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ON CERTAIN SUBCLASS OF p-VALENTLY BAZILEVIC FUNCTIONS

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Abstract. A certain subclass $B_1(p,n,\alpha,\beta)$ with $p,n \in N = \{1,2,\dots\}$, $\alpha > 0$ and $0 \le \beta < p$, of p-valently Bazilevic functions in the unit disc $U = \{z : |z| < 1\}$ is introduced. The object of the present paper is to derive some properties of the class $B_1(p,n,\alpha,\beta)$.

1. Introduction

Let $A(p,n)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k (p,n \in N = \{1,2,..\}, \tag{1.1}$$

which are analytic and p-valent in the unit disc $U = \{z : |z| < 1\}$. A function $f(z) \in A(p,n)$ is said to be in the class $S(p,n,\beta)$ of p-valently starlike functions of order $\beta(0 \le \beta < p)$ if it satisfies

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \beta \text{ and } \int_0^{2\pi} \text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} d\theta = 2\pi p \; . \tag{1.2}$$

The class $S(p,n,\beta)$ was studied recently by Owa [8] and Aouf et al. [1]. Also, we note that $S(p,n,0) = S^*(p,n)$ and $S(p,1,0) = S^*(p)$.

A function $f(z) \in A(p,n)$ is said to be in the class $B(p,n,\alpha,\beta)$ if it satisfies

$$\text{Re}\left\{\frac{z\, f'(z)}{f^{1-\alpha}(z)\, g^{\alpha}(z)}\right\} > \beta \tag{1.3}$$

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for some $\alpha(\alpha > 0)$, $\beta(0 \leq \beta < p)$, $g(z) \in S^*(p,n)$ and for all $z \in U$. Further, let $B_1(p,n,\alpha,\beta)$ be the subclass of $B(p,n,\alpha,\beta)$ for $g(z) = z^p \in S^*(p)$. Also we say that $f(z)$ in the class $B(p,n,\alpha,\beta)$ is a Bazilevic function of order β and type α (see [5]).

Remark 1. *(i) The classes $B(p,n,\alpha,\beta)$ and $B_1(p,n,\alpha,\beta)$ are the subclasses of p-valently Bazilevic functions in U.*

(ii) The classes $B(p,1,\alpha,\beta) = B(p,\alpha,\beta)$ and $B_1(p,1,\alpha,\beta) = B_1(p,\alpha,\beta)$ when $p = 1$ were studied by Owa [10] *and the class $B(p,\alpha,\beta)$ was studied by Nunokawa et al.* [5].

(iii) The class $B_1(1,n,\alpha,\beta)$ was studied by Owa [9].

(iv) The classes $B(1,1,\alpha,\beta) = B(\alpha,\beta)$ and $B_1(1,1,\alpha,\beta) = B_1(\alpha,\beta)$ when $p = n = 1$ were studied by Owa and Obradovic [11].

(v) The classes $B(1,1,\alpha,0) = B(\alpha)$ and $B_1(1,1,\alpha,0) = B_1(\alpha)$ when $p = n = 1$ and $\beta = 0$ were studied by Singh [12].

2. **Properties of the class** $B_1(p,n,\alpha,\beta)$

In order to establish our main result, we have to recall here the following lemma due to Miller and Mocanu [4].

Lemma 1. *Let $\varphi(u,v)$ be a complex valued function,*

$$\varphi : D \to C,\ D \subset C \times C = C^2 \qquad (C \text{ is the complex plane}),$$

and let $u = u_1 + iu_2$, $v = v_1 + iv_2$. Suppose that the function $\varphi(u,v)$ satisfies

(i) $\varphi(u,v)$ is continuous in D;

(ii) $(1,0) \in D$ and $\mathrm{Re}\{\varphi(1,0)\} > 0$;

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq -\frac{n}{2}(1+u_2^2)$, $\mathrm{Re}\{\varphi(iu_2,v_1)\} \leq 0$.

Let $q(z) = 1 + q_n z^n + q_{n+1} z^{n+1} + ...$ be regular in the unit disc U such that $(q(z), z q'(z)) \in D$ for all $z \in U$. If

$$\mathrm{Re}\{\varphi(iu_2,v_1)\} > 0 \quad (z \in U),$$

then

$$\mathrm{Re}\{q(z)\} > 0 \quad (z \in U).$$

Using the above lemma, we prove the following result.

Theorem 2. *If $f(z) \in B_1(p, n, \alpha, \beta)$, with $p, n \in N, \alpha > 0$ and $0 \leq \beta < p$, then*

$$\mathrm{Re}\left\{\frac{f(z)}{z^p}\right\}^{\alpha} > \frac{n + 2\alpha\beta}{n + 2\alpha\, p} \quad (z \in U). \tag{2.1}$$

Proof. We define the function $q(z)$ by

$$\left\{\frac{f(z)}{z^p}\right\}^{\alpha} = \delta + (1 - \delta)\, q(z) \tag{2.2}$$

with

$$\delta = \frac{n + 2\alpha\beta}{n + 2\alpha p} \quad . \tag{2.3}$$

Then, we see that $q(z) = 1 + q_n\, z^n + q_{n+1}\, z^{n+1} + ...$ is regular in U. It follows from (2.2) that

$$\frac{f^{'}(z)\, f^{\alpha-1}(z)}{z^{p\,\alpha-1}} - \beta = [p\delta + (p - p\delta)q(z)] + \frac{(1 - \delta)\, z\, q^{'}(z)}{\alpha}, \tag{2.4}$$

or

$$\mathrm{Re}\left\{\frac{f^{'}(z)\, f^{\alpha-1}(z)}{z^{p\,\alpha-1}} - \beta\right\}$$

$$= \mathrm{Re}\left\{p\delta - \beta + (p - p\delta)\, q(z) + \frac{(1 - \delta)\, z\, q^{'}(z)}{\alpha}\right\} > 0\ . \tag{2.5}$$

Now, setting $q(z) = u = u_1 + iu_2$, $z\, q^{'}(z) = v = v_1 + iv_2$, and

$$\varphi(u, v) = p\delta - \beta + (p - p\delta)u + \frac{(1 - \delta)v}{\alpha}\ , \tag{2.6}$$

it is easily seen that

(i) $\varphi(u, v)$ is continuous in $D = C \times C$

(ii) $(1, 0) \in D$ and $\mathrm{Re}\{\varphi(1, 0)\} = p - \beta > 0$, and

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq \frac{-1}{2}\, n(1 + u_2^2)$,

$$\begin{aligned} \mathrm{Re}\{\varphi(iu_2, v_1)\} &= p\delta - \beta + \frac{(1 - \delta)\, v_1}{\alpha} \\ &\leq p\delta - \beta - \frac{n(1 - \delta)(1 + u_2^2)}{2\alpha} \leq 0, \end{aligned}$$

for δ given by (2.3). Therefore the function $\varphi(u, v)$ satisfies the condition in Lemma 1. This implies that $\mathrm{Re}\{q(z)\} > 0$ $(z \in U)$, that is, that

$$\mathrm{Re}\left\{\frac{f(z)}{z^p}\right\}^{\alpha} > \delta = \frac{n + 2\alpha\beta}{n + 2\alpha p} \quad (z \in U). \tag{2.7}$$

This completes the proof of Theorem 1.

Putting $\beta = 0$ in Theorem 1, we have

Corollary 3. *If $f(z) \in B_1(p, n, \alpha, 0)$, with $p, n \in N$ and $\alpha > 0$, then*

$$\operatorname{Re}\left\{\frac{f(z)}{z^p}\right\}^{\alpha} > \frac{n}{n+2\alpha p} \quad (z \in U). \tag{2.8}$$

Further, making $\alpha = \frac{1}{p}$ in Corollary 1, we have

Corollary 4. *If $f(z) \in B_1(p, n, \frac{1}{p}, 0)$, then*

$$\operatorname{Re}\left\{\frac{f^{\frac{1}{p}}(z)}{z}\right\} > \frac{n}{n+2} \quad (z \in U). \tag{2.9}$$

Putting $\alpha = \frac{1}{2}$ in Theorem 1, we have

Corollary 5. *If $f(z) \in B_1(p, n, \frac{1}{2}, \beta)$, with p, $n \in N$ and $0 \leq \beta < p$, then*

$$\operatorname{Re}\sqrt{\frac{f(z)}{z^p}} > \frac{n+\beta}{n+p} \, (z \in U). \tag{2.10}$$

Remark 2. *(1) Putting $p = 1$ in Theorem 1, Corollary 1 and Corollary 2, respectively, we obtain the results obtained by Owa [9, Theorem1, Corollary1 and Corollary2, respectively].*

(2) Putting $p = \alpha = 1$ and $\beta = 0$ in Theorem1, we obtain the result obtained by Cho [2, Theorem2].

(3)Putting $n = 1$ in Theorem1, we obtain the result obtained by Owa [10, Lemma 4]. Owa [10] obtained this result by different method.

(4) Putting $n = 1$ in Corollary1 and Corollary2, respectively, we obtain the results obtained by Owa [10, Corollary3 and Corollary4, respectively].

(5) Putting $n = p = 1$ in Theorem1, we obtain the result obtained by Owa and Obradovic [11, Theorem4].

(6) Putting $n = p = 1$ in Corollary1, we obtain the result obtained by Owa and Obradovic [11 Corollary3] and Obradovic [7, Theorem3].

(7) Putting $n = p = 1$ and $\alpha = 1$ in Theorem1, we obtain the result obtained by Owa and Obradovic [11, Corollary4].

(8) Putting $n = p = \alpha = 1$ in Theorem1 and $\beta = 0$, we obtain the result obtained by Obradovic [6, Theorem2].

Theorem 6. *If $f(z) \in B_1(p, n, \alpha, \beta)$, with $p, n \in N, \alpha > 0$ and $0 \leq \beta < p$, then*

$$\mathrm{Re}\left\{\frac{f(z)}{z^p}\right\}^{\frac{\alpha}{2}} = \frac{n + \sqrt{n^2 + 4\alpha\beta(n + p\alpha)}}{2(n + p\alpha)} \quad (z \in U). \tag{2.11}$$

Proof. Defining the function $q(z)$ by

$$\left\{\frac{f(z)}{z^p}\right\}^{\frac{\alpha}{2}} = \delta + (1 - \delta)\, q(z) \tag{2.12}$$

with

$$\delta = \frac{n + \sqrt{n^2 + 4\alpha\beta(n + p\alpha)}}{2(n + p\alpha)}\ , \tag{2.13}$$

we easily see that $q(z) = 1 + q_n\, z^n + q_{n+1}\, z^{n+1} + ...$ is regular in U. Taking the differentiations of both sides in (2.12), we obtain that

$$\frac{f^{'}(z) f^{\alpha-1}(z)}{z^{p\,\alpha-1}} = p[\delta + (1-\delta)q(z)]^2 + \frac{2}{\alpha}(1-\delta)[\delta + (1-\delta)q(z)]\, z\, q^{'}(z), \tag{2.14}$$

that is, that

$$\mathrm{Re}\left\{\frac{f^{'}(z) f^{\alpha-1}(z)}{z^{p\,\alpha-1}} - \beta\right\} = \mathrm{Re}\{p[\delta + (1-\delta)q(z)]^2 + \frac{2}{\alpha}(1-\delta)[1 + (1-\delta)q(z)]\, z\, q^{'}(z) - \beta\} > 0 \quad (z \in U). \tag{2.15}$$

Taking $q(z) = u = u_1 + iu_2$ and $z\, q^{'}(z) = v = v_1 + iv_2$, we define the function $\varphi(u, v)$ by

$$\varphi(u, v) = p[\delta + (1-\delta)u]^2 + \frac{2}{\alpha}(1-\delta)[\delta + (1-\delta)u]v - \beta. \tag{2.16}$$

Then $\varphi(u, v)$ satisfies

(i) $\varphi(u, v)$ is continuous in $D = C \times C$;

(ii) $(1, 0) \in D$ and $\mathrm{Re}\{\varphi(1, 0)\} = p - \beta > 0$;

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leq \frac{-n}{2}\,(1 + u_2^2)$,

$$\begin{aligned} \mathrm{Re}\{\varphi(iu_2, v_1)\} &= p[\delta^2 - (1-\delta)^2\; u_2^2] + \frac{2}{\alpha}(1-\delta)\delta\; v_1 - \beta \\ &\leq p[\delta^2 - (1-\delta)^2\; u_2^2] - \beta - \frac{n}{\alpha}\,\delta(1-\delta)(1 + u_2^2) \leq 0\ . \end{aligned}$$

Thus the function $\varphi(u,v)$ satisfies the conditions in Lemma 1. Applying Lemma 1, we conclude that

$$\operatorname{Re}\left\{\frac{f(z)}{z^p}\right\}^{\frac{\alpha}{2}} > \delta = \frac{n+\sqrt{n^2+4\alpha\beta(n+p\alpha)}}{2(n+p\alpha)} \quad (z \in U). \tag{2.17}$$

This completes the proof of Theorem 2.

Putting $\beta = 0$ in Theorem 2, we have

Corollary 7. *If $f(z) \in B_1(p,n,\alpha,0)$, with p, $n \in N$ and $\alpha > 0$, then*

$$\operatorname{Re}\left\{\frac{f(z)}{z^p}\right\}^{\frac{\alpha}{2}} > \frac{n}{n+p\alpha} \quad (z \in U). \tag{2.18}$$

Putting $\alpha = 1$ in Theorem 2, we have

Corollary 8. ***[3, Theorem 2].*** *If $f(z) \in B_1(p,n,1,\beta)$, with p, $n \in N$ and $0 \le \beta < p$, then*

$$\operatorname{Re}\left\{\sqrt{\frac{f(z)}{z^p}}\right\} > \frac{n+\sqrt{n^2+4\beta(n+p)}}{2(n+p)} \quad (z \in U). \tag{2.19}$$

Putting $\alpha = 1$ and $\beta = 0$ in Theorem 2, we have

Corollary 9. ***[3, Corollary 3].*** *If $f(z) \in B_1(p,n,1,0)$, with $p,n \in N$, then*

$$\operatorname{Re}\left\{\sqrt{\frac{f(z)}{z^p}}\right\} > \frac{n}{n+p} \quad (z \in U). \tag{2.20}$$

Remark 3. *(i) Putting $p = 1$ in Theorem 2, we obtain the result obtained by Owa [9, Theorem 2].*

(ii) Putting $\alpha = p = 1$ in Theorem 2, we obtain the result obtained by Owa [9, Corollary 3].

(iii) Putting $\alpha = 2$, $p = 1$ and $\beta = 0$ in Theorem 2, we obtain the result obtained by Cho [2, Theorem 3].

Theorem 10. *If $f(z) \in B_1(p,n,\alpha,\beta)$, with p, $n \in N, \alpha > 0$ and $0 \le \beta < p$, then the function $G_1(z)$ defined by*

$$G_1^{\alpha+\gamma}(z) = z^{p\,\gamma} f^{\alpha}(z) \quad (\gamma \ge 0) \tag{2.21}$$

is in the class $B_1(p, n, \alpha+\gamma, \delta)$, where

$$\delta = \frac{1}{\alpha+\gamma}\,(\frac{p\,\gamma(n+2\alpha\,\beta)}{n+2p\,\alpha} + \alpha\;\beta). \qquad (2.22)$$

Proof. Noting that

$$\frac{(\alpha+\gamma)G_1'(z)}{1-(\alpha+\gamma)} = p\,\gamma\, z^{p\gamma-1} f^{\alpha}(z) + \alpha\, z^{p\gamma} f'(z)\, f^{\alpha-1}(z),$$

that is, that

$$(\alpha+\gamma)\,\frac{z\,G_1'(z)\,G_1^{(\alpha+\gamma)-1}(z)}{z^{p(\alpha+\gamma)}} = p\,\gamma\,(\frac{f(z)}{z^p})^{\alpha} + \frac{\alpha\;zf'(z)f^{\alpha-1}(z)}{z^{p\,\alpha}}\,. \qquad (2.23)$$

Therefore, it follows from Theorem 1 that

$$\begin{aligned}\text{Re}\left\{\frac{z\,G_1'(z)G_1^{(\alpha+\gamma)-1}(z)}{z^{p(\alpha+\gamma)}}\right\} &= \frac{1}{\alpha+\gamma}\;\text{Re}\left\{p\,\gamma(\frac{f(z)}{z^p})^{\alpha} + \frac{\alpha z f'(z)\, f^{\alpha-1}(z)}{z^{p\,\alpha}}\right\} \\ &> \frac{1}{\alpha+\gamma}\left\{p\,\gamma(\frac{n+2\alpha\,\beta}{n+2\alpha\,p}) + \alpha\,\beta\right\}\;.\end{aligned}$$

This completes the proof of Theorem 3.

Taking $\beta = 0$ in Theorem 3, we have

Corollary 11. *If $f(z) \in B_1(p, n, \alpha, 0)$, with $p,\ n \in N$ and $\alpha > 0$, then the function $G_1(z)$ defined by* (2.21) *is in the class $B_1(p, n, \alpha+\gamma, \delta)$, where*

$$\delta = \frac{p\,\gamma}{(\alpha+\gamma)(n+2p\,\alpha)}\;. \qquad (2.24)$$

Taking $p = 1$ in Theorem 3, we have

Corollary 12. *If $f(z) \in B_1(1, n, \alpha, \beta)$ with $n \in N$, $\alpha > 0$ and $0 \le \beta < 1$, then the function $G_2(z)$ defined by*

$$G_2^{\alpha+\gamma}(z) = z^{\,\gamma} f^{\alpha}(z) \quad (\gamma \ge 0) \qquad (2.25)$$

is in the class $B_1(1, n, \alpha+\gamma, \delta)$, where

$$\delta = \frac{1}{\alpha+\gamma}\,(\frac{\gamma(n+2\alpha\,\beta)}{n+2\alpha} + \alpha\,\beta). \qquad (2.26)$$

Remark 4. *Putting $n = 1$ in Theorem 3, Corollary 7 and Corollary 8, respectively, we obtain the results obtained by Owa [10, Theorem 2, Corollary 5 and Corollary 6, respectively].*

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