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RADIAL SOLUTIONS FOR SOME CLASSES OF ELLIPTIC BOUNDARY VALUE PROBLEMS

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Abstract. The aim of this paper is to present some existence and localization results of radial solutions for elliptic equations and systems on an annulus Ω of $\mathbb{R}^N$ $(N \geq 1)$ of radii a and b with $0 < a < b$. The main tool is Schauder's fixed point theorem.

1. **Introduction**

In this paper, we are concerned with the existence of radial solutions and their localization in a ball, for the elliptic boundary value problem

$$\begin{cases} -\Delta u = f\left(|x|, u\right) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

and the elliptic system

$$\begin{cases} -\Delta u = g\left(|x|, u, v\right) & \text{in } \Omega \\ -\Delta v = h\left(|x|, u, v\right) & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Here Ω is an annulus of $\mathbb{R}^N$ $(N \geq 1)$ of radii a and b with $0 < a < b$, $|x|$ is the Euclidean norm in $\mathbb{R}^N$, and $f : [a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ and $g,\ h : [a, b] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ are continuous functions. By a solution of problem (1.1) we mean a function $u \in C^1\left(\overline{\Omega}, \mathbb{R}\right)$ which satisfies (1.1) in the sense of distributions. A solution to problem (1.2) is a vector-valued function $(u, v) \in C^1\left(\overline{\Omega}, \mathbb{R}^2\right) := C^1\left(\overline{\Omega}, \mathbb{R}\right) \times C^1\left(\overline{\Omega}, \mathbb{R}\right)$ satisfying (1.2) in the sense of distributions.

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Radial solutions for elliptic boundary value problems has been discussed extensively in the literature; see [1], [2], [3], [5], [6] and the references therein.

In [2], using fixed point theorems of cone expansion/compression type, the lower and upper solution method and degree arguments, the authors study existence, non-existence and multiplicity of positive radial solutions.

The same authors in [3] deal with a class of second-order elliptic problems on a ball with non-homogeneous boundary condition. They obtain via a fixed point theorem the existence of at least three positive radial solutions.

The study of existence of positive radial solutions to a singular semilinear elliptic equation was investigated in [5]. Throughout, their nonlinearity is allowed to change sign and the singularity may occur.

In this paper, some existence and localization results of radial solutions for elliptic equations and systems on an annulus are presented. Our main tool in proving the existence of solutions to problems (1.1) and (1.2) is Schauder's fixed point theorem [4], [7].

2. **Existence result for Problem (1.1)**

Theorem 2.1. *Assume that for some $R > 0$, one of the following hypotheses is satisfied:*

$(\mathcal{H}1)$

$$|f(t,y)| \leq \alpha(t)F(y), \text{ for all } t \in [a,b] \text{ and } y \in \mathbb{R},$$

where the functions $\alpha \in L^1([a,b],\mathbb{R}_+)$ and $F \in C(\mathbb{R},\mathbb{R}_+)$ satisfy

$$|\alpha|_{L^1} \max_{|y|\leq R} F(y) \leq \frac{R}{b-a}\left(\frac{a}{b}\right)^{N-1};$$

$(\mathcal{H}2)$

$$|f(t,y)| \leq F(t,|y|), \text{ for all } t \in [a,b] \text{ and } y \in \mathbb{R},$$

for some $F \in C([a,b]\times\mathbb{R}_+,\mathbb{R}_+)$ nondecreasing with respect to its last variable, and with

$$\int_a^b F(s,R)\,ds \leq \frac{R}{b-a}\left(\frac{a}{b}\right)^{N-1}.$$

Then the boundary value problem (1.1) *has at least one radial solution with*

$$\|u\|_0 = \sup_{a\leq|x|\leq b} |u(x)| \leq R.$$

Proof. For $v \in C([a,b],\mathbb{R})$, let $u = Tv$ be the solution of

$$\begin{cases} -(r^{N-1}u')' = r^{N-1}f\left(r, v(r)\right), & a < r < b \\ u(a) = u(b) = 0 \end{cases}$$

where we have replaced $|x|$ by r. Then $T : C([a,b],\mathbb{R}) \longrightarrow C([a,b],\mathbb{R})$ is completely continuous and the fixed points of T are solutions of problem (1.1). One can write the expression of T as

$$Tu(r) = \int_a^r \left[\frac{1}{s^{N-1}}\int_s^\theta \tau^{N-1} f\left(\tau, u(\tau)\right)\, d\tau\right] ds$$

where θ is such that $\|u\|_0 = |u(\theta)|$.

Consider the closed ball:

$$B = \{u \in C([a,b],\mathbb{R}) :\ \|u\|_0 \leq R\}$$

where R is as in Assumptions $(\mathcal{H}1)$, $(\mathcal{H}2)$ and check that $T(B) \subset B$.

(a) Assume $(\mathcal{H}1)$. For any $u \in B$ and $r \in [a,b]$, we have

$$\begin{aligned} |Tu(r)| &\leq \int_a^b \left[\frac{1}{s^{N-1}}\int_a^b \tau^{N-1}|f\left(\tau, u(\tau)\right)|\, d\tau\right] ds \\ &\leq \left(\frac{b}{a}\right)^{N-1}(b-a)\int_a^b |f\left(s, u(s)\right)|\, ds \\ &\leq \left(\frac{b}{a}\right)^{N-1}(b-a)\int_a^b \alpha(s)F(u(s))\, ds \\ &\leq \left(\frac{b}{a}\right)^{N-1}(b-a)|\alpha|_{L^1}\max_{|y|\leq R} F(y) \\ &\leq R. \end{aligned}$$

Passing to the supremum, we obtain

$$\|Tu\|_0 \leq R.$$

(b) When $(\mathcal{H}2)$ holds, then

$$\begin{aligned}
|Tu(r)| &\leq \int_a^b \left[\frac{1}{s^{N-1}} \int_a^b \tau^{N-1} |f(\tau, u(\tau))| \, d\tau \right] ds \\
&\leq \left(\frac{b}{a} \right)^{N-1} (b-a) \int_a^b |f(s, u(s))| \, ds \\
&\leq \left(\frac{b}{a} \right)^{N-1} (b-a) \int_a^b F(s, |u(s)|) \, ds \\
&\leq \left(\frac{b}{a} \right)^{N-1} (b-a) \int_a^b F(s, R) \, ds \\
&\leq R.
\end{aligned}$$

Passing to the supremum, we obtain

$$\|Tu\|_0 \leq R.$$

Therefore, in both cases, the operator T maps the ball B into itself, ending the proof of our claim. Since T is completely continuous, the conclusion of Theorem 2.1 follows from Schauder's fixed point theorem. $\square$

3. **Existence results for Problem (1.2)**

In this section, we are concerned with the existence and localization of radial solutions to the Dirichlet problem (1.2) for elliptic systems.

Theorem 3.1. *Assume that for some $R > 0$ one of the following hypotheses is satisfied:*

$(\mathcal{H}3)$

$$|g(t, y, z)| \leq \beta(t) G(y, z), \text{ for all } t \in [a, b] \text{ and } y, z \in \mathbb{R}$$

and

$$|h(t, y, z)| \leq \gamma(t) H(y, z), \text{ for all } t \in [a, b] \text{ and } y, z \in \mathbb{R}$$

for some functions $\beta, \gamma \in L^1([a, b], \mathbb{R}_+)$ and $G, H \in C(\mathbb{R}^2, \mathbb{R}_+)$ with

$$|\beta|_{L^1} \max_{|y|, |z| \leq R} G(y, z) \leq \frac{R}{b-a} \left(\frac{a}{b} \right)^{N-1}$$

and

$$|\gamma|_{L^1} \max_{|y|,|z|\leq R} H(y,z) \leq \frac{R}{b-a}\left(\frac{a}{b}\right)^{N-1};$$

$(\mathcal{H}4)$

$$|g(t,y,z)| \leq G(t,|y|,|z|), \textit{for all } t \in [a,b] \textit{ and } y,z \in \mathbb{R}$$

and

$$|h(t,y,z)| \leq H(t,|y|,|z|), \ \textit{for all } t \in [a,b] \textit{ and } y,z \in \mathbb{R}$$

for some functions $G, H \in C([a,b]\times\mathbb{R}_+^2, \mathbb{R}_+)$ nondecreasing with respect to the last two arguments, and with

$$\int_a^b G(s,R,R)\,ds \leq \frac{R}{b-a}\left(\frac{a}{b}\right)^{N-1}$$

$$\int_a^b H(s,R,R)\,ds \leq \frac{R}{b-a}\left(\frac{a}{b}\right)^{N-1}.$$

Then the boundary value problem (1.2) *has at least one radial solution* (u,v) *with*

$$\|u\|_0 \leq R \ \textit{ and } \ \|v\|_0 \leq R.$$

Proof. We shall apply Schauder's fixed point theorem in the space $C([a,b],\mathbb{R}^2)$ endowed with the norm $\|(.,.)\|_0$ given by

$$\|(u,v)\|_0 = \|u\|_0 + \|v\|_0.$$

For $(\overline{u},\overline{v}) \in C([a,b],\mathbb{R}^2)$, let $(u,v) = T(\overline{u},\overline{v})$ be the solution of

$$\begin{cases} -(r^{N-1}u')' = r^{N-1}g\left(r,(\overline{u}(r),\overline{v}(r))\right), & a<r<b \\ -(r^{N-1}v')' = r^{N-1}h\left(r,(\overline{u}(r),\overline{v}(r))\right), & a<r<b \\ u(a)=u(b)=v(a)=v(b)=0. \end{cases}$$

Then $T : C([a,b],\mathbb{R}^2) \longrightarrow C([a,b],\mathbb{R}^2)$ is completely continuous and the fixed points of T are solutions of problem (1.2). One can write the expression of T as $T=(T_1,T_1)$, where

$$T_1(u,v)(r) = \int_a^r \left[\frac{1}{s^{N-1}}\int_s^{\theta_1} \tau^{N-1} g\left(\tau, u(\tau), v(\tau)\right)\,d\tau\right] ds,$$

$$T_2(u,v)(r) = \int_a^r \left[\frac{1}{s^{N-1}}\int_s^{\theta_2} \tau^{N-1} h\left(\tau, u(\tau), v(\tau)\right)\,d\tau\right] ds$$

and θ_1, θ_2 are such that $\|u\|_0 = |u(\theta_1)|$ and $\|v\|_0 = |v(\theta_2)|$.

Consider the closed, bounded and convex subset of $C([a,b],\mathbb{R}^2)$:

$$B=\left\{(u,v)\in C([a,b],\mathbb{R}^2):\ \|u\|_0\leq R, \|v\|_0\leq R\right\},$$

where R is as in Assumptions $(\mathcal{H}3)$, $(\mathcal{H}4)$, and check that $T(B)\subset B$.

(a) Assume $(\mathcal{H}3)$. For any $(u,v)\in B$ and $r\in[a,b]$, we have

$$\begin{aligned}
|T_1(u,v)(r)| &\leq \int_a^b\left[\frac{1}{s^{N-1}}\int_a^b \tau^{N-1}|g\left(\tau,u(\tau),v(\tau)\right)|\,d\tau\right]ds\\
&\leq \left(\frac{b}{a}\right)^{N-1}(b-a)\int_a^b |g\left(\tau,u(\tau),v(\tau)\right)|\,ds\\
&\leq \left(\frac{b}{a}\right)^{N-1}(b-a)\int_a^b \beta(s)G(u(s),v(s))\,ds\\
&\leq \left(\frac{b}{a}\right)^{N-1}(b-a)|\beta|_{L^1}\max_{|y|,|z|\leq R}G(y,z)\\
&\leq R.
\end{aligned}$$

Passing to the supremum, we obtain

$$\|T_1(u,v)\|_0\leq R.$$

Also

$$\begin{aligned}
|T_2(u,v)(r)| &\leq \int_a^b\left[\frac{1}{s^{N-1}}\int_a^b \tau^{N-1}|h\left(\tau,u(\tau),v(\tau)\right)|\,d\tau\right]ds\\
&\leq \left(\frac{b}{a}\right)^{N-1}(b-a)\int_a^b |h\left(\tau,u(\tau),v(\tau)\right)|\,ds\\
&\leq \left(\frac{b}{a}\right)^{N-1}(b-a)\int_a^b \gamma(s)H(u(s),v(s))\,ds\\
&\leq \left(\frac{b}{a}\right)^{N-1}(b-a)|\gamma|_{L^1}\max_{|y|,|z|\leq R}H(y,z)\\
&\leq R.
\end{aligned}$$

Hence

$$\|T_2(u,v)\|_0\leq R.$$

Therefore, the operator T maps the ball B into itself.

(b) Assume $(\mathcal{H}4)$. Then

$$\begin{aligned}
|T_1(u,v)(r)| &\leq \int_a^b \left[\frac{1}{s^{N-1}} \int_a^b \tau^{N-1} |g\left(\tau, u(\tau), v(\tau)\right)| \, d\tau \right] ds \\
&\leq \left(\frac{b}{a}\right)^{N-1} (b-a) \int_a^b |g\left(s, u(s), v(\tau)\right)| \, ds \\
&\leq \left(\frac{b}{a}\right)^{N-1} (b-a) \int_a^b G(s, |u(s)|, |v(s)|) \, ds \\
&\leq \left(\frac{b}{a}\right)^{N-1} (b-a) \int_a^b G(s, R, R) \, ds \\
&\leq R.
\end{aligned}$$

Hence

$$\|T_1(u,v)\|_0 \leq R.$$

Also

$$\begin{aligned}
|T_2(u,v)(r)| &\leq \int_a^b \left[\frac{1}{s^{N-1}} \int_a^b \tau^{N-1} |h\left(\tau, u(\tau), v(\tau)\right)| \, d\tau \right] ds \\
&\leq \left(\frac{b}{a}\right)^{N-1} (b-a) \int_a^b |h\left(s, u(s), v(s)\right)| \, ds \\
&\leq \left(\frac{b}{a}\right)^{N-1} (b-a) \int_a^b H(s, |u(s)|, |v(s)|) \, ds \\
&\leq \left(\frac{b}{a}\right)^{N-1} (b-a) \int_a^b H(s, R, R) \, ds \\
&\leq R.
\end{aligned}$$

Then

$$\|T_2(u,v)\|_0 \leq R.$$

Therefore, in both cases, the operator T maps the set B into itself, ending the proof of our claim. Since T is completely continuous, the conclusion of Theorem 3.1 follows from Schauder's fixed point theorem. □

References

[1] Shaohua Chen and Guozhen Lu, *Existence and nonexistence of positive radial solutions for a class of semilinear elliptic system*, Nonlinear Anal., **38** (1999), 919-932.

[2] do Ó, J.M., Lorca, S., and Ubilla, P., *Multiparameter elliptic equations in annular domains*, Progress in Nonlinear Differential Equations and Their Applications, Vol. **66** (2005), 233-246.

[3] do Ó, J.M., Lorca, S., and Ubilla, P., *Three positive radial solutions for elliptic equations in a ball*, Appl. Math. Letters, **18**(2005), 1163-1169.

[4] Granas, A, and Dugundji, J., *Fixed Point Theory*, Springer, New York, 2003.

[5] Lu, H., and Bai, Z., *Positive radial solutions of a singular elliptic equation with sign changing nonlinearities*, Applied Math. Letters, **19**(2006) 555-567.

[6] Schmitt, K., *Positive solutions of semilinear elliptic boundary value problems*, in "Topological Methods in Differential Equations and Inclusions", A. Granas and M. Frigon eds., Kluwer, Dordrecht, 1995, 447-500.

[7] Zeidler, E., *Nonlinear Functional Analysis*, Vol. 1, Fixed Point Theorems, Springer, Berlin, 1986.

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