STUDIA UNIV. "BABEŞ–BOLYAI", MATHEMATICA, Volume **LIII**, Number 1, March 2008

ON SOME INTEGRAL OPERATORS WHICH PRESERVE THE UNIVALENCE

VIRGIL PESCAR

Abstract. We study some integral operators and determine conditions for the univalence of these integral operators.

1. Introduction

Let A be the class of the functions $f(z)$ which are analytic in the open unit disc $U = \{z \in C : |z| < 1\}$ and $f(0) = f'(0) - 1 = 0$.

We denote by S the subclass of A consisting of functions $f(z) \in A$ which are univalent in U.

In this paper, we consider the integral operators

$$H_\beta(z) = \left\{ \beta \int_0^z [h(u)]^{\beta - 1}\, du \right\}^{\frac{1}{\beta}} \tag{1}$$

and

$$G_\beta(z) = \left\{ \beta \int_0^z u\,[g(u)]^{\beta - 2}\, du \right\}^{\frac{1}{\beta}} \tag{2}$$

for $h(z) \in S$, $g(z) \in S$ and $\beta \in C$.

2. Preliminary results

To discuss our integral operators, we need the following theorem.

Theorem 2.1 [1]. *Let α be a complex number,* $\operatorname{Re}\alpha > 0$ *and* $f \in A$. *If*

$$\frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1, \tag{3}$$

Received by the editors: 20.09.2006.

2000 *Mathematics Subject Classification.* 30C45.

Key words and phrases. univalent function, integral operator.

for all $z \in U$*, then for any complex number* β*,* $\operatorname{Re}\beta \geq \operatorname{Re}\alpha$ *the function*

$$F_\beta(z) = \left[\beta \int_0^z u^{\beta-1} f'(u)du\right]^{\frac{1}{\beta}} \tag{4}$$

is in the class S*.*

3. **Main results**

Theorem 3.1. *Let* α, β *be complex numbers and the function* $h \in S$*,*

$$h(z) = z + a_2 z^2 + \ldots.$$

If

$$\operatorname{Re}\,\beta \geq \operatorname{Re}\,\alpha > 0 \tag{j_1}$$

$$|\beta - 1| \leq \frac{\operatorname{Re}\,\alpha}{4} \quad \textit{for } \operatorname{Re}\,\alpha \in (0,1) \tag{j_2}$$

or

$$|\beta - 1| \leq \frac{1}{4} \quad \textit{for } \operatorname{Re}\,\alpha \in [1,\infty) \tag{j_3}$$

then the function

$$H_\beta(z) = \left\{\beta \int_0^z [h(u)]^{\beta-1}\, du\right\}^{\frac{1}{\beta}} \tag{5}$$

belongs to the class S*.*

Proof. From (5) we have

$$H_\beta(z) = \left\{\beta \int_0^z u^{\beta-1} \left[\frac{h(u)}{u}\right]^{\beta-1} du\right\}^{\frac{1}{\beta}} \tag{6}$$

The function $h(z)$ is regular and univalent, hence $\frac{h(z)}{z} \neq 0$ for all $z \in U$. We can choose the regular brach of the function $\left[\frac{h(z)}{z}\right]^{\beta-1}$, which is equal to 1 at the origin.

Let us consider the regular function in U, given by

$$f(z) = \int_0^z \left[\frac{h(u)}{u}\right]^{\beta-1} du. \tag{7}$$

Because $h \in S$, we obtain

$$\left|\frac{z\,h'(z)}{h(z)}\right| \leq \frac{1+|z|}{1-|z|} \tag{8}$$

for all $z \in U$.

We have

$$\left|\frac{z\,f''(z)}{f'(z)}\right| = |\beta - 1|\left|\frac{z\,h'(z)}{h(z)} - 1\right| \le |\beta - 1|\frac{2}{1-|z|}. \tag{9}$$

Now, we consider the cases

$i_1)$ Re $\alpha \ge 1$.

We observe that the function $p: [1,\infty) \to \mathbb{R}$,

$$p(x) = \frac{1-a^{2x}}{x} \quad (0 < a < 1) \tag{10}$$

is a decreasing function, and that, if we take $a = |z|$, $z \in U$, then

$$\frac{1-|z|^{2\mathrm{Re}\ \alpha}}{\mathrm{Re}\ \alpha} \le 1 - |z|^2 \tag{11}$$

for all $z \in U$.

From (11) and (9) we obtain

$$\frac{1-|z|^{2\mathrm{Re}\ \alpha}}{\mathrm{Re}\ \alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le 4\,|\beta - 1|. \tag{12}$$

From (12) and (j_3), we have

$$\frac{1-|z|^{2\mathrm{Re}\ \alpha}}{\mathrm{Re}\ \alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le 1 \tag{13}$$

for all $z \in U$.

$i_2)$ $0 < \mathrm{Re}\ \alpha < 1$.

The function $v: (0,1) \to \mathbb{R}$,

$$v(x) = 1 - a^{2x} \quad (0 < a < 1) \tag{14}$$

is a increasing function and for $a = |z|$, $z \in U$, we obtain

$$1 - |z|^{2\,\mathrm{Re}\ \alpha} \le 1 - |z|^2, \quad z \in U \tag{15}$$

for all $z \in U$.

From (9) and (15), we have

$$\frac{1-|z|^{2\mathrm{Re}\ \alpha}}{\mathrm{Re}\ \alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{4\,|\beta - 1|}{\mathrm{Re}\ \alpha} \tag{16}$$

for all $z \in U$.

Using the condition (j_2) and (16) we get

$$\frac{1-|z|^{2\text{Re }\alpha}}{\text{Re }\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{17}$$

for all $z \in U$.

Because $f'(z) = \left[\frac{h(z)}{z}\right]^{\beta-1}$, from Theorem 2.1 it results that the function $H_\beta(z)$ is regular and univalent in U.

Theorem 3.2. *Let* α, β *be complex numbers and the function* $g \in S$, $g(z) = z + a_2z^2 + \dots$.

If

$$\text{Re }\beta \geq \text{Re }\alpha > 0 \tag{p_1}$$

and

$$|\beta - 2| \leq \frac{\text{Re }\alpha}{4} \quad \textit{for } \text{Re }\alpha \in (0,1) \tag{p_2}$$

or

$$|\beta - 2| \leq \frac{1}{4} \quad \textit{for } \text{Re }\alpha \in [1,\infty) \tag{p_3}$$

then the function

$$G_\beta(z) = \left\{\beta \int_0^z u\left[g(u)\right]^{\beta-2} du\right\}^{\frac{1}{\beta}} \tag{18}$$

is in the class S.

Proof. We observe that

$$G_\beta(z) = \left\{\beta \int_0^z u^{\beta-1}\left[\frac{g(u)}{u}\right]^{\beta-2} du\right\}^{\frac{1}{\beta}} \tag{19}$$

We consider the regular function in U

$$\left[f(z) = \int_0^z \frac{g(u)}{u}\right]^{\beta-2} du$$

and by the same reasoning with a view to the Theorem 3.1. we conclude that the function $G_\beta(z)$ is in the class S in the conditions (p_1), (p_2) and (p_3).

References

[1] Pascu, N. N., *An improvement of Becker's univalence criterion*, Proceedings of the Commemorative Session Stoilov (Braşov, 1987), University of Braşov, 1987, 43-48.

[2] Pascu, N. N., Pescar, V., *On the integral operators of Kim-Merkes and Pfaltzgraff*, Mathematica, Tome **32**(55), No. 2, Cluj-Napoca, (1990), 185-192.

[3] Pescar, V., *New univalence criteria*, Transilvania University of Braşov, Braşov, 2002.

[4] Pommerenke, C., *Univalent functions*, Vandenkoeck Ruprecht in Göttingen, 1975.

"Transilvania" University of Braşov
Faculty of Mathematics and Computer Science
Department of Mathematics
2200 Braşov, Romania
E-mail address: virgilpescar@unitbv.ro