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REMARKS ON COMPUTING THE VALUE OF AN OPTION WITH BINOMIAL METHODS

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Dedicated to Professor D. D. Stancu on his 80^{th} birthday

Abstract. The purpose of this paper is to give a formula for computing the value of a financial option, using the binomial method.

1. Introduction

Binomial methods for valuing options and other derivative securities arise from discrete random walk models of the underlying security. This happens because the movement of asset prices is a random walk. It can be modeled, but any such model must incorporate a degree of randomness.

In valuating an option, the Black-Scholes formula is mostly used, the solution being obtained numerically, using the finite difference method, with serial and/or parallel algorithms (see [1], [2], [4]).

As is stated in [3] and [5], the binomial method is a particular case of the explicit finite difference method. Using this method, several serial and parallel algorithms are given. In what follows, we give a general formula for computing the value of an option, starting with discrete values at expiry date and using binomial methods.

2. Asset Price Random Walk

The theory of option pricing is based on the assumption that we do not know tomorrow's values of asset prices. We may use, anyway, the past history of the asset

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value, which tells us what are the likely jumps in asset price, what are their mean and variance and, more generally, what is the likely distribution of future asset prices.

It is known that asset prices move randomly. In order to model this movement, for each change in asset price, a **return** is associated, defined to be the change in the price divided by the original value (for more details, see [5]).

In order to get the equation which modeled this random walk, we consider that at time t, the asset price is S. In a small subsequence time interval, dt, the value S changes to $S+dS$. The corresponding return, $\frac{dS}{S}$, will be decomposed in two parts. One is predictable, deterministic, denoted by μdt, where μ is a measure of the average rate of growth of the asset price.

Note. In simple models, μ is taken to be a constant.

The second contribution to $\frac{dS}{S}$ models the random change in the asset price in response to external effects, such as unexpected news. It is represented by a random sample drawn from a normal distribution with mean zero and adds a term, σdX. Here, σ is a number called the **volatility**, which measures the standard deviation of the returns. The quantity dX is the sample from a normal distribution, with the mean zero and variance, dt.

We all this in mind, we obtain the stochastic differential equation

$$\frac{dS}{S} = \sigma dX + \mu dt \tag{2.1}$$

which is the mathematical representation of our simple recipe for generating asset prices.

3. Binomial Methods

3.1. Discrete random walks

In order to obtain binomial methods, we started from the idea that the continuous random walk given by (2.1) may be modeled by a discrete random walk, with the following properties:

• the asset price S changes only at the discrete times $\delta t, 2\delta t, 3\delta t, \ldots$ up to $M\delta t = T$, the expiry date of derivative security. We use δt instead of dt to denote the small but non-infinitesimal time-steps between movements in asset price.

• if the asset price is S^m at time step $m\delta t$ then at time $(m+1)\delta t$ it will take one of only two possible values; $uS^m > S^m$ or $vS^m > S^m$. It means that the asset price may move from S up to uS or down to vS. This is equivalent to the fact that there are only two returns $\dfrac{\delta S}{S}$ possible at each time step: $u - 1 > 0$ and $v - 1 < 0$, and these two returns are the same for all time steps.

• the probability, p, of S moving up to uS is known (as the probability $(1-p)$ of S moving down to vS).

Starting with a given value of the asset price (for example, to day's asset price) the remaining life-time of the derivative security is divided up into M time-steps of size $\delta t = (T-t)/M$. The asset price S is assumed to move only at times $m\delta t$ for $m = 1, 2, \ldots, M$. Then, a **tree** of all possible asset prices is created. This tree is constructed by starting with the given value S, generating the two possible asset prices (uS and vS) at the first time-step, then the three possible asset prices (u^2S, uvS and v^2S) at the second time-step, and so on, until the expiry time is reached.

Remark. We observe that after m time-steps, there are only $m+1$ possible asset prices.

3.2. Risk-neutral world

Another assumption in getting the binomial methods is a risk-neutral world. Under this circumstances, we may assume that the investitors are risk-neutral, and that the return from the underlying is the risk-free interest rate. Then, μ from (2.1), which does not appear into the Black-Scholes equation, is replaced by r, which appears in it and defined the interest rate.

So, in a risk-neutral world, equation (2.1) is replaced by

$$\frac{dS}{S} = \sigma dX + r dt. \tag{3.1}$$

The value of an option is then determined by calculating the present value of its expected return at expiry with the previous modification to the random walk. Having this in mind and, in addition, the fact that the present value of any amount at time T will be that amount discounted by multiplying by $e^{-r(T-t)}$ (for more details, see [5]), we may write the value V^m of the derivative security at time-step $m\delta t$ as the expected value of the security at time-step $(m+1)\delta t$ discounted by the risk-free interest rate r:

$$V^m = E(e^{-r\delta t} \cdot V^{m+1}) \tag{3.2}$$

Remark. Relation (3.2) is another way of interpreting the Black-Scholes formula.

3.3. How does a binomial method work

In a binomial method, we first build a tree of possible values of asset prices and their probabilities, given an initial asset price, then use this tree to determine the possible asset prices at expiry. The possible values of the security at expiry can then be calculated and, by working back, according with (3.2), the security can be valued.

In order to build up the tree of possible asset prices, we start at the current time $t = 0$. We assume that at this time we know the asset price, S_0^0. Then, at next time-step, δt, there are two possible asset prices: $S_1^1 = uS_0^0$ and $S_0^1 = vS_0^0$. At the following time-step, $2\delta t$, there are three possible asset prices: $S_2^2 = u^2 S_0^0$, $S_1^2 = uvS_0^0$ and $S_0^2 = v^2 S_0^0$. At the third time-step, $3\delta t$, the possible values are: $S_3^3 = u^3 S_0^0$, $S_2^3 = u^2 v S_0^0$, $S_1^3 = uv^2 S_0^0$ and $S_0^3 = v^3 S_0^0$, and so on.

At the m-th time-steps, $m\delta t$, there are $m+1$ possible values of the asset price,

$$S_n^m = u^n \cdot v^{m-n} \cdot S_0^0, \quad n = 0, 1, \ldots, m \tag{3.3}$$

Remark. In (3.3), S_n^m denotes the n-th possible value S at time-step $m\delta t$, whereas v^n and u^n denote v and u raised to the n-th power.

At the final time-step, $M\delta t$, we have $M+1$ possible values of the underlying asset, and we know all of them.

4. **Valuing the Option**

In what follows, we suppose that we know the payoff function for our derivative security and that it depends only on the values of the underlying asset at expiry. Then, we are able to value the option at expiry, i.e. time-step $M\delta t$. For example, for a call option, we find that

$$V_n^M = \max(S_n^m - E, 0), \quad n = 0, 1, \ldots, M \tag{4.1}$$

where E is the exercise price and V_n^M denotes the n-th possible value of the call at time-step M.

Then, we can find the expected value of the derivative security at the time-step prior to expiry, $(M-1)\delta t$, and for possible asset price S_n^{M-1}, $n = 0, 1, \ldots, M-1$, since we know the probability of an asset priced at S_n^{M-1} moving to S_{n+1}^M during a time-step is p, and the probability of it moving to S_n^M is $(1-p)$. Using the risk-neutral argument, we can calculate the value of the security at each possible asset price for the time-step $(M-1)$. Then, for $(M-2)$, and so on, back to time-step 0. This gives us the value of our option at the current time.

5. **The Case of European Option**

Let V_n^m denotes the value of the option at time-step $m\delta t$ and asset price S_n^m (where $0 \leq n \leq m$). According with (3.2), we calculate the expected value of the option at time-step $m\delta t$ from the values at time-step $(m+1)\delta t$ and discount in order to obtain the present value using the risk-free interest rate, r:

$$e^{r\delta t} \cdot V_n^m = p \cdot V_{n+1}^{m+1} + (1-p) \cdot V_n^{m+1} \tag{5.1}$$

which gives:

$$V_n^m = e^{-r\delta t}(p \cdot V_{n+1}^{m+1} + (1-p) \cdot V_n^{m+1}) \tag{5.2}$$

for every $n = 0, 1, \ldots, m$.

As we know the value of V_n^M, $n = 0, 1, \ldots, M$ from the payoff function, as in (4.1), we can, recursively, determine the values V_n^m for each $n = 0, 1, \ldots, m$, for $m < M$ to arrive at the current value of the option, V_0^0.

As in [5], the computation (5.2) may be permorned step by step, in M steps, to get the value V_0^0. We give another possible computation, based on the following theorem:

Theorem 1. *The value of the option at time-step* m*,* $0 \leq m \leq M$*,* V_n^m*, for every* $0 \leq n \leq m$ *can be calculated using only the values at expiry time,* V_n^M*,* $0 \leq n \leq m$*, according with the formula:*

$$C_n^m = \sum_{n=0}^{m} A_n \cdot V_n^M,$$

for every $0 \leq m \leq M$*, where* A_n*,* $0 \leq n \leq m$ *are the binomial coefficients of* $(\alpha+\beta)^m$*, where* $\alpha = e^{-r\delta t}p$ *and* $\beta = e^{-r\delta t}(1-p)$*.*

Proof. Using the notation α and β for the coefficients in (5.2), we have

$$V_n^m = \alpha V_{n+1}^{m+1} + \beta V_n^{m+1}, \tag{5.3}$$

for fixed m, $(m < M)$ and $0 \leq n \leq m$, or, in matriceal form:

$$\begin{bmatrix} V_0^m \\ V_1^m \\ \vdots \\ V_m^m \end{bmatrix} = \alpha \begin{bmatrix} V_1^{m+1} \\ V_2^{m+1} \\ \vdots \\ V_{m+1}^{m+1} \end{bmatrix} + \beta \begin{bmatrix} V_0^{m+1} \\ V_1^{m+1} \\ \vdots \\ V_m^{m+1} \end{bmatrix} \tag{5.4}$$

Knowing the values V_n^M, $n = 0, 1, \ldots, M$, we may compute the value V_n^{M-1}:

$$V_n^{M-1} = \alpha V_{n+1}^M + \beta V_n^M, \quad n = 0, 1, \ldots, M-1.$$

Then, at the step $(M-2)$, we get:

$$V_n^{M-2} = \alpha^2 V_{n+2}^M + \alpha\beta V_{n+1}^M + \beta^2 V_n^M, \quad n = 0, \ldots, M-2$$

and

$$V_n^{M-3} = \alpha^3 V_{n+3}^M + \alpha^2\beta V_{n+2}^M + \alpha\beta^2 V_{n+1}^M + \beta^3 V_n^m, \quad n = 0, \ldots, M-3$$

and so on, finally:

$$V_0^0 = \sum_{i=0}^{M} A_i \cdot V_i^M$$

where A_i are the binomial coefficients of $(\alpha+\beta)^M$. □

6. Conclusions

This method of computing the value of an option is more economical from time and memory space point of view than a serial computation made step by step, according with the step-time m. Our result indicates the resemblance of the binomial method with the finite-differences way of computation. The speed of computation can also be reduced by parallel calculus.

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