OPTIMAL DYNAMIC PORTFOLIOS UNDER A TAIL CONDITIONAL EXPECTATION CONSTRAINT

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Abstract. We consider a portfolio problem when a tail conditional expectation constraint is imposed. The financial market is composed of n risky assets driven by geometric Brownian motion and one risk-free asset. The tail conditional expectation is derived, re-calculated at short intervals of time and imposed continuously. The method of Lagrange multipliers is combined with the Hamilton-Jacobi-Bellman equation to insert the constraint into the resolution framework. A numerical method is applied to obtain an approximate solution to the problem. We find that the imposition of the tail conditional expectation constraint when risky assets evolve following a log-normal distribution, curbs investment in the risky assets and increases consumption.

1. Introduction

In recent years particular stress has been laid on the substitution of variance as a risk measure in the standard Markowitz [11] (1952) mean-variance problem. Since it makes no distinction between positive and negative deviations from the mean, variance is a good measure of risk only for distributions that are (approximately) symmetric around the mean such as the normal distribution or more generally, elliptical distributions (see e.g., McNeil, Frey and Embrechts [12] (2004)). However, in most cases such as in portfolios containing options, we are dealing with wealth distributions

that are highly skewed. It is thus more reasonable to consider asymmetric risk measures since individuals are typically loss averse. In this regard, Value-at-Risk (VaR) has emerged as the industry standard as regulatory authorities enforced the use of VaR which is a downside risk measure (see, e.g., Jorion [9] (1997)).

Despite its widespread acceptance, VaR is known to possess unappealing features. Artzner et al. [3] (1999) proposed an axiomatic foundation for risk measures, by identifying four properties that a reasonable risk measure should satisfy and providing a characterization of the risk measures satisfying these properties, which they called coherent risk measures. Tail conditional expectation (TCE) is one of such so-called coherent risk measures (see Rockafellar and Uryasev [14] (2002)). Going by these axioms, VaR is not coherent.

Our focus in this paper is the dynamic portfolio and consumption choice of a trader subject to a risk limit specified in terms of TCE. Yiu [15] (2004) has successfully controlled risky investment by imposing VaR as a dynamic constraint, with a model that applies the VaR constraint over time and emphasizes the repeated re-calculations of the VaR like in practice. He expresses the belief that other risk measures imposed in the same way will achieve similar results. We close that gap here by experimenting with the TCE constraint and extending the utility maximization to cover consumption and terminal wealth. This problem has not yet received adequate attention in the existing literature. We show through numerical simulations by applying an algorithm similar to that in Yiu [15] (2004) that the introduction of a TCE constraint reduces investment in risky assets and increases consumption.

The rest of this paper is structured as follows. In section 2, we model the financial market and describe the portfolio dynamic. Section 3 derives the Value-at-Risk and tail conditional expectation constraints, while section 4 makes precise the optimal control problem to be solved. Section 5 develops the solution of the problem by using the Lagrange technique to combine the Hamilton-Jacobi-Bellman (HJB) equation and the TCE constraint. In section 6, a numerical algorithm is presented to obtain an approximate solution to the TCE-constrained problem. Section 7 presents simulations and section 8 concludes the paper.

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2. **The model**

We consider a standard Black-Scholes type market (see, e.g., Korn [10] (1997) for relevant definitions) consisting of one riskless bond and n risky stocks. The financial market is continuous-time with a finite time horizon [0,T].

Uncertainty in the financial market is modeled by a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, equipped with a filtration that is a non-decreasing family $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ of sub-σ-fields of $\mathcal{F}$

$$\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F} \quad \forall \quad 0 \le s < t < \infty.$$

It is assumed throughout this paper that all inequalities as well as equalities hold $\mathbf{P}$- almost surely. Moreover, it is assumed that all stated processes are well defined without giving any regularity conditions ensuring this. The riskfree rate $r = r_t$ of the riskless asset (bond) S^0 is supposed to evolve according to

$$dS_t^0 = r S_t^0 dt, \ S_0^0 = s. \tag{1}$$

For the risky assets (stocks), for which the prices will be denoted by $S_t = (S_t^1, \ldots, S_t^n)$ for some $n \in \mathbb{N}$, the basic evolution model is that of a log-normal diffusion process.

$$\frac{dS_t^i}{S_t^i} = \mu^i dt + \sum_{j=1}^{k} \sigma^{ij} dW_t^j \tag{2}$$

$$S_0^i = s^i, \ i = 1, \ldots, n \ \forall \ t \in [0, T],$$

where, for some $k \in \mathbb{N}$, $W_t = [W_t^1, \ldots, W_t^k]'$, with the symbol $(')$ standing for transpose, is a k-dimensional Wiener process, i.e., a vector of k independent one-dimensional Wiener processes.

The n-vector $\mu = \mu_t = (\mu_t^1, \ldots, \mu_t^n)'$, contains the expected instantaneous rates of return and the $n \times k$-matrix $\sigma = \sigma_t = \sigma_t^{ij}$, $(i = 1, \ldots, n, \ j = 1, \ldots, k)$ measures the instantaneous sensitivities of the risky asset prices with respect to exogenous shocks so that the $(n \times n)$-matrix $\sigma \sigma'$ contains the variance and covariance rates of instantaneous rates of return. μ and σ must be adapted to the information filtration $\mathbb{F} = (\mathcal{F}_t)$.

An agent invests according to an investment strategy that can be described by the $(n + 1)$-dimensional, $\mathcal{F}_t$-predictable process

$$x_t = (x_t^0, x_t^1, \ldots, x_t^n), \tag{3}$$

where x_t^i, $(i = 1, \ldots, n)$ denotes the number of shares of asset i held in the portfolio at time t ($i = 0$ refers to the bond). The process x describes an investor's portfolio as carried forward through time. The value of the investor's wealth at time t is then

$$V_t^x = x_t^0 S_t^0 + \sum_{i=1}^n x_t^i S_t^i, \tag{4}$$

where $x_t^i S_t^i$ represents the amount invested in asset i at time t.

Equivalently, one may consider the vector

$$\theta_t = (\theta_t^1, \ldots, \theta_t^n),$$

where

$$\theta_t^i = \frac{x_t^i S_t^i}{V_t^x}, \quad (i = 1, \ldots, n)$$

denotes the fraction of wealth invested in the risky asset i at time t.

Let therefore θ_t^i be the proportion of the investor's wealth in the risky security i at time t, for $i = 1 \ldots n$, with the remainder $1 - \sum_{j=1}^n \theta_t^i$ invested in the risk-free asset. Let also c_t be the instantaneous consumption rate. It is assumed that $\theta_t^1, \ldots, \theta_t^n$ and c_t are admissible and $\mathcal{F}_t$- adapted control processes. That is, θ_t^i and c_t are non-anticipative functions that satisfy the condition of bounded variation $\int_0^T \sum_{1=1}^n (\theta_t^i)^2 < \infty$ and $\int_0^T c_t^2 < \infty$ respectively, for an investment time horizon $T < \infty$. The corresponding portfolio value process reads

$$dV_t^\theta = V_t^\theta \left[\left(1 - \sum_{i=1}^n \theta_t^i \right) \frac{dS_t^0}{S_t^0} + \sum_{i=1}^n \theta_t^i \frac{dS_t^i}{S_t^i} \right] - c_t dt, \ V_t^\theta = v$$

$$= V_t^\theta \left[\left(r + \sum_{i=1}^n \theta_t^i (\mu^i - r) \right) dt + \sum_{i=1}^n \sum_{j=1}^k \theta_t^i \sigma^{i,j} dW_t^j \right] - c_t dt, \ V_t^\theta = v. \tag{5}$$

To have a better exposition, we adopt a matrix expression: denote $\sigma = [\sigma^{i,j}]$, $\theta_t = [\theta_t^1 \ldots \theta_t^n]'$, $\mu - r = [\mu^1 - r \ldots \mu^n - r]'$ and $W_t = [W_t^1 \ldots W_t^k]'$, so that σ is an

$n \times k$ matrix, $\mu - r$ and θ_t are n-dimensional column vectors and W_t is a k-dimensional column vector. Hence equation (5) can be rewritten as

$$dV_t^\theta = V_t^\theta \left[(r_t + \theta_t'(\mu_t - r)) \, dt + \theta_t' \sigma_t dW_t \right] - c_t dt, \; V_t^\theta = v. \tag{6}$$

Thus,

$$V_t^\theta = \left(1 - \sum_{i=1}^n \theta_t^i \right) V_t^\theta + \sum_{i=1}^n \theta_t^i V_t^\theta. \tag{7}$$

We have adopted an incomplete market asset pricing setting of He and Pearson [7] (1991). To eliminate redundant assets, we assume that σ is of full row rank-that is, $\sigma\sigma'$ is an invertible matrix.

3. The tail conditional expectation (TCE) constraint

Here we start by defining Value-at-risk since the subsequent definition of tail conditional expectation will depend on it.

Definition 1. *(Value-at-Risk)*

Given some probability level $\alpha \in (0,1)$, the time t wealth benchmark Υ_t and horizon Δt, the Value-at-Risk of time t wealth V_t at the confidence level $(1-\alpha)$, denoted VaR_t^α, is given by the smallest number L such that the probability that the loss $G_{t+\Delta t} := \Upsilon_{t+\Delta t} - V_{t+\Delta t}$ exceeds L is no larger than α.

$$VaR_t^\alpha = \inf \left\{ L \geq 0 : \mathbf{P}(G_{t+\Delta t} \geq L | \mathcal{F}_t) \leq \alpha \right\} := (Q_t^\alpha)^-, \tag{8}$$

where

$$Q_t^\alpha = \sup \left\{ L \in \mathbb{R} : \mathbf{P}((V_{t+\Delta t}^\theta - \Upsilon_{t+\Delta t}) \leq L | \mathcal{F}_t) \leq \alpha \right\} \tag{9}$$

is the quantile of the projected wealth surplus at the horizon $t + \Delta t$ and $x^- = \max[0, -x]$.

Thus $VaR_t^\alpha = 0$ for $Q_t^\alpha > 0$. VaR_t^α is therefore the loss of wealth with respect to a benchmark $\Upsilon_{t+\Delta t}$ at the horizon Δt which could be exceeded only with a small conditional probability α if the current portfolio θ_t were kept unchanged. Typical values for the probability level α are $\alpha = 0.05$ or $\alpha = 0.01$. In market risk management the time horizon Δt is usually one or ten days.

Proposition 1. *(Computation of Value-at-Risk)*

We have

$$VaR_t^\alpha = (Q_t^\alpha)^- = \left(V_t^\theta \exp\left[\Phi^{-1}(\alpha)\|\theta_t'\sigma\|\sqrt{\Delta t} \right.\right.$$
$$\left.\left. + \left(\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta} - \frac{1}{2}\|\theta_t'\sigma\|^2 \right)\Delta t \right] - \Upsilon_{t+\Delta t} \right)^-, \quad (10)$$

where $\Phi(\cdot)$ *and* $\Phi^{-1}(\cdot)$ *denote the normal distribution and the inverse distribution functions respectively, and* $\|\cdot\|$. *stands for norm.*

Proof. The distribution of wealth at time $t + \Delta t$ is approached by

$$V_{t+\Delta t}^\theta = V_t^\theta \exp\left[(\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta} - \frac{1}{2}\|\theta_t'\sigma\|^2)\Delta t + \theta_t'\sigma(W_{t+\Delta t} - W_t) \right], \quad (11)$$

This follows immediately from (6) and Itô's Lemma (see Korn [10] (1997)), if we consider that given a portfolio $\{\theta_t, c_t\}$ and the associated portfolio value V_t at time t, the random variable $V_{t+\Delta t}(V_t, t)$ would be the future value of the portfolio at time $t + \Delta t$ with the portfolio weights being kept constant between time t and time $t + \Delta t$.

In accordance with expression (9) on the definition of VaR_t^α, we have

$$\mathbf{P}\left((V_{t+\Delta t}^\theta - \Upsilon_{t+\Delta t}) \le L|\mathcal{F}_t \right)$$

$$= \mathbf{P}\left(V_t^\theta \exp\left[(\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta} - \frac{1}{2}\|\theta_t'\sigma\|^2)\Delta t + \theta_t'\sigma(W_{t+\Delta t} - W_t) \right] \right.$$
$$\left. -\Upsilon_{t+\Delta t} \le L|\mathcal{F}_t \right)$$

$$= \mathbf{P}\left(\exp\left[(\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta} - \frac{1}{2}\|\theta_t'\sigma\|^2)\Delta t + \theta_t'\sigma(W_{t+\Delta t} - W_t) \right] \right.$$
$$\left. \le \frac{L + \Upsilon_{t+\Delta t}}{V_t^\theta}|\mathcal{F}_t \right)$$

$$= \mathbf{P} \left(\frac{\theta_t' \sigma (W_{t+\Delta t} - W_t)}{\|\theta_t' \sigma\| \sqrt{\Delta t}} \right.$$

$$\left. \leq \frac{\ln \left(\frac{L + \Upsilon_{t+\Delta t}}{V_t^\theta} \right) - \left(\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta} - \frac{1}{2} \|\theta_t' \sigma\|^2 \right) \Delta t}{\|\theta_t' \sigma\| \sqrt{\Delta t}} | \mathcal{F}_t \right)$$

$$= \Phi \left(\frac{\ln \left(\frac{L + \Upsilon_{t+\Delta t}}{V_t^\theta} \right) - \left(\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta} - \frac{1}{2} \|\theta_t' \sigma\|^2 \right) \Delta t}{\|\theta_t' \sigma\| \sqrt{\Delta t}} \right),$$

where $\Phi(\cdot)$ is the cumulative distribution function of a standard normal random variable, given that the random variable $\theta_t' \sigma (W_{t+\Delta t} - W_t)$ is conditionally normally distributed with zero mean and variance $\|\theta_t' \sigma\|^2 \Delta t$. Thus,

$$\mathbf{P} \left((V_{t+\Delta t}^\theta - \Upsilon_{t+\Delta t}) \leq L | \mathcal{F}_t \right) \leq \alpha$$

$$\Longleftrightarrow \Phi \left(\frac{\ln \left(\frac{L + \Upsilon_{t+\Delta t}}{V_t^\theta} \right) - \left(\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta} - \frac{1}{2} \|\theta_t' \sigma\|^2 \right) \Delta t}{\|\theta_t' \sigma\| \sqrt{\Delta t}} \right) \leq \alpha$$

$$\Longleftrightarrow \ln \left(\frac{L + \Upsilon_{t+\Delta t}}{V_t^\theta} \right) \leq \Phi^{-1}(\alpha) \|\theta_t' \sigma\| \sqrt{\Delta t} + \left(\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta} - \frac{1}{2} \|\theta_t' \sigma\|^2 \right) \Delta t$$

$$\Longleftrightarrow L \leq V_t^\theta \exp \left[\Phi^{-1}(\alpha) \|\theta_t' \sigma\| \sqrt{\Delta t} + \left(\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta} - \frac{1}{2} \|\theta_t' \sigma\|^2 \right) \Delta t \right]$$
$$- \Upsilon_{t+\Delta t},$$

which implies

$$Q_t^\alpha = V_t^\theta \exp \left[\Phi^{-1}(\alpha) \|\theta_t' \sigma\| \sqrt{\Delta t} + \left(\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta} - \frac{1}{2} \|\theta_t' \sigma\|^2 \right) \Delta t \right] - \Upsilon_{t+\Delta t}.$$

Therefore,

$$VaR_t^\alpha = (Q_t^\alpha)^- = -V_t^\theta \exp \left[\Phi^{-1}(\alpha) \|\theta_t' \sigma\| \sqrt{\Delta t} \right.$$
$$\left. + \left(\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta} - \frac{1}{2} \|\theta_t' \sigma\|^2 \right) \Delta t \right] + \Upsilon_{t+\Delta t}.$$

$$\square$$

Tail conditional expectation is closely related to the Value-at-Risk concept, but overcomes some of the conceptual deficiencies of Value-at-Risk (cf. Rockafellar

and Uryasev [14] (2002)). In particular, it is a coherent risk measure (see Artzner et al. [2] (1997)).

Definition 2. *(Tail conditional expectation)*
Consider distribution of the loss $G_{t+\Delta t} := \Upsilon_{t+\Delta t} - V_{t+\Delta t}$ represented by a continuous distribution function $F_{G_{t+\Delta t}}$ with $\int_{\mathbb{R}} |G_{t+\Delta t}| dF(G_{t+\Delta t}) < \infty$. Then the TCE_t^α at confidence level $(1 - \alpha)$ is defined as

$$TCE_t^\alpha = \mathbb{E}_t \left\{ (\Upsilon_{t+\Delta t} - V_{t+\Delta t}^\theta) \geq VaR_t^\alpha | \mathcal{F}_t \right\}.$$

$$TCE_t^\alpha = \frac{\mathbb{E}_t \left\{ (\Upsilon_{t+\Delta t} - V_{t+\Delta t}^\theta) I((\Upsilon_{t+\Delta t} - V_{t+\Delta t}) \geq -Q_t^\alpha) | \mathcal{F}_t \right\}^+}{\alpha},$$

where $I(A)$ is the indicator function of the set A and $x^+ = \max[0, x]$.

In other words, the tail conditional expectation of wealth V_t at time t is the conditional expected value of the loss exceeding $(Q_t^\alpha)^-$. Again, given the log-normal distribution of asset returns, the TCE_t^α can be explicitly computed as can be seen in the following proposition.

Proposition 2. *(Computation of tail conditional expectation)*
We have

$$TCE_t^\alpha = \frac{\alpha \Upsilon_{t+\Delta t} - V_t \left[\exp \left((\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta}) \Delta t \right) \Phi \left(\Phi^{-1}(\alpha) - \|\theta_t'\sigma\|\sqrt{\Delta t} \right) \right]}{\alpha}.$$

where $\Phi(\cdot)$ and $\Phi^{-1}(\cdot)$ denote the normal distribution and the inverse distribution functions.

Proof.

$$\mathbb{E} \left\{ (\Upsilon_{t+\Delta t} - V_{t+\Delta t}^\theta) I(\Upsilon_{t+\Delta t} - V_{t+\Delta t} \geq -Q_t^\alpha) | \mathcal{F}_t \right\}$$

$$= \mathbb{E} \left\{ \left(\Upsilon_{t+\Delta t} - V_t^\theta \exp \left[(\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta} - \frac{1}{2}\|\theta_t'\sigma\|^2) \Delta t \right. \right. \right.$$
$$\left. \left. \left. + \theta_t'\sigma(W_{t+\Delta t} - W_t)] \right) I(\Upsilon_{t+\Delta t} - V_{t+\Delta t} \geq -Q_t^\alpha) | \mathcal{F}_t \right\} \quad (12)$$

The argument of the indicator function is evaluated as follows

$$\Upsilon_{t+\Delta t} - V_t^\theta \exp\left[(\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta} - \frac{1}{2}\|\theta_t'\sigma\|^2)\Delta t + \theta_t'\sigma(W_{t+\Delta t} - W_t) \right]$$

$$\geq -V_t^\theta \exp\left[\Phi^{-1}(\alpha)\|\theta_t'\sigma\|\sqrt{\Delta t} + \left(\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta} - \frac{1}{2}\|\theta_t'\sigma\|^2\right)\Delta t \right]$$

$$+ \Upsilon_{t+\Delta t}$$

$$\Rightarrow -V_t^\theta \exp\left[\theta_t'\sigma(W_{t+\Delta t} - W_t)\right] \geq -V_t^\theta \exp\left[\Phi^{-1}(\alpha)\|\theta_t'\sigma\|\sqrt{\Delta t}\right]$$

$$= \frac{\theta_t'\sigma(W_{t+\Delta t} - W_t)}{\|\theta_t'\sigma\|\sqrt{\Delta t}} \leq \Phi^{-1}(\alpha).$$

Therefore (12) becomes

$$\mathbb{E}\left\{ \left(\Upsilon_{t+\Delta t} - V_t \exp\left[(\theta_t'(\mu - r) + r - \frac{c_t}{V_t} - \frac{1}{2}\|\theta_t'\sigma\|^2)\Delta t \right.\right.\right.$$

$$\left.\left.\left. +\theta_t'\sigma(W_{t+\Delta t} - W_t)\right]\right) I \left(\frac{\theta_t'\sigma(W_{t+\Delta t} - W_t)}{\|\theta_t'\sigma\|\sqrt{\Delta t}} \leq \Phi^{-1}(\alpha) \right) |\mathcal{F}_t \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\Phi^{-1}(\alpha)} \left(\Upsilon_{t+\Delta t} - V_t^\theta \exp\left[(\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta} \right.\right.$$

$$\left.\left. -\frac{1}{2}\|\theta_t'\sigma\|^2)\Delta t + \theta_t'\sigma x\sqrt{\Delta t} \right] \right) \exp\frac{-x}{2} dx$$

$$= \alpha\Upsilon_{t+\Delta t} - V_t^\theta \left[\exp((\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta})\Delta t) \right.$$

$$\left. \cdot \int_{-\infty}^{\Phi^{-1}(\alpha)} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x - \|\theta_t'\sigma\|\sqrt{\Delta t})^2}{2} \right) dx \right]$$

We calculate the integral by change of variables and obtain

$$= \alpha\Upsilon_{t+\Delta t} - V_t^\theta \left[\exp\left((\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta})\Delta t \right) \Phi\left(\Phi^{-1}(\alpha) - \|\theta_t'\sigma\|\sqrt{\Delta t} \right) \right].$$

Dividing by α, we obtain the Tail Conditional Expectation as

$$TCE_t^\alpha = \frac{\alpha\Upsilon_{t+\Delta t} - V_t^\theta \left[\exp\left((\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta})\Delta t \right) \Phi\left(\Phi^{-1}(\alpha) - \|\theta_t'\sigma\|\sqrt{\Delta t} \right) \right]}{\alpha}.$$

$$\square$$

4. Problem statement

We seek the optimal asset and consumption allocation that maximizes (over all allowable $\{\theta_t, c_t\}$) the expected utility of discounted terminal wealth at time T and consumption over the entire horizon $[0, T]$, for a risk averse investor who limits his risk by imposing an upper bound on the TCE.

In mathematical terms the final optimal control problem with TCE constraint is

$$\max_{\{\theta,\ c\} \in A(v)} \mathbb{E}_{0, V_0} \left\{ \int_0^T e^{-\rho s} U^1(c_s, s) ds + e^{-\rho T} U^2(V_T, T) \right\}, \tag{13}$$

subject to the wealth dynamics

$$dV_t^\theta = \left[V_t^\theta (\theta_t'(\mu - r) + r) \right] dt - c_t dt + V_t^\theta \theta_t' \sigma dW_t, \ V_0^\theta = v$$

and the TCE constraint

$$\frac{1}{\alpha} \left(\alpha \Upsilon_{t+\Delta t} - V_t^\theta \left[\exp \left((\theta_t'(\mu - r) + r - \frac{c_t}{V_t^\theta}) \Delta t \right) \right. \right.$$
$$\left. \left. \cdot \Phi \left(\Phi^{-1}(\alpha) - \|\theta_t' \sigma\| \sqrt{\Delta t} \right) \right] \right) \le \varepsilon, \ \forall \ t \in [0, T), \tag{14}$$

where $\mathbb{E}$ denotes the expectation operator, given $V_t^\theta = v$ (and given the chosen consumption and investment strategies), U^1 and U^2 are twice differentiable, increasing, concave utility functions (CRRA), ε is an upper bound on TCE and $\rho > 0$ is the rate at which consumption and terminal wealth are discounted. Furthermore, we let

$$U(x) = U^1(x) = U^2(x) = \frac{x^{1-\gamma}}{1 - \gamma},$$

where $\gamma \in (0, \infty) \backslash \{1\}$.

5. Optimality conditions

In applying the dynamic programming approach we solve the HJB equation associated with the utility maximization problem (13). From Fleming and Rishel [6]

(1975) we have that the corresponding HJB equation is given by

$$\rho J(v,t) = \sup_{c_t \geq 0,\ \theta_t \in \mathbb{R}^n} \left\{ U(c_t) + J_t(v,t) + J_v(v,t)\left(v[\theta_t'(\mu - r) + r] - c_t\right) \right.$$
$$\left. + \frac{1}{2}J_{vv}(v,t)v^2\theta_t'\sigma\sigma'\theta_t \right\}, \quad (15)$$

subject to the terminal condition

$$J(v,T) = U(v),$$

where J, the value function is given by

$$J(v,t) = \max_{\{\theta,\ c\} \in A(v)} \mathbb{E}_{t,V_t} \left\{ \int_t^T e^{-\rho s} U(c_s,s)ds + e^{-\rho T} U(V_T,T) \right\}, \quad (16)$$

where subscripts on J denote partial derivatives and $V_t^\theta = v$, the wealth realization at time t.

In solving the HJB equation (15), the static optimization problem

$$\max_{c_t \geq 0,\ \theta_t \in \mathbb{R}^n} \left\{ U(c_t) + J_v(v,t)\left(v[\theta_t'(\mu - r) + r] - c_t\right) + \frac{1}{2}J_{vv}(v,t)v^2\theta_t'\sigma\sigma'\theta_t \right\}, \quad (17)$$

subject to the TCE constraint (14) can be tackled separately to reduce the HJB equation (15) to a nonlinear partial differential equation of J only.

Introducing the Lagrange function $\mathcal{L}(\cdot)$ as

$$\mathcal{L}\left(\theta(v,t),c(v,t),\lambda(v,t)\right) = J_v(v,t)\left(v\left[\theta_t'(\mu - r) + r - c_t\right]\right)$$
$$+ \frac{1}{2}v^2\|\theta_t'\sigma\|^2 J_{vv}(v,t) + U(c_t) - \lambda(v,t)\left(\alpha\Upsilon_{t+\Delta t}\right.$$
$$\left. - v\left[\exp\left(\left(\theta_t'(\mu - r) + r - \frac{c_t}{v}\right)\Delta t\right) \cdot \Phi\left(\Phi^{-1}(\alpha) - \|\theta_t'\sigma\|\sqrt{\Delta t}\right)\right] - \varepsilon_1\right), \quad (18)$$

where λ is the Lagrange multiplier and $\varepsilon_1 = \varepsilon \cdot \alpha$ and the first-order necessary conditions with respect to θ, c and λ respectively of the static optimization problem (18)

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are given by

$$vJ_v(\mu - r) + \frac{1}{2}J_{vv}v^2\sigma\sigma'\theta_t + \lambda(v,t)v\left[(\mu - r)\Delta t \exp\left((\theta_t'(\mu - r) + r - \frac{c_t}{v})\Delta t\right)\right.$$

$$\cdot\Phi\left(\Phi^{-1}(\alpha) - \|\theta_t'\sigma\|\sqrt{\Delta t}\right) - \exp\left((\theta_t'(\mu - r) + r - \frac{c_t}{v})\Delta t\right)\cdot\frac{\sqrt{\Delta t}}{2}\frac{\sigma\sigma'\theta_t}{\|\theta_t'\sigma\|}$$

$$\left.\cdot\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}(\Phi^{-1}(\alpha) - \|\theta_t'\sigma\|\sqrt{\Delta t})^2\right]\right] = 0, \quad (19)$$

whereby $\pi \approx 3.14159$ and we have applied the product law of differentiation and the fundamental theorem of calculus in deriving the latter first-order derivative.

$$U_c(c_t) + \lambda(v,t)\Delta t \cdot \Phi\left(\Phi^{-1}(\alpha) - \|\theta_t'\sigma\|\sqrt{\Delta t}\right)$$

$$\cdot\left[\exp\left((\theta_t'(\mu - r) + r - \frac{c_t}{v})\Delta t\right)\cdot\Phi\left(\Phi^{-1}(\alpha) - \|\theta_t'\sigma\|\sqrt{\Delta t}\right)\right] = J_v(v,t), \quad (20)$$

where U_c is the first-order derivative of U with respect to c and

$$H(v,t) = \alpha\Upsilon_{t+\Delta t} + v\left[\exp\left((\theta_t'(\mu - r) + r - \frac{c_t}{v})\Delta t\right)\right.$$

$$\left.\cdot\Phi\left(\Phi^{-1}(\alpha) - \|\theta_t'\sigma\|\sqrt{\Delta t}\right)\right] + \varepsilon_1 = 0, \quad (21)$$

while the complimentary slackness condition is given as

$$\lambda(v,t)H(v,t) = 0, \quad (22)$$

$$\lambda(v,t) \geq 0.$$

Simultaneous resolution of these first-order conditions yields the optimal solutions θ^{opt}, c^{opt} and λ^{opt}. Substituting these into (15) gives the partial differential equation

$$-\rho J(v,t) + \frac{(c^{opt}(v,t))^{1-\gamma}}{1-\gamma} + J_t(v,t) + J_v(v,t)\left(v[(\theta^{opt}(v,t))'(\mu - r) + r]\right.$$

$$\left.-c^{opt}(v,t)\right) + \frac{1}{2}J_{vv}(v,t)v^2(\theta^{opt}(v,t))'\sigma\sigma'(\theta^{opt}(v,t)) = 0, \quad (23)$$

with terminal condition

$$J(v,T) = \frac{v^{1-\gamma}}{1-\gamma},$$

which can then be solved for the optimal value function $J^{opt}(v,t)$. Because of the non-linearity in θ^{opt} and c^{opt}, the first-order conditions together with the HJB equation

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are a non-linear system so the stochastic differential equation (23) has no analytic solution and numerical methods such as Newton's method or Sequential Quadratic Programming (SQP)(see, e.g., Nocedal and Wright [13] (1999)) are required to solve for $\theta^{opt}(v,t)$, $c^{opt}(v,t)$, $\lambda^{opt}(v,t)$ and $J^{opt}(v,t)$ iteratively.

6. Numerical method

We use an iterative algorithm similar to that of Yiu [15] (2004) which yields a $\mathcal{C}^{2,1}$ approximation $\widehat{J}$ of the exact solution J. $\{\widehat{\theta}_t, \widehat{c}_t\}$ is the investment strategy related to $\widehat{J}$.

When the optimal solution strictly satisfies the TCE constraint (14), the Lagrange multiplier $\lambda(v,t)$ is zero. If the constraint is active, the multiplier is positive.

First, we divide the domain of resolution into a grid of $n_v \times n_t$ mesh points. Iterations are indexed by k.

1. For each point (t,v), with $t \in [0, \Delta t, \ldots, n_t \Delta t]$ and $v \in [0, \Delta v, \ldots, n_v \Delta v]$, we compute the value function $\widehat{J}^{k=0} = J(v,t)$ and the optimal strategy $\{\theta_t^{opt}, c_t^{opt}\}$ of the unconstrained problem. All Lagrange multipliers are set to zero, $\lambda_{t,v}^{k=0} = 0$. This solution is the starting point of the algorithm.

2. For all points of the grid, the constraint is checked. If the constraint is not active ($TCE_t^\alpha < \varepsilon$), the multiplier is zero $\lambda_{t,v}^{k+1} = 0$ and $\{\theta_t^{k+1}, c_t^{k+1}\}$ is the solution of a similar equation to that of the unconstrained case.

$$\lambda_{t,v}^{k+1} = 0,$$

$$\theta_t^{k+1} = -\frac{\widehat{J}_v}{v\widehat{J}_{vv}}(\mu - r)(\sigma^T \sigma)^{-1},$$

$$\widehat{U}_c(c_t^{k+1}) = \widehat{J}_v.$$

If the VaR_t^α constraint is active, ($VaR_t^\alpha \geq \varepsilon$), we solve a nonlinear system in $\lambda_{t,v}^{k+1}$, $\widehat{\theta}_t^{j+1}$ and $\widehat{c}_t^{j+1}$. This nonlinear system is composed of the first-order necessary conditions of the static optimization problem

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(18). That system is numerically solved by the sequential quadratic programming method (see Nocedal and Wright [13] (1999)).

3. The last stage consists in the calculation of the value function $\widehat{J}^{k+1}$ according to the investment/consumption strategy $\{\widehat{\theta}_t^{k+1}, \widehat{c}_t^{k+1}\}$ as detailed below this algorithm.

4. Return to step 2 with $k = k+1$ until the error at time t from wealth level v, $\epsilon_{t,v}$, satisfies $|\epsilon_{t,v}| < 1 \cdot e^{-5}$, where

$$\epsilon_{t,v} = \hat{J}_t - \rho \hat{J}(v,t) + \hat{J}_v \left(v[(\hat{\theta}_t^{opt})'(\mu - r) + r] - c_t^{opt} \right)$$

$$+ \frac{1}{2} v^2 \|(\hat{\theta}_t^{opt})'\sigma\|^2 \hat{J}_{vv} + U(c_t^{opt}).$$

For the numerical solution of the partial differential equation (23) to obtain the value function we use the trial function

$$J(v,t) = f(t) \frac{v^{1-\gamma}}{1-\gamma}, \qquad f(T) = 1,$$

such that

$$\begin{aligned} J_t &= f'(t) \frac{1}{1-\gamma} v^{1-\gamma} \\ J_v &= f(t) v^{-\gamma} \\ J_{vv} &= -\gamma f(t) v^{-(\gamma+1)}. \end{aligned}$$

Substituting these partials in (23) and dividing by $v^{1-\gamma}$, after some tedious computation, we obtain the ordinary differential equation

$$f'(t) = -\kappa(\theta^{opt}(v,t), c^{opt}(v,t), v) f(t) - B(c^{opt}(v,t), v), \tag{24}$$

whereby

$$\kappa(\theta^{opt}(v,t), c^{opt}(v,t), v) = (1-\gamma) \left(\frac{-\rho}{1-\gamma} + (\theta^{opt}(v,t))'(\mu - r) \right.$$

$$\left. -c^{opt}(v,t) v^{-1} - \frac{1}{2} v^2 (\theta^{opt}(v,t))' \sigma \sigma'(\theta^{opt}(v,t)) \right)$$

and

$$B(c^{opt}(v,t), v) = (c^{opt}(v,t))^{1-\gamma} v^{\gamma-1},$$

with terminal condition

$$f(T) = 1.$$

The function f in equation (24) is computed numerically by the Euler-Cauchy method (See Isaacson and Keller [8] (1994)).

We have implemented the above algorithm to illustrate the optimal portfolio of the preceding section with examples. To this end, we have written a program in MATLAB 7.0 to carry out the procedure and run it on a personal computer with an Intel Pentium IV processor. We assume that $n = 2$. That is, the market is composed of two risky stocks and a risk-free bond. Table 1 shows the parameters for the portfolio optimization problem and the underlying Black-Scholes model of the financial market. We achieve convergence in 300 seconds after three iterations.

Parameter	Value
Stock (S^1)	$\mu = 4\%$, $\sigma^{11} = 5\%$, $\sigma^{12} = 5\%$
Stock (S^2)	$\mu = 6\%$, $\sigma^{21} = 5\%$, $\sigma^{22} = 20\%$
Bond (S^0)	$r = 3\%$
Investment horizon	$t \in [0, 1]$
State of wealth	$v \in [0, 20]$
Shortfall probability	$\alpha = 1\%$
Value-at-Risk horizon	$\Delta t = \frac{1}{48} \approx 7$ days
No. of wealth mesh points	$N_v = 81$
Mesh size for wealth	$\Delta v = \frac{20}{80} = 0.25$
Utility function	$U(x) = \frac{x^{1-\gamma}}{1-\gamma}$, $\gamma = 0.9$

TABLE 1. Parameters for the consumption and investment portfolio optimization problem.

Wealth benchmark, Υ_t	Bound, ε
conditional expectation	0.3
Money market	1.0

TABLE 2. Bounds and benchmarks for the TCE-constrained problem.

7. Simulations

We consider the tail conditional expectation of the wealth surplus $V_t - \Upsilon_{t+\Delta t}$ with respect to the benchmark $\Upsilon_{t+\Delta t}$ such that it satisfies

$$TCE_t^\alpha(V_{t+\Delta t} - \Upsilon_{t+\Delta t}) \leq \varepsilon,$$

where ε comes from table 2. That is, the TCE is re-evaluated at each discrete time step (TCE horizon) Δt and kept below the upper bound ε, by making use of conditioning information. Figures 1 and 2 show in the right panel the amount of wealth invested in the risky assets with and without the TCE constraint, plotted against the possible wealth realization at different times. The left panel shows the value function.
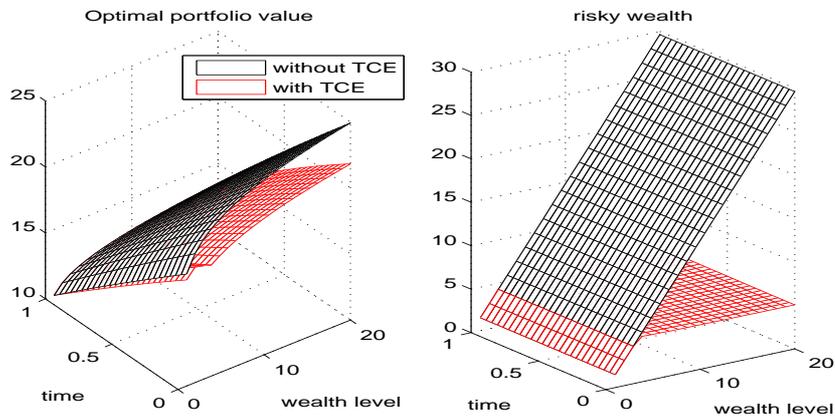

FIGURE 1. TCE when benchmark is the conditional expected wealth plotted against wealth at various times of the investment horizon. In red, TCE $\leq \varepsilon = 0.3$.

In Figure 1 the shortfall benchmark is taken to be the conditional expected wealth $\Upsilon_{t+\Delta t} = \mathbb{E}_t\{V_{t+\Delta t}\}$, given as

$$\Upsilon_{t+\Delta t} = \mathbb{E}_t\{V_{t+\Delta t}\} = V_t \exp\left[\left(\theta_t'(\mu - r) + r - \frac{c_t}{V_t}\right)\Delta t\right], \tag{25}$$

while in Figure 2 it is the investment in the risk-free bond $\Upsilon_{t+\Delta t} = V_t e^{r\Delta t}$.
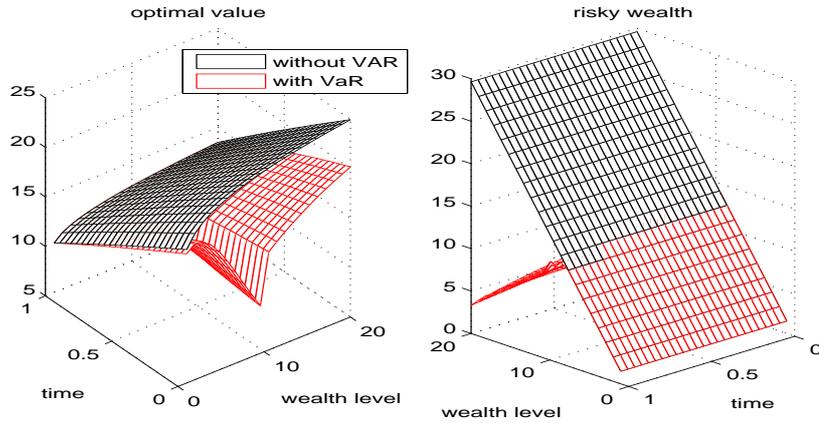

FIGURE 2. Effect of the TCE constraint when benchmark is investment in the bond.

As can be observed from the images, as the wealth level increases, so does the investment in risky assets. This results from the property of constant relative risk aversion of the utility function. A good control over the investment in the risky assets has been achieved and the proportions invested in the risky assets are reduced in order to fulfil the TCE constraint. In particular, when the constraint is not active, the optimal portfolio follows the unconstrained solution; as the portfolio value increases, the TCE constraint becomes active and allocates less to the risky assets. Figure 3 reveals to us that the local minimum (around wealth level 10) observed in the left panel of Figure 2 comes as a result of a sudden increase in the consumption rate once the constraint becomes active. The left panel of Figure 1 suggests that this increase in consumption is more subtle when we take as wealth benchmark, the conditional expected wealth.
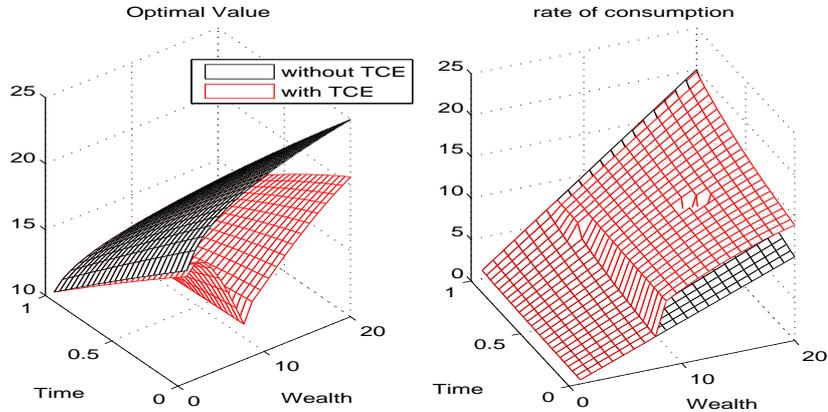
FIGURE 3. Effect of the TCE constraint on consumption when benchmark is investment in the bond.

The value function of the constrained problem is identical to that of the unconstrained one when the Lagrange multipliers are null, whereas it is inferior when the constraint is active.

8. Concluding remarks

Using a CRRA utility function, we have investigated how a bound imposed on TCE affects the optimal portfolio choice and consumption. In so doing, we have used dynamic wealth benchmarks - conditional expected wealth and investment in riskless stocks, whereby the TCE was re-evaluated at short intervals along the investment horizon. We deduce from our observations that the constraint reduces risky investment. Moreover, part of the wealth hitherto invested in risky assets is diverted to consumption when the constraint is tight.

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