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LOGARITHMIC MODIFICATION OF THE JACOBI WEIGHT FUNCTION

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Dedicated to Professor D. D. Stancu on his 80^{th} *birthday*

Abstract. In this paper we are interested in a logarithmic modification of the Jacobi weight function, i.e., we study the following moment functional $\mathcal{L}^{\alpha,\beta}(p) = \int_{-1}^{1} p(x)(1-x)^\alpha(1+x)^\beta \log(1-x^2)\,dx$, $p \in \mathcal{P}$, where $\alpha, \beta > -1$ and $\mathcal{P}$ is the space of all algebraic polynomials. We give the recurrence relations for the modified moments $\mu_k = \mathcal{L}^{\alpha,\beta}(q_k)$, $k \in \mathbb{N}_0$, in the cases when q_k is a sequence of monic Chebyshev polynomials of the first and second kind. In particular, when $\alpha = \beta = \ell - 1/2$, $\ell \in \mathbb{N}_0$, we derive explicit formulae for the modified moments. As an application of these modified moments, the numerical construction of coefficients in the three-term recurrence relation for polynomials orthogonal with respect the functional $\mathcal{L}^{\alpha,\beta}$ and the corresponding Gaussian quadratures are presented.

1. **Introduction**

We consider the moment functional

$$\mathcal{L}^{\alpha,\beta}(p) = \int_{-1}^{1} p(x)(1-x)^\alpha(1+x)^\beta \log(1-x^2)\,dx, \quad \alpha, \beta > -1, \tag{1.1}$$

on the space of all algebraic polynomials $\mathcal{P}$. In (1.1) we recognize the Jacobi weight function $w^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$. Recently, in [1], there appeared an interest in a construction of numerical methods for integration of an integral which appears in the moment functional $\mathcal{L}^{\pm 1/2, \pm 1/2}$.

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In this paper we give a stable numerical procedure which can be used for the construction of polynomials orthogonal with respect to the moment functional $\mathcal{L}^{\alpha,\beta}$, as well as a stable numerical method for the corresponding quadrature rules for computing the mentioned integrals (Section 3).

These procedures are enabled by finding the recurrence relations for modified moments $\mathcal{L}^{\alpha,\beta}(q_k)$, $k \in \mathbb{N}_0$, with respect to the polynomial sequences $q_k = T_k$ and $q_k = U_k$, where T_k and U_k are the monic Chebyshev polynomials of the first and second kind, respectively (Section 2). In the case when $\alpha = \beta = \ell - 1/2$, $\ell \in \mathbb{N}_0$, we derive explicit formulae for these modified moments. The procedure for finding modified moments are loosely connected with an earlier work of Piessens (see [8]).

2. **Modified moments**

First we introduce the modified moments of the Jacobi weight function with respect to the monic Chebyshev polynomials T_n and U_n, $n \in \mathbb{N}_0$, of the first and the second kind, respectively. We use the following notation

$$m_n^{\alpha,\beta} = \int_{-1}^{1} w^{\alpha,\beta}(x) T_n(x)\, dx \tag{2.2}$$

and

$$e_n^{\alpha,\beta} = \int_{-1}^{1} w^{\alpha,\beta}(x) U_n(x)\, dx. \tag{2.3}$$

We have the following lemma.

Lemma 2.1. *Modified moments of the Jacobi weight, given in* (2.2) *and* (2.3)*, satisfy the following recurrence relations*

$$m_{n+1}^{\alpha,\beta} = \frac{\beta-\alpha}{n+\alpha+\beta+2} m_n^{\alpha,\beta} + \frac{1}{4}\frac{n-\alpha-\beta-2}{n+\alpha+\beta+2}(1+\delta_{n-1,0}) m_{n-1}^{\alpha,\beta}, \quad n \in \mathbb{N},$$

and

$$e_{n+1}^{\alpha,\beta} = \frac{\beta-\alpha}{n+\alpha+\beta+2} e_n^{\alpha,\beta} + \frac{1}{4}\frac{n-\alpha-\beta}{n+\alpha+\beta+2} e_{n-1}^{\alpha,\beta}, \quad n \in \mathbb{N}.$$

In both cases initial conditions are the same

$$\begin{aligned} m_0^{\alpha,\beta} &= e_0^{\alpha,\beta} = 2^{\alpha+\beta+1} B(1+\alpha, 1+\beta), \\ m_1^{\alpha,\beta} &= e_1^{\alpha,\beta} = 2^{\alpha+\beta+1} \frac{\beta-\alpha}{\alpha+\beta+2} B(1+\alpha, 1+\beta). \end{aligned}$$

Proof. We are going to need the following identity (see [2, p. 142])

$$(1-x^2)T_k'(x) = -kT_{k+1}(x) + \frac{k(1+\delta_{k-1,0})}{4}T_{k-1}(x), \quad k \in \mathbb{N},$$

satisfied by the monic Chebyshev polynomials of the first kind. Here, $\delta_{k,m}$ is the Kronecker's delta. Integrating this identity with respect to the Jacobi weight, and using an integration by parts, we have

$$\begin{aligned} -km_{k+1}^{\alpha,\beta} \quad + \quad \frac{k(1+\delta_{k-1,0})}{4}m_{k-1}^{\alpha,\beta} &= \int_{-1}^{1}(1-x^2)w^{\alpha,\beta}(x)T_k'(x)dx \\ &= -\int_{-1}^{1}(-x(\alpha+\beta+2)+\beta-\alpha)w^{\alpha,\beta}(x)T_k(x)dx \\ &= -(\beta-\alpha)m_k^{\alpha,\beta} + (\alpha+\beta+2)\left(m_{k+1}^{\alpha,\beta} + \frac{1+\delta_{k-1,0}}{4}m_{k-1}^{\alpha,\beta}\right), \end{aligned}$$

where we used the three-term recurrence relation for the monic Chebyshev polynomials of the first kind

$$T_{k+1}(x) = xT_k(x) - \frac{1+\delta_{k-1,0}}{4}T_{k-1}(x).$$

Similarly, for the monic Chebyshev polynomials of the second kind we have the following identity (see [2, p. 144])

$$(1-x^2)U_k'(x) = -kU_{k+1}(x) + \frac{k+2}{4}U_{k-1}(x), \quad k \in \mathbb{N}.$$

Integrating this identity with respect to the Jacobi weight we get

$$\begin{aligned} -ke_{k+1}^{\alpha,\beta} \quad + \quad \frac{k+1}{4}e_{k-1}^{\alpha,\beta} &= \int_{-1}^{1}(1-x^2)w^{\alpha,\beta}(x)U_k'(x)dx \\ &= -\int_{-1}^{1}(-x(\alpha+\beta+2)+\beta-\alpha)w^{\alpha,\beta}(x)U_k(x)dx \\ &= -(\beta-\alpha)e_k^{\alpha,\beta} + (\alpha+\beta+2)\left(e_{k+1}^{\alpha,\beta} + \frac{1}{4}e_{k-1}^{\alpha,\beta}\right), \end{aligned}$$

where we used the three-term recurrence relation for the monic Chebyshev polynomials of the second kind

$$U_{n+1}(x) = xU_n(x) - \frac{1}{4}U_{n-1}(x).$$

Regarding the initial conditions, $m_0^{\alpha,\beta}$ and $m_1^{\alpha,\beta}$ are first two moments of the Jacobi weight function. $\square$

Now, for the functional $\mathcal{L}^{\alpha,\beta}$ given by (1.1), we introduce the modified moments with respect to the monic Chebyshev polynomials in the following forms

$$\mu_n^{\alpha,\beta} = \int_{-1}^{1} w^{\alpha,\beta}(x)\log(1-x^2)T_n(x)dx, \quad n \in \mathbb{N}_0, \tag{2.4}$$

and

$$\eta_n^{\alpha,\beta} = \int_{-1}^{1} w^{\alpha,\beta}(x)\log(1-x^2)U_n(x)dx, \quad n \in \mathbb{N}_0, \tag{2.5}$$

where T_n and U_n, $n \in \mathbb{N}$, are sequences of the monic Chebyshev polynomials of the first and second kind, respectively.

Theorem 2.1. *The sequences of the modified moments $\mu_n^{\alpha,\beta}$ and $\eta_n^{\alpha,\beta}$, $n \in \mathbb{N}$, satisfy the following recurrence relations*

$$\begin{aligned}
\mu_{n+1}^{\alpha,\beta} &= \frac{\beta-\alpha}{n+\alpha+\beta+2}\mu_n^{\alpha,\beta} + \frac{1+\delta_{n-1,0}}{4}\frac{n-\alpha-\beta-2}{n+\alpha+\beta+2}\mu_{n-1}^{\alpha,\beta} \\
&\qquad - \frac{2}{n+\alpha+\beta+2}\left(m_{n+1}^{\alpha,\beta} + \frac{1+\delta_{n-1,0}}{4}m_{n-1}^{\alpha,\beta}\right), \\
\mu_1^{\alpha,\beta} &= \frac{\beta-\alpha}{\alpha+\beta+2}\mu_0^{\alpha,\beta} - \frac{2}{\alpha+\beta+2}m_1^{\alpha,\beta}
\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}
\eta_{n+1}^{\alpha,\beta} &= \frac{\beta-\alpha}{n+\alpha+\beta+2}\eta_n^{\alpha,\beta} + \frac{1}{4}\frac{n-\alpha-\beta-2}{n+\alpha+\beta+2}\eta_{n-1}^{\alpha,\beta} \\
&\qquad - \frac{2}{n+\alpha+\beta+2}\left(e_{n+1}^{\alpha,\beta} + \frac{1}{4}e_{n-1}^{\alpha,\beta}\right), \\
\eta_1^{\alpha,\beta} &= \frac{\beta-\alpha}{\alpha+\beta+2}\eta_0^{\alpha,\beta} - \frac{2}{\alpha+\beta+2}e_1^{\alpha,\beta}.
\end{aligned} \tag{2.7}$$

Proof. Using the same identities as in the proof of Lemma 2.1, we have

$$\begin{aligned}
-k\mu_{k+1}^{\alpha,\beta} &+ \frac{k(1+\delta_{k-1,0})}{4}\mu_{k-1}^{\alpha,\beta} = \int_{-1}^{1}(1-x^2)w^{\alpha,\beta}(x)\log(1-x^2)T_k'(x)\,dx \\
&= -\int_{-1}^{1}[-x(\alpha+\beta+2)+\beta-\alpha]w^{\alpha,\beta}(x)\log(1-x^2)T_k(x)\,dx \\
&\quad +2\int_{-1}^{1}w^{\alpha,\beta}(x)xT_k(x)\,dx \\
&= (\alpha+\beta+2)\left(\mu_{k+1}^{\alpha+\beta}+\frac{1+\delta_{k-1,0}}{4}\mu_{k-1}^{\alpha,\beta}\right)-(\beta-\alpha)\mu_k^{\alpha,\beta} \\
&\quad +2\left(m_{k+1}^{\alpha,\beta}+\frac{1+\delta_{k-1,0}}{4}m_{k-1}^{\alpha,\beta}\right).
\end{aligned}$$

Similarly, for the monic Chebyshev polynomials of the second kind we have

$$\begin{aligned}
-k\eta_{k+1}^{\alpha,\beta} &+ \frac{k+2}{4}\eta_{k-1}^{\alpha,\beta} = \int_{-1}^{1}(1-x^2)w^{\alpha,\beta}(x)\log(1-x^2)U_k'(x)\,dx \\
&= -\int_{-1}^{1}[-x(\alpha+\beta+2)+\beta-\alpha]w^{\alpha,\beta}(x)\log(1-x^2)U_k(x)\,dx \\
&\quad +2\int_{-1}^{1}w^{\alpha,\beta}(x)xU_k(x)\,dx \\
&= (\alpha+\beta+2)\left(\eta_{k+1}^{\alpha+\beta}+\frac{1}{4}\eta_{k-1}^{\alpha,\beta}\right)-(\beta-\alpha)\eta_k^{\alpha,\beta}+2\left(e_{k+1}^{\alpha,\beta}+\frac{1}{4}e_{k-1}^{\alpha,\beta}\right),
\end{aligned}$$

which gives (2.7).

For the first moment and the Chebyshev polynomials of the first kind, we have

$$\begin{aligned}
(\beta-\alpha)\mu_0^{\alpha,\beta}-(\alpha+\beta+2)\mu_1^{\alpha,\beta} &= \int_{-1}^{1}(\beta-\alpha-x(\alpha+\beta+2))w^{\alpha,\beta}(x)\log(1-x^2)dx \\
&= \int_{-1}^{1}(w^{\alpha+1,\beta+1}(x))'\log(1-x^2)dx = 2\int_{-1}^{1}w^{\alpha,\beta}(x)xdx = 2m_1^{\alpha,\beta},
\end{aligned}$$

and the proof transfers verbatim to the case of the Chebyshev polynomials of the second kind. □

Especially for $\alpha = \beta = -1/2$, it was shown (see [1], [3]) that

$$\mu_n^{-1/2,-1/2} = \int_{-1}^{1} \frac{\log(1-x^2)}{\sqrt{1-x^2}} T_n(x)dx = \begin{cases} -2\pi \log 2, & n = 0, \\ -\dfrac{\pi}{2^{n-1}n}, & n \neq 0. \end{cases} \tag{2.8}$$

Actually, the explicit expressions can be given in the case $\alpha = \beta = \ell - 1/2$, $\ell \in \mathbb{N}_0$.

Theorem 2.2. *If $\alpha = \beta = \ell - 1/2$, $\ell \in \mathbb{N}_0$, then*

$$\begin{aligned} \mu_n^{\ell-1/2,\ell-1/2} &= \frac{2^{1-n-2\ell}}{1+\delta_{n,0}} \left[\sum_{k=0}^{\ell-1} (-1)^{\ell-k} \binom{2\ell}{k} \left(\frac{1+\delta_{2(\ell-k)+n,0}}{2^{1-2(\ell-k)-n}} \mu_{2(\ell-k)+n}^{-1/2,-1/2} \right.\right. \\ &+ \left.\left. \frac{1+\delta_{|2(\ell-k)-n|,0}}{2^{1-|2(\ell-k)-n|}} \mu_{|2(\ell-k)-n|}^{-1/2,-1/2} \right) + \binom{2\ell}{\ell} \frac{1+\delta_{n,0}}{2^{1-n}} \mu_n^{-1/2,-1/2} \right], \end{aligned}$$

and if $\ell \in \mathbb{N}$ we have

$$\begin{aligned} \eta_n^{\ell-1/2,\ell-1/2} = \frac{1}{2^{n+2\ell-1}} \sum_{k=0}^{\ell-1} (-1)^{\ell+k-1} \binom{2\ell-1}{k} & \left(\frac{1+\delta_{|n-2(\ell-k)+2|,0}}{2^{1-|n-2(\ell-k)+2|}} \mu_{|n-2(\ell-k)+2|}^{-1/2,-1/2} \right. \\ & \left. - \frac{1+\delta_{n+2(\ell-k),0}}{2^{1-n-2(\ell-k)}} \mu_{n+2(\ell-k)}^{-1/2,-1/2} \right), \end{aligned}$$

for $n \in \mathbb{N}_0$.

Proof. In order to prove these formulas we interpret the equation (2.8) into the following form

$$\begin{aligned} \mu_n^{-1/2,-1/2} &= \frac{1}{2^{n-2}(1+\delta_{n,0})} \int_0^{\pi} \sin^{2(-1/2)+1} \phi \log \sin \phi \cos n\phi \, d\phi \\ &= \frac{1}{2^{n-2}(1+\delta_{n,0})} \int_0^{\pi} \log \sin \phi \cos n\phi \, d\phi, \end{aligned}$$

which can be obtained from the previous using the substitution $x = \cos \phi$. With the same substitution, we get

$$\begin{aligned} \mu_n^{\ell-1/2,\ell-1/2} &= \frac{2^{2-n}}{1+\delta_{n,0}} \int_0^{\pi} \sin^{2(\ell-1/2)+1} \phi \log \sin \phi \cos n\phi \, d\phi \\ &= \frac{2^{2-n}}{1+\delta_{n,0}} \int_{-1}^{1} \log \sin \phi \cos n\phi \frac{1}{2^{2\ell}} \left[\sum_{k=0}^{\ell-1} 2 \binom{2n}{k} \cos \nu\phi + \binom{2n}{n} \right] d\phi \\ &= \frac{2^{1-n-2\ell}}{1+\delta_{n,0}} \left[\sum_{k=0}^{\ell-1} (-1)^{\ell-k} \binom{2\ell}{k} \left(\frac{1+\delta_{\nu+n,0}}{2^{1-\nu-n}} \mu_{\nu+n}^{-1/2,-1/2} \right.\right. \\ &+ \left.\left. \frac{1+\delta_{|\nu-n|,0}}{2^{1-|\nu-n|}} \mu_{|\nu-n|}^{-1/2,-1/2} \right) + \binom{2\ell}{\ell} \frac{1+\delta_{n,0}}{2^{1-n}} \mu_n^{-1/2,-1/2} \right], \end{aligned}$$

and also

$$\begin{aligned}
\eta_n^{\ell-1/2,\ell-1/2} &= \frac{1}{2^{n-1}}\int_0^\pi \sin^{2(\ell-1/2)}\phi\log\sin\phi\sin(n+1)\phi\,d\phi \\
&= \frac{1}{2^{n+2\ell-2}}\int_0^\pi \log\sin\phi\sin(n+1)\phi\left[\sum_{k=0}^{\ell-1}(-1)^{\ell+k-1}\binom{2\ell-1}{k}\sin(\nu-1)\phi\right]d\phi \\
&= \frac{1}{2^{n+2\ell-1}}\sum_{k=0}^{\ell-1}(-1)^{\ell+k-1}\binom{2\ell-1}{k}\left(\frac{1+\delta_{|n-\nu+2|,0}}{2^{1-|n-\nu+2|}}\mu_{|n-\nu+2|}^{-1/2,-1/2}\right. \\
&\quad \left. -\frac{1+\delta_{n+\nu,0}}{2^{1-n-\nu}}\mu_{n+\nu}^{-1/2,-1/2}\right),
\end{aligned}$$

where $\nu = 2(\ell-k)$. In the previous derivations we used identities

$$2^{n-1}(1+\delta_{n,0})T_n(\cos\phi) = \cos n\phi, \quad 2^n U_n(\cos\phi) = \frac{\sin(n+1)\phi}{\sin\phi}, \quad n\in\mathbb{N}_0$$

(see [2, pp. 140-145]). □

3. **Numerical construction**

The monic polynomials $\pi_k(x)$, $k\in\mathbb{N}_0$, orthogonal with respect to the functional $\mathcal{L}^{\alpha,\beta}$ given by (1.1), satisfy the three-term recurrence relation

$$\begin{aligned}
\pi_{k+1}(x) &= (x-\alpha_k)\pi_k(x) - \beta_k\pi_{k-1}(x), \quad k=0,1,\ldots, \\
\pi_0(x) &= 1, \quad \pi_{-1}(x) = 0,
\end{aligned} \tag{3.9}$$

with $\alpha_k\in\mathbb{R}$ and $\beta_k>0$. Let $\mu_k = \mathcal{L}^{\alpha,\beta}(x^k)$, $k\in\mathbb{N}_0$, be the corresponding moments. The first $2n$ moments $\mu_0,\mu_1,\ldots,\mu_{2n-1}$ uniquely determine the first n recurrence coefficients $\alpha_k=\alpha_k(\mathcal{L}^{\alpha,\beta})$ and $\beta_k=\beta_k(\mathcal{L}^{\alpha,\beta})$, $k=0,1,\ldots,n-1$, in (3.9). However, the corresponding map

$$[\mu_0\ \mu_1\ \mu_2\ \ldots\ \mu_{2n-1}]^{\mathrm{T}} \mapsto [\alpha_0\ \beta_0\ \alpha_1\ \beta_1\ \ldots\ \alpha_{n-1}\ \beta_{n-1}]^{\mathrm{T}}$$

is severely ill-conditioned when n is large. Namely, this map is very sensitive with respect to small perturbations in moment information (the first $2n$ moments). An analysis of such maps in details can be found in the recent book of Gautschi [4, Chapter 2].

For the numerical construction of the coefficients α_k and β_k in (3.9), for $k \le n-1$, we use the modified Chebyshev algorithm (see [6], [2, pp. 112-115], [4, pp. 76-78]). In fact, it is a generalization from ordinary to modified moments of an algorithm due to Chebyshev. Thus, instead of ordinary moments μ_k, $k = 0, 1, \dots, 2n-1$, we use the so-called *modified moments* $M_k = \mathcal{L}^{\alpha,\beta}(q_k)$, where $\{q_k(x)\}_{k\in\mathbb{N}_0}$ $(\deg q_k(x) = k)$ is a given system of polynomials chosen to be close in some sense to the desired orthogonal polynomials $\{\pi_k\}_{k\in\mathbb{N}_0}$. Then, the corresponding map

$$[M_0\ M_1\ M_2\ \dots\ M_{2n-1}]^{\mathrm{T}} \mapsto [\alpha_0\ \beta_0\ \alpha_1\ \beta_1\ \dots\ \alpha_{n-1}\ \beta_{n-1}]^{\mathrm{T}},$$

can become remarkably well-conditioned, especially for measures supported on a finite interval as is our case.

We suppose that the polynomials q_k are also monic and satisfy a three-term recurrence relation

$$q_{k+1}(x) = (x - a_k)q_k(x) - b_k q_{k-1}(x), \quad k = 0, 1, \dots,$$

where $q_{-1}(x) = 0$ and $q_0(x) = 1$, with given coefficients $a_n \in \mathbb{R}$ and $b_k \ge 0$. In the case $a_k = b_k = 0$, we have the monomials $q_k(x) = x^k$, and m_k reduce to the ordinary moments μ_k $(k \in \mathbb{N}_0)$.

Following Gautschi [4, pp. 76-78], we introduce the "mixed moments"

$$\sigma_{k,i} = \mathcal{L}^{\alpha,\beta}(\pi_k(x)q_i(x)), \quad k, i \ge -1. \tag{3.10}$$

Here, $\sigma_{0,i} = M_i$, $\sigma_{-1,i} = 0$ and, because of orthogonality, $\sigma_{k,i} = 0$ for $k > i$. Also, we take $\sigma_{0,0} = M_0 =: \beta_0$.

Starting with

$$\alpha_0 = a_0 + \frac{M_1}{M_0}, \quad \beta_0 = M_0,$$

the mixed moments (3.10) and the recursive coefficients α_k and β_k can be generated, for $k = 1, \dots, n-1$, by

$$\sigma_{k,i} = \sigma_{k-1,i+1} - (\alpha_{k-1} - a_i)\sigma_{k-1,i} - \beta_{k-1}\sigma_{k-2,i} + b_i\sigma_{k-1,i-1}, \quad i = k, \dots, 2n-k-1,$$

and

$$\alpha_k = a_k + \frac{\sigma_{k,k+1}}{\sigma_{k,k}} - \frac{\sigma_{k-1,k}}{\sigma_{k-1,k-1}}, \quad \beta_k = \frac{\sigma_{k,k}}{\sigma_{k-1,k-1}}.$$

k	$\alpha=\beta=-1/2$	$\alpha=\beta=1/2$	$\alpha=-\beta=1/2$	
0	−.435517218060720426(1)	−.606789763508705511	−.860673760222240852	−.435517218060720426(1)
1	.860673760222240851	.573587431195261228	.527113772343850128	.119914438687149513
2	.464736588514111009(−1)	.114461454116468030	−.323288979995422790	.222297722364958557
3	.437750434111890820	.391111576176891121	.252285050300644864	.225398401276919416
4	.138826525876246256	.161829438079109437	−.195426513976275507	.240891954171019052
5	.357255952055384943	.340206037337296702	.167009536268718202	.240001913598555923
6	.173196501068578500	.184754218459783853	−.140316170540163753	.245508439602626979
7	.325070388999947606	.316285630106789601	.124992947351188716	.244627540142048162
8	.191292682755600884	.198234729047124326	−.109507820167391439	.247327535832805290
9	.307742549214515765	.302391236700917694	.999144211328118150(−1)	.246657779751021059
10	.202476815568105879	.207103795178056862	−.898095844978158012(−1)	.248227380967697643
11	.296913379675872664	.293312118423073353	.832344255494742355(−1)	.247724032418561568
12	.210077692873599117	.213380332231185856	−.761239117883410348(−1)	.248737809281559934
13	.289504244019526891	.286914939447949928	.713341949682585194(−1)	.248351917023193687
14	.215580744479691408	.218055524644931387	−.660604287327404782(−1)	.249055199532678131
15	.284115953377671865	.282164324339524842	.624147238273606532(−1)	.248752298245571638
16	.219749600512164189	.221672621153416525	−.583483509979231300(−1)	.249266028855671693
17	.280020972151339655	.278497059977941169	.554798452588170684(−1)	.249023034384907651
18	.223017214719124101	.224554192048122145	−.522493851724157680(−1)	.249413227082215333
19	.276803577220537913	.275580500898679617	.499330667579243568(−1)	.249214543935549266
20	.225647434140755260	.226903820710798416	−.473051538327220806(−1)	.249520079948034282
21	.274208974543520496	.273205540289618306	.453952986211696832(−1)	.249354937727147130
22	.227810241668448622	.228856321932349324	−.432159773487912543(−1)	.249600111094014629
23	.272072299281140578	.271234133616303370	.416140361779129718(−1)	.249460889421264035
24	.229620097438390398	.230504511774221198	−.397776456662248980(−1)	.249661613626413084
25	.270282157440446096	.269571475997677721	.384145741637719542(−1)	.249542794493506471
26	.231156901833905767	.231914377880235599	−.368461961856252644(−1)	.249709903829893082
27	.268760574065860863	.268150301959223885	.356721760118687286(−1)	.249607405942853795
28	.232478125947355157	.233134118216279428	−.343172170827991634(−1)	.249748517822303506
29	.267451335299078592	.266921562814687583	.332953957694100852(−1)	.249659264304746552
30	.233626167045277497	.234199756690383015	−.321131355841375670(−1)	.249779882175776878
31	.266312892274520582	.265848650925761851	.312156637119852377(−1)	.249701513312081518
32	.234632987213776614	.235138758136264563	−.301751136273002383(−1)	.249805707166477196
33	.265313871763564801	.264903690385864234	.293805522152342180(−1)	.249736385250931688
34	.235523138170630016	.235972427512013477	−.284577133254071905(−1)	.249827226457090460
35	.264430140837420667	.264065080137475243	.277492886748764822(−1)	.249765500444632627
36	.236315791462598761	.236717544989121286	−.269252872049970366(−1)	.249845348133479749
37	.263642832194118322	.263315822453103279	.262896866353189157(−1)	.249790057958688233
38	.237026135817784364	.237387506514949344	−.255494758068136499(−1)	.249860752688816095
39	.262936982321762994	.262642358706969972	.249759982534029898(−1)	.249810960433893287

TABLE 3.1. Three-term recurrence coefficients for the linear functionals $\mathcal{L}^{-1/2,-1/2}$, $\mathcal{L}^{1/2,1/2}$ and $\mathcal{L}^{1/2,-1/2}$

Using Theorem 2.1 and Lemma 2.1, we calculate the modified moments of the functionals $\mathcal{L}^{\alpha,\beta}$. Using an implementation of the modified Chebyshev algorithm given in [7] we can construct the three-term recurrence coefficients of the monic polynomials π_k orthogonal with respect to $\mathcal{L}^{\alpha,\beta}$. In Table 3.1 we present the coefficients β_k for $k \le 39$ for polynomials orthogonal with respect the functionals $\mathcal{L}^{-1/2,-1/2}$ (second column) and $\mathcal{L}^{1/2,1/2}$ (third column). Numbers in parenthesis indicate decimal exponents. Note that $\alpha_k = 0$, $k \in \mathbb{N}_0$, due to the symmetry of the weights. Also, we give the coefficients α_k and β_k, $k \le 39$ (columns four and five in the same table), for polynomials orthogonal with respect to the linear functional $\mathcal{L}^{1/2,-1/2}$. For the computation of the integral $\mu_0^{\alpha,\beta}$ which is needed to start the computation according to recurrence relations given in Theorem 2.1, we refer to [5].

We report that computations are completely numerically stable, i.e., using this algorithm the precision of results are practically the same as the precision of the input data.

Finally, we are in the position to give an example. We consider the computation of the integral

$$\begin{aligned} I &= \int_{-1}^{1} \sqrt{\frac{1-x}{1+x}} \frac{4}{1+4x^2} \log(1-x^2)\, dx \qquad (3.11) \\ &= -4.15464458276047008962153413668307918164\dots\,. \end{aligned}$$

The construction of Gaussian quadrature rules for the linear functional $\mathcal{L}^{1/2,-1/2}$ can be performed numerically stable using Q-algorithm (see [9]) with three-term recurrence coefficients given in Table 3.1. Table 3.2 holds relative errors of the application of Gaussian quadrature rules with 10, 20, 30 and 40 points, as we inspect the convergence is evident.

n	10	20	30	40
rel. err.	1(−5)	5(−10)	3(−14)	m.p.

TABLE 3.2. Relative error in the computation of the integral (3.11), using Gaussian quadrature rules with n nodes

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