after some manipulations we obtain

$$\frac{1}{2}\left|\alpha_1(t)-\alpha_2(t)\right|^2_{L^2(\Omega)} \le C\int_0^t \left|\omega_1(s)-\omega_2(s)\right|_{L^2(\Omega)} \left|\alpha_1(s)-\alpha_2(s)\right|_{L^2(\Omega)}\ ds$$

$$+C\int_0^t \left|\alpha_1(s)-\alpha_2(s)\right|^2_{L^2(\Omega)}\ ds.$$

Applying Gronwall's inequality to the previous inequality yields

$$\left|\alpha_1(t)-\alpha_2(t)\right|_{L^2(\Omega)} \le C\int_0^t \left|\omega_1(s)-\omega_2(s)\right|_{L^2(\Omega)}\ ds. \tag{4.31}$$

Substituting (4.29), (4.30) and (4.31) in (4.24), we obtain

$$\left|\mathcal{L}(\boldsymbol{\eta}_1,\omega_1)-\mathcal{L}(\boldsymbol{\eta}_2,\omega_2)\right|_{\mathcal{H}\times L^2(\Omega)}$$

$$\le C\int_0^t \left|(\boldsymbol{\eta}_1,\omega_1)(s)-(\boldsymbol{\eta}_2,\omega_2)(s)\right|_{\mathcal{H}\times L^2(\Omega)}\ ds. \tag{4.32}$$

Lemma 4.7 is a consequence of the result (4.32) and Banach's fixed point Theorem. □

Now, we have all ingredients to solve QV.

Lemma 9. *There exists a unique solution* $(\mathbf{u},\boldsymbol{\sigma},\alpha)$ *of problem PV satisfying* $\mathbf{u}\in C(0,T;V)$, $\boldsymbol{\sigma}\in C(0,T;\mathcal{H}_1)$, $\alpha\in W^{1,2}\left(0,T;L^2(\Omega)\right)\cap L^2\left(0,T;H^1(\Omega)\right)$.

Proof. Let $(\boldsymbol{\eta}^*,\omega^*)\in L^2(0,T;\mathcal{H}\times L^2(\Omega))$ be the fixed point of $\mathcal{L}$ given by (4.24), by Lemma 4.5, we deduce that $(\mathbf{u}_\eta,\boldsymbol{\sigma}_\eta)=(\mathbf{u}_{\eta\theta^*},\boldsymbol{\sigma}_{\eta\theta^*})\in C(0,T;V)\times C(0,T;\mathcal{H}_1)$ is the unique solution of QV_η. Since $\mathcal{L}(\boldsymbol{\eta}^*,\omega^*)=(\boldsymbol{\eta}^*,\omega^*)$, from the relations (4.4), (4.5), (4.6) and Lemma 4.6 we obtain that $(\mathbf{u},\boldsymbol{\sigma},\alpha)=(\mathbf{u}_{\eta^*\theta^*},\boldsymbol{\sigma}_{\eta^*\theta^*},\alpha_{\omega^*})$ is the unique solution of QV. The regularity of the solution follows from Lemma 4.6. The uniqueness of the solution results from the uniqueness of the fixed point of the operator $\mathcal{L}$. □

Theorem 4.1 is now a consequence of Lemma 4.2 and Lemma 4.8.

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