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ON α-CONVEX ANALYTIC FUNCTIONS DEFINED BY GENERALIZED RUSCHEWEYH DERIVATIVES OPERATOR

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Abstract. In this paper we introduce a class of alpha-convex functions by using the generalised Ruscheweyh derivative operator. We study properties of this class and give a theorem about the image of a function from this class through the Bernardi integral operator.

1. **Introduction**

Let $\mathcal{H}=\mathcal{H}(U)$ denote the class of functions analytic in $U=\{z\in\mathbb{C}:|z|<1\}$. For n a positive integer and $a\in\mathbb{C}$, let

$$\mathcal{H}[a,n]=\{f\in\mathcal{H}:f(z)=a+a_nz^n+...\}.$$

We also consider the class

$$\mathcal{A}=\left\{f\in\mathcal{H}:f(z)=z+a_2z^2+...\right\}.$$

We denote by $\mathcal{Q}$ the set of functions f that are analytic and injective on $\overline{U}\setminus E(f)$, where

$$E(f)=\left\{\zeta\in\partial U:\lim_{z\to\zeta}f(z)=\infty\right\},$$

and are such that $f'(\zeta)\neq 0$ for $\zeta\in\partial U\setminus E(f)$.

Since the functions considered in this paper and conditions on them are defined uniformly in the unit disk U, we shall omit the requirement "$z\in U$".

We use the terms of subordination and superordination, so we review here those definitions. Let $f,F\in\mathcal{H}$. The function f is said to be *subordinate* to F, or

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F is said to be *superordinate* to f, if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, and such that $f(z) = F(w(z))$. In such a case we write $f \prec F$ or $f(z) \prec F(z)$. If F is univalent, then $f \prec F$ if and only if $f(0) = F(0)$ and $f(U) \subset F(U)$.

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$, let h be a univalent function in U and $q \in \mathcal{Q}$. In [7], the authors considered the problem of determining conditions on admissible functions ψ such that

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) \prec h(z) \tag{1}$$

implies $p(z) \prec q(z)$, for all functions $p \in \mathcal{H}[a,n]$ that satisfy the differential subordination (1). Moreover, they found conditions such that the function q is the "smallest" function with this property, called the best dominant of the subordination (1).

Let $\varphi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$, let $h \in \mathcal{H}$ and $q \in \mathcal{H}[a,n]$. Recently, in [8], the authors studied the dual problem and determined conditions on φ such that

$$h(z) \prec \varphi\left(p(z), zp'(z), z^2p''(z); z\right) \tag{2}$$

implies $q(z) \prec p(z)$, for all functions $p \in \mathcal{Q}$ that satisfy the above differential superordination. Moreover, they found conditions such that the function q is the "largest" function with this property, called the best subordinant of the superodination (2).

In the present paper we shall also need a recent generalization of the Ruscheweyh derivatives. This was introduced in the paper [3].

Let $f \in \mathcal{A}$, $\lambda \geq 0$ and $m \in \mathbb{R}$, $m > -1$, then we consider

$$\mathcal{D}_\lambda^m f(z) = \frac{z}{(1-z)^{m+1}} * \mathcal{D}_\lambda f(z),\ z \in U,$$

where $\mathcal{D}_\lambda f(z) = (1-\lambda) f(z) + \lambda z f'(z)$, $z \in U$.

If $f \in \mathcal{A}$, $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$, $z \in U$, we obtain the power series expansion of the form

$$\mathcal{D}_\lambda^m f(z) = z + \sum_{n=2}^{\infty} [1 + (n-1)\lambda] \frac{(m+1)_{n-1}}{(1)_{n-1}} a_n z^n,\ z \in U,$$

where $(a)_n$ is the Pochhammer symbol, given by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & \text{for } n = 0 \\ a(a+1)(a+2)\dots(a+n-1), & \text{for } n \in \mathbb{N}^*. \end{cases}$$

In the case $m \in \mathbb{N}$, we have

$$\mathcal{D}_{\lambda}^{m} f(z) = \frac{z\left(z^{m-1}\mathcal{D}_{\lambda} f(z)\right)^{(m)}}{m!}, \; z \in U,$$

and for $\lambda = 0$ we obtain the m-th Ruscheweyh derivative introduced in [12], $\mathcal{D}_0^m = \mathcal{D}^m$.

We next introduce the two classes of α-convex functions by using the generalized Ruscheweyh derivatives.

Definition 1.1. *Let q be a univalent function in U, with $q(0) = 1$ and such that $D = q(U)$ is a convex domain from the right half-plane. We consider $\alpha \in [0,1]$, $\lambda \geq 0$ and $m \in \mathbb{N}^*$. The function $f \in \mathcal{A}$ is said to be in the class*

(i) $M_{\alpha}(m,\lambda,q)$, *if*

$$J(\alpha,m,\lambda,f;z) = (1-\alpha)\frac{z\left(\mathcal{D}_{\lambda}^{m} f(z)\right)'}{\mathcal{D}_{\lambda}^{m} f(z)} + \alpha\frac{\left(z\left(\mathcal{D}_{\lambda}^{m} f(z)\right)'\right)'}{\left(\mathcal{D}_{\lambda}^{m} f(z)\right)'} \prec q(z),$$

for $z \in U$, or, equivalently,

$$J(\alpha,m,\lambda,f;z) = (1-\alpha)\frac{z\left(\mathcal{D}_{\lambda}^{m} f(z)\right)'}{\mathcal{D}_{\lambda}^{m} f(z)} + \alpha\left(1 + \frac{z\left(\mathcal{D}_{\lambda}^{m} f(z)\right)''}{\left(\mathcal{D}_{\lambda}^{m} f(z)\right)'}\right) \prec q(z).$$

(ii) $\overline{M}_{\alpha}(m,\lambda,q)$, *if*

$$q(z) \prec J(\alpha,m,\lambda,f;z).$$

Subclasses of $M_{\alpha}(m,\lambda,q)$ were studied by several authors, out of which we mention

$$M_0(0,0,q) = S^*(q),$$
$$M_{\alpha}(0,0,q) = M_{\alpha}(q),$$
$$M_0(0,0,q_{\gamma}) = S^*(\gamma), \text{ where } q_{\gamma}(z) = \frac{1+(1-2\gamma)z}{1-z}, \; 0 \leq \gamma < 1,$$
$$M_{\alpha}(0,0,\varphi) = M_{\alpha}, \text{ for } \varphi(z) = \frac{1+z}{1-z},$$
$$M_0(m,0,\varphi) = R_n,$$
$$M_0(m,0,q_{\gamma}) = R_n(\gamma).$$

The class $S^*(q)$ was introduced by W. Ma and D. Minda in [5], the class $M_\alpha(q)$ was studied by V. Ravichandran and M. Darus in [11], M_α is the class of α-convex functions introduced by P.T. Mocanu in [10], R_n is the class defined by R. Singh and S. Singh in [13], and $R_n(\gamma)$ makes the object of the papers of O.P. Ahuja, [1] and [2].

We shall use the following notations

$$S^*_{m,\lambda}(q) = \left\{ f \in \mathcal{A} : \frac{z\left(\mathcal{D}^m_\lambda f(z)\right)'}{\mathcal{D}^m_\lambda f(z)} \prec q(z), z \in U \right\}$$

and

$$\overline{S}^*_{m,\lambda}(q) = \left\{ f \in \mathcal{A} : q(z) \prec \frac{z\left(\mathcal{D}^m_\lambda f(z)\right)'}{\mathcal{D}^m_\lambda f(z)}, z \in U \right\}.$$

2. **Preliminaries**

In our present investigation we shall need the folllowing results concerning Briot-Bouquet differential subordinations, and generalizations of Briot-Bouquet differential subordinations and superordinations.

Theorem 2.1 ([4]). *Let $\beta, \gamma \in \mathbb{C}$, $\beta \neq 0$ and consider the convex function h, such that*

$$\operatorname{Re}\left[\beta h(z) + \gamma\right] > 0,\ z \in U.$$

If $p \in \mathcal{H}[h(0), n]$, then

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \Rightarrow p(z) \prec h(z).$$

Theorem 2.2 ([6]). *Let q be a univalent function in U and consider θ and φ to be analytic functions in a domain $D \supset q(U)$, such that $\varphi(w) \neq 0$, for all $w \in q(U)$. We denote by $Q(z) = zq'(z) \cdot \varphi[q(z)]$, $h(z) = \theta[q(z)] + Q(z)$ and assume that*

(i) *h is convex, or*
(ii) *Q is starlike.*

We further suppose that

(iii) $\operatorname{Re} \dfrac{zh'(z)}{Q(z)} = \operatorname{Re}\left[\dfrac{\theta'[q(z)]}{\varphi[q(z)]} + \dfrac{zQ'(z)}{Q(z)}\right] > 0.$

If p is an analytic function in U, with $p(0) = q(0)$, $p(U) \subseteq D$ and such that

$$\theta[p(z)] + zp'(z)\varphi[p(z)] \prec \theta[q(z)] + zq'(z)\varphi[q(z)] = h(z)$$

then

$$p(z) \prec q(z)$$

and q is the best dominant.

Theorem 2.3 ([9])**.** *Let θ, φ be analytic functions in a domain D and consider q a univalent function in U, such that $q(0) = a$, $q(U) \subset D$. We define $Q(z) = zq'(z) \cdot \varphi[q(z)]$, $h(z) = \theta[q(z)] + Q(z)$ and suppose that*

(i) $\operatorname{Re}\left[\dfrac{\theta'[q(z)]}{\varphi[q(z)]}\right] > 0$ *and*
(ii) *Q is starlike.*

If $p \in \mathcal{H}[a,1] \cap \mathcal{Q}$, $p(U) \subset D$ and $\theta[p(z)] + zp'(z) \cdot \varphi[p(z)]$ is univalent in U, then

$$\theta[q(z)] + zq'(z) \cdot \varphi[q(z)] \prec \theta[p(z)] + zp'(z) \cdot \varphi[p(z)] \Rightarrow q(z) \prec p(z)$$

and q is the best subordinant.

3. Main results

Theorem 3.1. *Let $\alpha \in [0,1]$. Then $f \in M_\alpha(m, \lambda, q)$ if and only if the function g defined by*

$$g(z) = \mathcal{D}_\lambda^m f(z)\left[\frac{z\left(\mathcal{D}_\lambda^m f(z)\right)'}{\mathcal{D}_\lambda^m f(z)}\right]^\alpha, \ z \in U$$

belongs to $S^*_{m,\lambda}(q)$. The branch of the power function is chosen such that

$$\left.\left[\frac{z\left(\mathcal{D}_\lambda^m f(z)\right)'}{\mathcal{D}_\lambda^m f(z)}\right]^\alpha\right|_{z=0} = 1.$$

Proof. We calculate the logarithmic derivative of g and obtain

$$\frac{zg'(z)}{g(z)} = \frac{z\left(\mathcal{D}_\lambda^m f(z)\right)'}{\mathcal{D}_\lambda^m f(z)} + \alpha\left[1 + \frac{z\left(\mathcal{D}_\lambda^m f(z)\right)''}{\left(\mathcal{D}_\lambda^m f(z)\right)'} - \frac{z\left(\mathcal{D}_\lambda^m f(z)\right)'}{\mathcal{D}_\lambda^m f(z)}\right],$$

or

$$\frac{zg'(z)}{g(z)} = J(\alpha, m, \lambda, f; z).$$

The equivalence from the hypothesis is immediately verified. □

Theorem 3.2. *If the function f belongs to the class $M_\alpha(m,\lambda,q)$, for a given $\alpha \in (0,1]$, then $f \in S^*_{m,\lambda}(q)$.*

Proof. We define the function p to be given by

$$p(z) = \frac{z\left(\mathcal{D}^m_\lambda f(z)\right)'}{\mathcal{D}^m_\lambda f(z)}.$$

The logarithmic derivative of p is

$$\frac{zp'(z)}{p(z)} = 1 + \frac{z\left(\mathcal{D}^m_\lambda f(z)\right)''}{\left(\mathcal{D}^m_\lambda f(z)\right)'} - \frac{z\left(\mathcal{D}^m_\lambda f(z)\right)'}{\mathcal{D}^m_\lambda f(z)},$$

thus

$$p(z) + \alpha\frac{zp'(z)}{p(z)} = J(\alpha, m, \lambda, f; z).$$

Because $f \in M_\alpha(m,\lambda,q)$, we get

$$p(z) + \alpha\frac{zp'(z)}{p(z)} \prec q(z).$$

The function q was supposed to be convex and we also assumed that the image $q(U)$ is in the right half-plane. We have $\alpha \in (0,1]$, and therefore

$$\operatorname{Re}\left[\frac{1}{\alpha}q(z)\right] > 0, z \in U.$$

By applying Theorem 2.1 for $\beta = \dfrac{1}{\alpha}$ and $\gamma = 0$ we conclude that $p(z) \prec q(z)$, and thus $f \in S^*_{m,\lambda}(q)$. □

Let a be a complex number such that $\operatorname{Re} a > 0$ and $f \in \mathcal{A}$. We also consider the Bernardi integral operator given by

$$F(f)(z) = \frac{1+a}{z^a}\int_0^z f(t)\, t^{a-1} dt. \tag{3}$$

Theorem 3.3. *If $f \in M_\alpha(m,\lambda,q)$, then $F \in S^*_{m,\lambda}(q)$.*

Proof. We calculate the derivative of F from the relation (3) and obtain

$$(1+a) f(z) = aF(z) + zF'(z). \tag{4}$$

Then we apply the generalized Ruscheweyh derivatives operator to both terms in (4), and we get

$$(1+a)\,\mathcal{D}^m_\lambda f(z) = a\mathcal{D}^m_\lambda F(z) + \mathcal{D}^m_\lambda\left(zF'(z)\right). \tag{5}$$

If the analytic function f has a Taylor series expansion of the form

$$f(z) = z + \sum_{n=2}^{n} a_n z^n,$$

then

$$F(z) = z + \sum_{n=2}^{n} b_n z^n,$$

where $b_n = \dfrac{1+a}{a+n} a_n$, for all $n \geq 2$. We have

$$\mathcal{D}_\lambda^m (zF'(z)) = z + \sum_{n=2}^{\infty} [1 + \lambda (n-1)] C(m,n) n b_n z^n.$$

By applying the generalized Ruscheweyh derivatives operator to F, we obtain

$$\mathcal{D}_\lambda^m F(z) = z + \sum_{n=2}^{\infty} [1 + \lambda (n-1)] C(m,n) b_n z^n,$$

and from here we conclude that

$$z (\mathcal{D}_\lambda^m F(z))' = z + \sum_{n=2}^{\infty} [1 + \lambda (n-1)] C(m,n) n b_n z^n.$$

Therefore the following equality

$$\mathcal{D}_\lambda^m (zF'(z)) = z (\mathcal{D}_\lambda^m F(z))'$$

is satisfied. The equation (5) becomes

$$(1+a) \mathcal{D}_\lambda^m f(z) = a \mathcal{D}_\lambda^m F(z) + z (\mathcal{D}_\lambda^m F(z))'. \tag{6}$$

We calculate the derivative of both terms in (6), we multiply with z and obtain

$$(1+a) z (\mathcal{D}_\lambda^m f(z))' = (a+1) z (\mathcal{D}_\lambda^m F(z))' + z^2 (\mathcal{D}_\lambda^m F(z))''. \tag{7}$$

We divide the identity (7) to the relation (6) and have

$$\frac{z (\mathcal{D}_\lambda^m f(z))'}{\mathcal{D}_\lambda^m f(z)} = \frac{z (\mathcal{D}_\lambda^m F(z))'}{\mathcal{D}_\lambda^m F(z)} \frac{a + 1 + \dfrac{z (\mathcal{D}_\lambda^m F(z))''}{(\mathcal{D}_\lambda^m F(z))'}}{a + \dfrac{z (\mathcal{D}_\lambda^m F(z))'}{\mathcal{D}_\lambda^m F(z)}},$$

or, by using the notation

$$P(z) := \frac{z (\mathcal{D}_\lambda^m F(z))'}{\mathcal{D}_\lambda^m F(z)} \text{ and } p(z) := \frac{z (\mathcal{D}_\lambda^m f(z))'}{\mathcal{D}_\lambda^m f(z)},$$

$$p(z) = P(z)\frac{a + \dfrac{zP'(z)}{P(z)} + P(z)}{a + P(z)} = P(z) + \frac{zP'(z)}{a + P(z)},$$

Because $f \in M_\alpha(m, \lambda, q)$, by applying Theorem cat o fi, we get $f \in S^*_{m,\lambda}(q)$, or

$$p(z) = \frac{z\left(\mathcal{D}^m_\lambda f(z)\right)'}{\mathcal{D}^m_\lambda f(z)} \prec q(z).$$

The subordination

$$P(z) + \frac{zP'(z)}{a + P(z)} \prec q(z)$$

holds. Because q is a convex function and $\operatorname{Re}[a + q(z)] > 0$, from Theorem 2.1 with $\beta = 1$ and $\gamma = a$ we can conclude that

$$P(z) \prec q(z),$$

or

$$\frac{z\left(\mathcal{D}^m_\lambda F(z)\right)'}{\mathcal{D}^m_\lambda F(z)} \prec q(z),$$

and thus $F \in S^*_{m,\lambda}(q)$. □

Theorem 3.4. *Let q be a convex function in$U$, with $q(0) = 1$ and $\operatorname{Re} q(z) > 0$. We consider $Q(z) = \alpha\dfrac{zq'(z)}{q(z)}$ and $h(z) = q(z) + Q(z)$, $z \in U$. If Q is a convex function in U and $f \in M_\alpha(m, \lambda, h)$ for an $\alpha \in (0, 1]$, then $f \in S^*_{m,\lambda}(q)$.*

Proof. We choose the functions θ and φ to be $\theta(w) = w$, $\varphi(w) = \dfrac{\alpha}{w}$ and notice that the hypothesis of Theorem 2.2 are satisfied. It follows that $\dfrac{z\left(\mathcal{D}^m_\lambda f(z)\right)'}{\mathcal{D}^m_\lambda f(z)} \prec q(z)$ and q is the best dominant. Therefore $f \in S^*_{m,\lambda}(q)$. □

Theorem 3.5. *Let q be a convex function in U, with $q(0) = 1$ and $\operatorname{Re} q(z) > 0$. We consider $Q(z) = \alpha\dfrac{zq'(z)}{q(z)}$ and $h(z) = q(z) + Q(z)$, $z \in U$. If Q is convex in U, f belongs to the class $\overline{M}_\alpha(m, \lambda, h)$ for an $\alpha \in (0, 1]$, $\dfrac{z\left(\mathcal{D}^m_\lambda f(z)\right)'}{\mathcal{D}^m_\lambda f(z)} \in \mathcal{H}[1, 1] \cap \mathcal{Q}$ and $J(\alpha, m, \lambda, f; z)$ is univalent in U, then $f \in \overline{S}^*_{m,\lambda}(q)$.*

Proof. We choose $\theta(w) = w$, $\varphi(w) = \dfrac{\alpha}{w}$ and notice that the conditions of theorem 2.3 are satisfied. It follows that $q(z) \prec \dfrac{z\left(\mathcal{D}^m_\lambda f(z)\right)'}{\mathcal{D}^m_\lambda f(z)}$ and q is the best subordinant. Therefore $f \in \overline{S}^*_{m,\lambda}(q)$. □

Corollary 3.6. *For $k = 1, 2$, let q_k be two convex functions in U, with $q_k(0) = 1$ and $\operatorname{Re} q_k(z) > 0$. We consider $Q_k(z) = \alpha \frac{z q_k'(z)}{q_k(z)}$ and $h_k(z) = q_k(z) + Q_k(z)$, $z \in U$. If Q_k are convex in U, $f \in \overline{M}_\alpha(m, \lambda, h_1) \cap M_\alpha(m, \lambda, h_2)$ for an $\alpha \in (0, 1]$, $\frac{z(\mathcal{D}_\lambda^m f(z))'}{\mathcal{D}_\lambda^m f(z)} \in \mathcal{H}[1,1] \cap \mathcal{Q}$ and $J(\alpha, m, \lambda, f; z)$ is univalent in U, then $f \in \overline{S}^*_{m,\lambda}(q_1) \cap S^*_{m,\lambda}(q_2)$.*

We will give an example by taking $q_1(z) = 1 + \beta z$, $\beta \in \mathbb{C}^*$, $|\beta| \leq 1$ and $q_2(z) = 1 + z$, $z \in U$. The functions $Q_1(z) = \frac{\beta z}{1 + \beta z}$, $Q_2(z) = \frac{z}{1+z}$, $z \in U$ are convex in this case, and $h_1(z) = 1 + \beta z + \frac{\beta z}{1 + \beta z}$, $h_2(z) = 1 + z + \frac{z}{1+z}$, $z \in U$ are also convex and have positive real part.

Example 3.7. *Let $\beta \in \mathbb{C}^*$, $|\beta| \leq 1$, and $f \in \mathcal{A}$ such that*

$$1 + \beta z + \frac{\beta z}{1 + \beta z} \prec J(\alpha, m, \lambda, f; z) \prec 1 + z + \frac{z}{1+z},\ z \in U.$$

Then

$$1 + \beta z \prec \frac{z(\mathcal{D}_\lambda^m f(z))'}{\mathcal{D}_\lambda^m f(z)} \prec 1 + z,\ z \in U.$$

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