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FIXED POINT STRUCTURES WITH THE COMMON FIXED POINT PROPERTY: MULTIVALUED OPERATORS

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Dedicated to Professor Gheorghe Coman at his 70^{th} *anniversary*

Abstract. The concept of fixed point structure with the common fixed point property is extended to multivalued operators. In the terms of this concept some common fixed point theorems are given.

1. **Introduction**

In this paper we follow the notations and terminologies in I.A. Rus [10] and [12].
Let X be a nonempty set and $T, Q : X \to P(X)$ two multivalued operators.
In the present paper we shall consider the following problems:

Problem A. In which conditions we have that:

$$F_T \neq \emptyset, \quad F_Q \neq \emptyset, \quad T \circ Q = Q \circ T \ \Rightarrow\ F_T \cap F_Q \neq \emptyset?$$

Problem B. In which conditions we have that:

$$(SF)_T \neq \emptyset, \quad (SF)_Q \neq \emptyset, \quad T \circ Q = Q \circ T \ \Rightarrow\ (SF)_T \cap (SF)_Q \neq \emptyset?$$

The aim of this paper is to study these problems in terms of the fixed point structures ([10]).

We recall that if $T : X \to P(X)$ is a multivalued operator then we shall denote:

$$F_T := \{x \in X \mid x \in T(x)\};$$

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$$(SF)_T := \{x \in X \mid T(x) = \{x\}\};$$

$$I(T) := \{A \subset X \mid T(A) \subset A\}.$$

2. **Fixed point structures with the common fixed point property**

Definition 2.1. A fixed point structure $(X, S(X), M^0)$ on a set X (see [10]) is with the common fixed point property iff:

$$Y \in S(X), \quad T, Q \in M^0(Y), \quad T \circ Q = Q \circ T \;\Rightarrow\; F_T \cap F_Q \neq \emptyset.$$

Definition 2.2. A strict fixed point structure $(X, S(X), M^0)$ on a set X (see [10]) is with the common strict fixed point property iff:

$$Y \in S(X), \quad T, Q \in M^0(Y), \quad T \circ Q = Q \circ T \;\Rightarrow\; (SF)_T \cap (SF)_Q \neq \emptyset.$$

Remark 2.1. For the case of singlevalued operators see I.A. Rus [11].

Remark 2.2. For the common fixed point theorems in terms of the fixed point structures see A. Muntean [8] and A. Sîntămărian [14].

Remark 2.3. For the common fixed point theorems for the generalized commuting operators (weakly commuting, R-weakly commuting, compatible, δ-compatible,...) see G.F. Jungck [5], O. Hadzic [3], O. Hadzic and Lj. Gajic [4], B.E. Rhoades [9], A. Ahmad and M. Imdad [1], M.A. Ahmed [2], T. Kamran [6], H. Kaneko [7],...

Example 2.1. The trivial fixed point structure is a fixed point structure with the common fixed point property.

Example 2.2. Let (X, d) be a complete metric space, $S(X) := P_{cl}(X)$ and $M^0(Y) := \{T : Y \to P_{cl}(Y) \mid T$ is a multivalued contraction with $(SF)_T \neq \emptyset\}$. The triple $(X, P_{cl}(X), M^0)$ is a strict fixed point structure with the common strict fixed point property.

Indeed, from the Theorem 3.2 in [12] it follows that $(X, P_{cl}(X), M^0)$ is a strict fixed point structure. Let $Y \in P_{cl}(X)$, $T, Q \in M^0(Y)$ such that $T \circ Q = Q \circ T$. We have $F_T = (SF)_T = \{x^*\}$ and $F_Q = (SF)_Q = \{y^*\}$. From $T \circ Q = Q \circ T$ it follows that $x^* = y^*$.

Remark 2.4. For other examples see I.A. Rus [11], A. Muntean [8] and A. Sîntămărian [14].

Remark 2.5. To give examples of fixed point structures with the common fixed point property is one of the basic open problem of the common fixed point theory.

3. (θ, φ)-contraction pairs

Let X be a nonempty set, $Y \subset X$, $Z \subset P(X)$ and $\theta : Z \to \mathbb{R}_+$.

Definition 3.1. A pair of operators $T, Q : Y \to P(Y)$ is a (θ, φ)-contraction pair iff:

(i) $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function;

(ii) $A \in P(Y) \cap Z$ implies that $T(A) \cup Q(A) \in Z$;

(iii) $\theta(T(A) \cup Q(A)) \leq \varphi(\theta(A)),\ \forall\ A \in I(T) \cap I(Q) \cap Z$.

We have the following general common fixed point principles.

Theorem 3.1. *Let $(X, S(X), M^0)$ be a fixed point structure with the common fixed point property and (θ, η) a compatible pair with this fixed point structure. Let $Y \in \eta(Z)$ and $T, Q \in M^0(Y)$. We suppose that:*

(i) $\theta|_{\eta(Z)}$ has the intersection property;

(ii) $T \circ Q = Q \circ T$;

(iii) the pair (T, Q) is a (θ, φ)-contraction pair.

Then, $F_T \cap F_Q \neq \emptyset$.

Proof. Let $Y_1 := \eta(T(Y) \cup Q(Y)), \ldots, Y_{n+1} = \eta(T(Y_n) \cup Q(Y_n))$, $n \in \mathbb{N}$. First of all we remark that $Y_n \in I(T) \cap I(Q)$, $\forall\ n \in \mathbb{N}$. From the conditions (ii) and (iii) we have that

$$\theta(Y_{n+1}) = \theta(\eta(T(Y_n) \cup Q(Y_n))) = \theta(T(Y_n) \cup Q(Y_n))$$

$$\leq \varphi(\theta(Y_n)) \leq \cdots \leq \varphi^{n+1}(\theta(Y)) \to 0 \text{ as } n \to \infty.$$

From the condition (i) it follows that

$$Y_\infty := \bigcap_{n \in \mathbb{N}} Y_n \neq \emptyset \quad \text{and} \quad \theta(Y_\infty) = 0.$$

Now, we remark that $\eta(Y_\infty) = Y_\infty$ and $Y_\infty \in I(T) \cap I(Q)$. From the definition of the fixed point structure it follows that $Y_\infty \in S(X)$ and from Definition 2.1 the operators $T|_{Y_\infty}$ and $Q|_{Y_\infty}$ have a common fixed point. So, $F_T \cap F_Q \neq \emptyset$.

In a similar way we have

Theorem 3.2. *Let $(X, S(X), M^0)$ be a strict fixed point structure with the common fixed point property and (θ, η) a compatible pair with $(X, S(X), M^0)$. Let $Y \in \eta(Z)$ and $T, Q \in M^0(Y)$. We suppose that:*

(i) $\theta|_{\eta(Z)}$ has the intersection property;

(ii) $T \circ Q = Q \circ T$;

(iii) the pair (T, Q) is a (θ, φ)-contraction pair.

Then, $(SF)_T \cap (SF)_Q \neq \emptyset$.

4. θ-condensing pairs

Let X be a nonempty set, $Y \subset X$, $\theta : Z \to \mathbb{R}_+$ and $Z \subset P(X)$.

Definition 4.1. A pair $T, Q : Y \to P(Y)$ is a θ-condensing pair iff:

(i) $A_i \in Z$, $i \in I$, $\bigcap\limits_{i\in I} A_i \neq \emptyset \;\Rightarrow\; \bigcap\limits_{i\in I} A_i \in Z$;

(ii) $A \in P(Y) \cap Z \;\Rightarrow\; T(A) \cup Q(A) \in Z$;

(iii) $\theta(T(A) \cup Q(A)) < \theta(A)$, for all $A \in I(T) \cap I(Q) \cap Z$ such that $\theta(A) \neq \emptyset$.

We have

Theorem 4.1. *Let $(X, S(X), M^0)$ be a f.p.s. with the common fixed point property and (θ, η) a compatible pair with this fixed point structure. Let $Y \in \eta(Z)$ and $T, Q \in M^0(Y)$.*

We suppose that:

(i) $x \in Y$, $A \in Z$ imply $A \cup \{x\} \in Z$ and $\theta(A \cup \{x\}) = \theta(A)$;

(ii) $T \circ Q = Q \circ T$;

(iii) the pair (T, Q) is θ-condensing pair.

Then, $F_T \cap F_Q \neq \emptyset$.

Proof. Let $x_0 \in Y$. By Lemma 2.3 in [14] there exists $A_0 \subset Y$ such that $x_0 \in A_0$, $A_0 \in F_\eta \cap I(T) \cap I(Q)$ and $\eta(T(A_0) \cup Q(A_0) \cup \{x_0\}) = A_0$. From the condition (iii) it follows that $\theta(A_0) = 0$. But $\eta(A_0) = A_0$ and $\theta(A_0) = 0$ imply that $A_0 \in S(X)$. From the Definition 2.1 the operators $T|_{A_0}$ and $Q|_{A_0}$ have a common fixed point. So, $F_T \cap F_Q \neq \emptyset$.

In a similar way we have

Theorem 4.2. *Let $(X, S(X), M^0)$ be a strict fixed point structure with the common strict fixed point property and (θ, η) a compatible pair with this fixed point structure. Let $Y \in \eta(Z)$ and $T, Q \in M^0(Y)$. We suppose that:*

(i) $x \in Y,\ A \in Z$ imply $A \cup \{x\} \in Z$ and $\theta(A \cup \{x\}) = \theta(A)$;

(ii) $T \circ Q = Q \circ T$;

(iii) the pair (T, Q) is θ-condensing pair.

Then, $(SF)_T \cap (SF)_Q \neq \emptyset$.

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