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HOMOGENIZATION WITH MULTIPLE SCALE EXPANSION ON SELFSIMILAR STRUCTURES

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Dedicated to Professor Gheorghe Coman at his 70^{th} *anniversary*

Abstract. The homogenization theory is devoted to analysis of partial differential equations with rapidly oscillating coefficients. Let $\mathcal{A}^k$ be a given partial differential operator and we consider the equation $\mathcal{A}^k u^k = f$, together with the appropriate boundary initial conditions. Here $k \in \mathbb{N}$ and $f \in H^1(\mathbb{R}^n)$. We are interested in studying the solutions of this system in the limit as $k \to \infty$.

The homogenization theory is devoted to analysis of partial differential equations with rapidly oscillating coefficients. Let $\mathcal{A}^\epsilon$ be a given partial differential operator and we consider the equation

$$\mathcal{A}^\epsilon u^\epsilon = f,$$

together with the appropriate boundary initial conditions. Here ϵ is a small parameter $\epsilon << 1$, associated with the oscillations. We are interested in studying the solutions of this system in the limit as $\epsilon \to 0$. The homogenization theory study the following issues:

- Convergence to a limit.
- Characterization of the limiting process.

$$\mathcal{A}u = f$$

- Explicit analytical construction of $\mathcal{A}$.

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- Properties of the limiting equation.

This type of equation models various physical problems. As examples we mention composite materials, flow in porous media, atmospheric turbulence. A common feature of all these problems is that phenomena occur at various length and times scales. In the classical homogenization theory the structure are periodic. This implies that the coefficients of the corresponding PDE which model the physical phenomenon under investigation are periodic.

In this article we will quit the periodicity assumption. The coefficients are generated by an iterated function system. We will give a generalization of classical homogenization theory. We will use some basic notions from the fractal geometry as the invariant set and the Hutchinson's invariant measure regarding iterating function systems. Usually the solutions of the limiting equations live on selfsimilar fractals.

1. **Setting of the problem**

Consider the similarities $\varphi_1, ..., \varphi_m : \mathbb{R}^n \to \mathbb{R}^n$ with the scale factors $r_1, ..., r_n \in]0, 1[$, respectively. Suppose there exists an open, bounded set $O \subset \mathbb{R}^n$ such that $\varphi_i(O) \subset O$ and $\varphi_i(O) \cap \varphi_j(O) = \emptyset \, (i \neq j)$. For $i_1, ..., i_k \in \{1, ..., m\}$ denote σ the word $i_1...i_k$, and let $|\sigma| = k$ be the lenght of σ. Let $\varphi_\sigma = \varphi_{i_1} \circ ... \circ \varphi_{i_k}$ and $r_\sigma = r_{i_1}...r_{i_k}$.

For $A \subseteq \mathbb{R}^n$ put

$$F(A) := \varphi_1(A) \cup ... \cup \varphi_m(A),$$

$$F^1 := F, \, F^k := F \circ F^{k-1} \,\, (k \in \mathbb{N}), \text{and } K := \cap_{k \geq 1} F^k(\overline{O}).$$

Then we have

$$F(K) = K \subseteq \overline{O}$$

and

$$\lim_{k \to \infty} F^k(A) = K,$$

for every compact subset $A \subset \mathbb{R}^n$, where the limit is understood in sense of the Hausdorff metric. K is the unique nonempty compact set which is invariant under F.

Moreover, the Hausdorff dimension of K is the solution of

$$\sum_{i=1}^{m} r_i^s = 1.$$

If $\mathcal{H}^s(K)$ denotes the s-dimensional Hausdorff measure of K, then $0 < \mathcal{H}^s(K) < +\infty$.

Define $\mathcal{M}$ to be the set of positive, Borel regular measures μ on $\mathbb{R}^n$ having bounded support and finite mass. Put "spt" for support.

Define

$$\mathcal{M}^1 := \{\mu \in \mathcal{M} :\ \mu(\mathbb{R}^n) = 1\}.$$

Let

$$C(\mathbb{R}^n) := \{f : \mathbb{R}^n \to \mathbb{R} :\ f \text{ is continuous}\}.$$

For $\mu \in \mathcal{M}$, $\psi \in C(\mathbb{R}^n)$, define

$$\mu(\psi) := \int_{\mathbb{R}^n} \psi d\mu.$$

If $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, then we define the push forward measure $\varphi_\# : \mathcal{M} \to \mathcal{M}$ by

$$\varphi_\# \mu(A) := \mu(\varphi^{-1}(A)),\ A \subseteq \mathbb{R}^n,$$

equivalently

$$\varphi_\# \mu(\psi) := \mu(\psi \circ \varphi),\ \psi \in \mathbb{C}(\mathbb{R}^n).$$

We define the weak topology on $\mathcal{M}$ by taking as a sub-basis all sets of the form $\{\mu :\ a < \mu(\varphi) < b\}$, for arbitrary real $a < b$ and arbitrary $\varphi \in C(\mathbb{R}^n)$. We have $\mu_i \to \mu$ in the weak topology iff $\mu_i(\varphi) \to \mu(\varphi)$ for all $\varphi \in C(\mathbb{R}^n)$.

For $\mu, \nu \in \mathcal{M}^1$ let

$$L(\mu, \nu) := \sup\{\mu(\varphi) - \nu(\varphi) : \varphi \in C(\mathbb{R}^n),\ Lip\varphi \leq 1\}.$$

Then L is a metric on $\mathcal{M}^1$ and the metric topology coincide with the weak topology on $\mathcal{M}^1$. Moreover, the metric space $(\mathcal{M}^1, L)$ is complete (see [6]).

If $\nu \in \mathcal{M}^1$, let

$$G(\nu) := \sum_{i=1}^{m} r_i^s \varphi_{i\#} \nu.$$

Thus

$$G(\nu)(\psi)=\sum_{i=1}^{m} r_i^s \nu(\psi\circ\varphi_i),\ \psi\in C(\mathbb{R}^n),$$

and $G:\mathcal{M}^1\to\mathcal{M}^1$ is a contraction map. Consequently, there exists a unique measure $\overline{\mu}\in\mathcal{M}^1$, the *invariant measure on* K, such that

$$G(\overline{\mu})=\overline{\mu}\quad\text{ and }\quad spt\overline{\mu}=K.$$

For $\nu\in\mathcal{M}^1$ put

$$G^1(\nu):=G(\nu),\ G^k(\nu)=G\circ G^{k-1}(\nu),\ k\in\mathbb{N}.$$

It is easy to see that

$$G^k(\nu)=\sum_{|\sigma|=k} r_\sigma^s\varphi_{\sigma\#}(\nu),$$

and

$$\overline{\mu}=\lim_{k\to\infty}G^k(\nu)$$

in the sense of L metric.

Moreover,

$$\overline{\mu}=(\mathcal{H}^s(K))^{-1}\mathcal{H}^s|_K.$$

For all these properties we refer to Hutchinson [6].

Let $\Omega\subset\mathbb{R}^n$ be a nonempty bounded open domain with smooth boundary. Let ν be the Lebesgue measure on Ω divided by the Lebesque measure of Ω.

Let us suppose that the functions φ_i are composed by homotheties and translations, and $\Omega\subseteq O$. For the word σ consider the set

$$\Omega_\sigma:=\varphi_\sigma(\Omega).$$

In this paper we will develop the method of homogenization through multiple scales for the Dirichlet problem

$$\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}\left(\varphi_\sigma^{-1}(x)\right)\frac{\partial u^k(x)}{\partial x_j}\right)=f(x),\qquad\text{for } x\in\Omega_\sigma,\ \text{ for all } \sigma \text{ with } |\sigma|=k,\tag{1.1}$$

$$u^k(x)=0,\qquad\text{for } x\in\partial\Omega\sigma.\tag{1.2}$$

We are interested in studying the solutions u^k of (1.1) in the limit as $k \to \infty$. In particular, we would like to understand if the limit exists and what kind of properties does the limit u satisfy.

We will assume that the coefficients $a_{ij} : \mathbb{R}^n \to \mathbb{R}$ are smooth and uniformly elliptic on Ω, i.e. there exists $\alpha > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(y)\xi_i\xi_j \geq \alpha|\xi|^2, \; \forall y \in \mathbb{R}^n, \; \xi \in \mathbb{R}^n. \tag{1.3}$$

It is to accentuate, that the coefficients a_{ij} have not to be periodic. We will also assume that the function f is smooth and independent of k.

Example 1.1. *(Homogenization for periodic structures) Let* $Y = [0,1]^n$ *be the unit cell and suppose that the coefficients* a_{ij} *are* 1*-periodic, i.e.*

$$a_{ij}(y + e_k) = a_{ij}(y), \; i, j, k = 1, ..., n, \; y \in Y.$$

Define $\varphi_1(x) = \frac{x}{2}$. *For* $m = 2^n$ *and* $i \in \{2.,,,.m\}$ *we define the functions* $\varphi_i(x)$ *as translations of* φ_1 *by sumes of unit vectors such that* $Y = \cup_{i=1}^{n} \varphi_i(Y)$. *We can choose* $O = [0,1]^n$. *In this case, for* $|\sigma| = k$ *and* $\epsilon = \frac{1}{2^k}$, *we have* $a_{ij}(\varphi_\sigma^{-1}(x)) = a_{ij}(\frac{x}{\epsilon})$. *Therefore, in this case, problem (1.1) redeces to the homogenization problem for periodic structures (see* [4]*). Insteed of* $]0,1[^n$ *we can choose* O *as an open cub containing* Ω. *In this case the functions* φ_i *will be translated by a corresponding constant vector.*

Example 1.2. *(Homogenization on Sierpinski gasket) Let* $n = 2$, $m = 3$, *and* q_1, q_2, q_3 *be the vertices of an equilateral triangle. The functions* φ_i *are defined by*

$$\varphi_i(x) = \frac{1}{2}(x - q_i) + q_i, \; i = 1, 2, 3.$$

The Sierpinski gasket is the unique nonempty compact set $K \subset \mathbb{R}^2$ *such that*

$$K = \varphi_1(K) \cup \varphi_2(K) \cup \varphi_3(K).$$

The set K *is one of the simplest examples of a self-similar fractal. Its Hausdorff dimension is* $\frac{\log 3}{\log 2}$. *In this case* $r_1 = r_2 = r_3 = \frac{1}{2}$ *and* O *can be choosen as the interior* *of the triangle $q_1q_2q_3$. If we take $\Omega = O$, then the problem of homogenization reduces to the study of the limit of u^k on the Sierpinsku gasket.*

2. **The multiple scales expansion**

By the iterated functions system properties, for $\sigma \neq \sigma'$, $|\sigma| = |\sigma'|$ we have:

$$\Omega_\sigma \cap \Omega_{\sigma'} = \emptyset.$$

The idea behind the method of multiple scales is to assume that the solution u^k is of the form

$$u^k(x) = u_0(x, \varphi_\sigma^{-1}(x)) + r_\sigma u_1(x, \varphi_\sigma^{-1}(x)) + r_\sigma^2 u_2(x, \varphi_\sigma^{-1}(x)) + ..., \text{ for } x \in \Omega_\sigma \quad (2.4)$$

Using the two-scale convergence method, the validity of this expansion will be justified later. Anyway, since $\lim_{|\sigma|\to\infty} r_\sigma = 0$, from physical point of view it is reasonable to expect that the solution of (1.1) is of the (2.4) form, since there are different lenght scales in our problem and the above expansion takes this fact explicitly into account. The variables x and $y = \varphi_\sigma^{-1}(x)$ represent the "slow" (macroscopic) and "fast" (microscopic) scales of the problem respectively. For big $|\sigma|$ the variable y changes much more rapidly then x. We can think of x as being a constant, when looking at the problem at the microscopic scale. So we can treat x and y as independent variables.

The fact that $y = \varphi_\sigma^{-1}(x)$ implies that the partial derivatives with respect to x_j become:

$$\frac{\partial}{\partial x_j} \to \frac{\partial}{\partial x_j} + \frac{1}{r_\sigma}\frac{\partial}{\partial y_j},\ j = 1, ..., n.$$

Using this we can write the differential operator

$$\mathcal{A}^\sigma = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij} \circ \varphi_\sigma^{-1} \frac{\partial}{\partial x_j}\right) \quad (2.5)$$

in the form

$$\mathcal{A}^\sigma = r_\sigma^{-2}\mathcal{A}_0 + r_\sigma^{-1}\mathcal{A}_1 + \mathcal{A}_2, \quad (2.6)$$

where

$$\mathcal{A}_0 : \quad = \quad -\sum_{i,j=1}^{n} \frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial}{\partial y_j}\right), \tag{2.7}$$

$$\mathcal{A}_1 : \quad = \quad -\sum_{i,j=1}^{n} \left[\frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial}{\partial x_j}\right) + \frac{\partial}{\partial x_i}\left(a_{ij}(y)\frac{\partial}{\partial y_j}\right)\right], \tag{2.8}$$

$$\mathcal{A}_2 : \quad = \quad -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(y)\frac{\partial}{\partial x_j}\right). \tag{2.9}$$

Now, equation (1.1), on account of (2.6), becomes

$$\left(r_\sigma^{-2}\mathcal{A}_0 + r_\sigma^{-1}\mathcal{A}_1 + \mathcal{A}_2\right) u^k(x) = f(x), \text{ for } x \in \Omega\sigma \text{ and } |\sigma| = k. \tag{2.10}$$

If we substitute (2.4) into (2.10) we obtain the following sequence of problems:

$$\mathcal{A}_0 u_0 \quad = \quad 0 \tag{2.11}$$

$$\mathcal{A}_0 u_1 \quad = \quad -\mathcal{A}_1 u_0 \tag{2.12}$$

$$\mathcal{A}_0 u_2 \quad = \quad -\mathcal{A}_1 u_1 - \mathcal{A}_2 u_0. \tag{2.13}$$

These equations are of the form:

$$\mathcal{A}_0 u = h \tag{2.14}$$

Here $u = u(x, y)$ and $h = h(x, y)$. However x enters merely as a parameter since $\mathcal{A}_0$ is a uniformly elliptic partial differential operator with respect to y. This equation admits a unique, smooth solution if and only if the right hand side averages to 0 on Ω:

$$\int_\Omega h(x, y)d\nu(y) = 0, \tag{2.15}$$

This solvability condition is a consequence of the Fredholm alternative (see [5]).

By (1.3) the only solutions of the homogenous equation $\mathcal{A}_0 u_0 = 0$ are constants in y: $u_0(x, y) = u(x)$. This means that the first term in the multiple scales expansion is independent of the fast scales represented by y. Consequently, we can derive a homogenized equation for $u(x)$ which is independent of the microscopic scales.

In view of relation $u_0(x,y) = u(x)$ the equation (2.12) becomes:

$$\mathcal{A}_0 u_1 = -\sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial y_i} \frac{\partial u}{\partial x_j}. \tag{2.16}$$

Suppose

$$\sum_{i=1}^{n} \int_{\Omega} \frac{\partial a_{ij}(y)}{\partial y_i} d\nu(y) = 0, \text{ for all } j = 1, ..., n. \tag{2.17}$$

In this case, the solvability condition is satisfied, and equation (2.16) is well posed: it admits a unique solution, up to constants in y. To solve this equation we will use the separation of variable tehnique. To this end, we look for a solution of the form:

$$u_1(x,y) = -\sum_{j=1}^{n} \chi^j(y) \frac{\partial u(x)}{\partial x_j} + \overline{u}_1(x). \tag{2.18}$$

Substituting (2.18) into (2.16) we obtain:

$$\mathcal{A}_0 \chi^j(y) = -\sum_{i=1}^{n} \frac{\partial a_{ij}(y)}{\partial y_i}, \text{ for } j = 1, ..., n. \tag{2.19}$$

This is called the *cell problem* and $\chi^j(y)$ is the *first order corrector field*. Condition (2.17) implies that the problem is well posed. We remark that at this moment $\overline{u}_1$ is undetermined.

Now we consider equation (2.13). The function f being independent of y, then the solvability condition implies:

$$\int_{\Omega} [\mathcal{A}_1 u_1(x,y) + \mathcal{A}_2 u(x)]\, d\nu(y) = f(x). \tag{2.20}$$

We have:

$$\begin{aligned}
&\int_{\Omega} [\mathcal{A}_1 u_1(x,y) + \mathcal{A}_2 u(x)]\, d\nu(y) \\
&= \sum_{i,j=1}^{n} \int_{\Omega} \left[-\frac{\partial}{\partial y_i} \left(a_{ij}(y) \frac{\partial u_1}{\partial x_j} \right) - \frac{\partial}{\partial x_i} \left(a_{ij}(y) \frac{\partial u_1}{\partial y_j} \right) \right] d\nu(y) \\
&\quad + \sum_{i,j=1}^{n} \int_{\Omega} -\frac{\partial}{\partial x_i} \left(a_{ij}(y) \frac{\partial u(x)}{\partial x_j} \right) d\nu(y) \\
&= \sum_{i,j,l=1}^{n} \int_{\Omega} \frac{\partial}{\partial y_i} \left[a_{ij}(y) \chi^l(y) \frac{\partial^2 u(x)}{\partial x_l \partial x_j} \right] d\nu(y)
\end{aligned}$$

$$+\sum_{i,j,l=1}^{n}\frac{\partial^2 u(x)}{\partial x_i \partial x_l}\int_\Omega a_{ij}(y)\frac{\partial \chi^l(y)}{\partial y_j}d\nu(y)-\sum_{i,j=1}^{n}\frac{\partial^2 u(x)}{\partial x_i \partial x_j}\int_\Omega a_{ij}(y)d\nu(y)$$

$$=\sum_{i,l=1}^{n}\frac{\partial^2 u(x)}{\partial x_i \partial x_l}\int_\Omega\left\{\sum_{j=1}^{n}\left[a_{ij}(y)\frac{\partial \chi^l(y)}{\partial y_j}+\chi^l(y)\frac{\partial a_{ji}(y)}{\partial y_j}+a_{ji}(y)\frac{\partial \chi^l(y)}{\partial y_j}\right]-a_{il}(y)\right\}d\nu(y)$$

Denote

$$\overline{a}_{il}:=-\int_\Omega\left\{\sum_{j=1}^{n}\left[(a_{ij}(z)+a_{ji}(z))\frac{\partial \chi^l(z)}{\partial z_j}+\chi^l(z)\frac{\partial a_{ji}(z)}{\partial y_j}\right]-a_{il}(z)\right\}d\nu(z).$$

By this notation the homogenized equation will be the following:

$$-\sum_{i,l=1}^{n}\overline{a}_{il}\frac{\partial^2 u(x)}{\partial x_i \partial x_l} = f(x),\ \text{for } x\in\Omega_\sigma \tag{2.21}$$

$$u(x) = 0,\ \text{for } x\in\partial\Omega_\sigma, \tag{2.22}$$

and for all σ.

3. **Two scale convergence**

In this section we will recall the homogenization procedure given by [7]. The notion of two-scale convergence introduced by Nguetseng [8],[9] and developed futher by Allaire [1],[2] was modified by Kolumbán for iterating function system inthe following way:

Denote $C_b(\mathbb{R}^n)$ the set of bounded and continuous real functions defined on $\mathbb{R}^n$. We will also use the space $L^2_\sigma(\Omega, C_b(\mathbb{R}^n))$, which is the set of all measurable functions $u:\Omega\to C_b(\mathbb{R}^n)$ such that $\| u \|\in L^2(\Omega)$. The norm of this space is

$$\| u \|_{L^2(\Omega,C_b(\mathbb{R}^n))}=\left[\int_\Omega |\sup_{y\in\mathbb{R}^n} u(x,y)|^2dx\right]^{\frac{1}{2}}.$$

Theorem 3.1. *(Oscillations lemma,* [7]*) Let ν be the Lebesgue measure restricted to Ω and let $\Phi\in L^2(O,C_b(\mathbb{R}^n))$. Then the following convergence result holds:*

$$\lim_{k\to\infty}\sum_{|\sigma|=k} r_\sigma^s\int_\Omega \Phi(\varphi_\sigma(z),z)d\nu(z)=\int_K\left[\int_\Omega \Phi(x,y)d\nu(y)\right]d\overline{\mu}(x), \tag{3.23}$$

where K is the invariant set and $\overline{\mu}$ is the invariant measure of the iterated function system $\{\varphi_1,...,\varphi_m\}$.

Let u^k be a sequence in $L^2(O)$. We say that u^k two-scale converges weakly to $u_0 \in L^2(K \times \Omega)$ and write $u^k \xrightarrow{2} u_0$ if for every function $\Phi \in L^2(O, C_b(\mathbb{R}^n))$ we have

$$\lim_{k\to\infty} \sum_{|\sigma|=k} r_\sigma^s \int_\Omega u^k(\varphi_\sigma(x))\Phi(\varphi_\sigma(x), x)d\nu = \int_K \int_\Omega u_0(x,y)\Phi(x,y)d\nu(y)d\overline{\mu}(x). \quad (3.24)$$

Two-scale convergence implies a kind of weak convergence in $L^2(\Omega)$. In fact we have the following lemma:

Lemma 3.1. *([7]) Let u^k be in $L^2(\cup_{|\sigma|=k}\Omega_\sigma)$ which two-scale converges weakly to u_0. Then, for all $\Phi \in L^2(\Omega)$,*

$$\lim_{k\to\infty} \sum_{|\sigma|=k} r_\sigma^s \int_\Omega u^k(\varphi_\sigma(x))\Phi(\varphi_\sigma(x)d\nu(x) = \int_K \left(\int_\Omega u_0(x,y)d\nu(y)\right)\Phi(x)d\overline{\mu}(x). \quad (3.25)$$

The first result over the two scale expansion is the following lema:

Lemma 3.2. *Let $u^k \in L^2(\Omega)$ be a function which admits the two-scale expansion*

$$u^k(x) = u_0(x, \varphi_\sigma^{-1}(x)) + r_\sigma u_1(x, \varphi_\sigma^{-1}(x)) + ..., \text{ for } x \in \Omega_\sigma,$$

where $u_j \in L^2(\Omega, C_b(\mathbb{R}^n))$, $j \in \{0,1\}$. Then $u^k \xrightarrow{2} u_0$.

Proof. For $\Phi \in L^2(\Omega, C_b(\mathbb{R}^n))$ we have:

$$\begin{aligned}
&\sum_{|\sigma|=k} r_\sigma^s \int_{\Omega_\sigma} u^k(x)\Phi(x,\varphi_\sigma^{-1}(x))d\nu_\sigma(x) = \\
= &\sum_{|\sigma|=k} r_\sigma^s \int_\Omega u^k(\varphi_\sigma(x))\Phi(\varphi_\sigma(x),x)d\nu(x) = \\
= &\sum_{|\sigma|=k} r_\sigma^s \int_\Omega u_0(\varphi_\sigma(x),x)\Phi(\varphi_\sigma(x),x)d\nu(x) + \\
+ &\sum_{|\sigma|=k} r_\sigma^{s+1} \int_\Omega u_1(\varphi_\sigma(x),x)\Phi(\varphi_\sigma(x),x)d\nu(x) + ...
\end{aligned}$$

The first term converges to $\int_K \int_\Omega u_0(x,y)d\nu(y)d\overline{\mu}(x)$.

Using Cauchy-Schwartz inequality, we obtain:

$$
\begin{aligned}
&| \sum_{|\sigma|=k} r_\sigma^{s+1} \int_\Omega u_1(\varphi_\sigma(x), x)\Phi(\varphi_\sigma(x), x))d\nu(x)| \leq \\
\leq &\sum_{|\sigma|=k} r_\sigma^{s+1} \parallel u_1(\varphi(\cdot), \cdot) \parallel\parallel \Phi(\varphi(\cdot), \cdot) \parallel_{L^2(\Omega\times\Omega)} \leq \\
\leq &\parallel u_1 \parallel_{L^2(\Omega\times\Omega)} \parallel \Phi \parallel_{L^2(\Omega\times\Omega)} \sum_{|\sigma|=k} r_\sigma^{s+1} \to 0
\end{aligned}
$$

So

$$
\int_\Omega u^k(x)\Phi(x, \varphi_\sigma^{-1}(x))dx \to \int_K \int_\Omega u_0(x, y)\Phi(x, y)d\nu(y)d\overline{\mu}(x).
$$

Hence u^k two-scale converges weakly to u_0. □

We will use the following compactness result as a criteria which enable to conclude that a given sequence is weakly two-scale convergent.

Theorem 3.2. *([7]) Let $u^k \in L^2(\cup_{|\sigma|=k}\Omega_\sigma)$, $k \in \mathbb{N}$, and let*

$$
a_k := \left[\sum_{|\sigma|=k} r_\sigma^s \int_\Omega [u^k \circ \varphi_\sigma(x)]^2 d\nu(x) \right]^{\frac{1}{2}}.
$$

If the sequence (a_k) is bounded then there exists a subsequence of (u^k) which two-scale converges weakly to a function $u_0 \in L^2_{\overline{\mu}\otimes\nu}(K \times \Omega)$.

The weakly two-scale convergence defined is still a weak type of convergence, since it is defined in terms of the product of a sequence u^k with an appropriate test function. We also define a notion of strong two-scale convergence.

Let u^k be a sequence in $L^2(\Omega)$. We say that u^k two-scale converges strongly to $u_0 \in L^2_{O\times\Omega}$ and write $u^k \xrightarrow{2} u_0$ if

$$
\lim_{k\to\infty} \sum_{|\sigma|=k} r_\sigma^s \int_\Omega |u^k(\varphi_\sigma(x))|^2 d\nu(x) = \int_K \int_\Omega |u_0(x, y)|^2 d\nu(y)d\overline{\mu}(x) \tag{3.26}
$$

Although every strongly two-scale convergent sequence is also weakly two-scale convergent, the converse is not true.

As it is always the case with weak convergence, the limit of the product of two-scale convergent sequences is not in general the product of the limits. However, we can pass to the limit when we one of the two sequence is strongly two-scale convergent.

The next theorem can be proved as the similar result in the classical homogenization theory (see [2]).

Theorem 3.3. *Suppose* $u^k \xrightarrow{2} u_0$ *and* $v^k \xrightarrow{2} v_0$. *Then* $u^k v^k \xrightarrow{2} u_0 v_0$.

For simplicity in the following we suppose $n = 1$ and $O = \Omega =]a, b[$, $[a, b] \subseteq \mathbb{R}$. Let μ be an atomless finite Borel measure on $[a, b]$. Further let $K := spt\mu$ with $a, b \in K$ and $L_2 := L^2_\mu(K)$ be the separable Hilbert space with scalar product $\langle f, g\rangle = \int_a^b fg d\mu$. We define

$$\mathcal{D}_1^\mu := \{f \in L_2 : \exists f' \in L_2 \text{ with } f(x) = f(a) + \int_a^x f'(y)d\mu(y),\ x \in spt\mu\}.$$

Proposition 3.1. $\mathcal{D}_1^\mu \subset C(K)$, *i.e. every function in* $\mathcal{D}_1^\mu$ *is continuous on* K, *and the function* f' *defined above is unique in* L_2.

So we can introduce the $\mu-$ derivative of f. The μ-derivative of f on $\mathcal{D}_1^\mu$ is

$$\nabla^\mu f = \frac{df}{d\mu}.$$

In the case $\mu = \nu$, where ν denotes Lebesgue measure on $\mathbb{R}$, $\mathcal{D}_1^\mu$ coincides with the Sobolev space W_2^1. As in the classical Lebesgue case the $\mu-$Dirichlet form on $\mathcal{D}_1^\mu$ is defined as

$$\mathcal{E}^\mu(f, g) := \langle \nabla^\mu f, \nabla^\mu g\rangle$$

Denote

$$\mathcal{D}_2^\mu := \{f \in \mathcal{D}_1^\mu :\ \nabla^\mu f \in \mathcal{D}_1^\mu\}.$$

The $\mu-$Laplace operator from $\mathcal{D}_2^\mu$ is given by

$$\Delta^\mu f := \nabla^\mu(\nabla^\mu f) = f'.$$

Remark 3.1.

$$\mathcal{D}_2^\mu = \Bigg\{ f \in L_2 :\ \exists f', f'' \in L_2 \textit{ with } f(x) = f(a) + \int_a^x f'(y)d\mu(y),\ x \in K,$$
$$f'(y) = f'(a) + \int_a^y f''(z)d\mu(z),\ y \in K \Bigg\}.$$

Using Fubini's theorem we have the following representation of $f \in \mathcal{D}_2^\mu$*:*

$$f(x) = f(a) + \nabla^\mu f(a)\mu([a, x]) + \int_a^x \mu([y, x])\Delta^\mu f(y)d\mu(y),\ x \in K.$$

Proposition 3.2. *For any $c, d \in K$ with $c \leq d$ and $f, g \in \mathcal{D}_1^\mu$ we have*

$$\int_c^d (\nabla^\mu f) g d\mu = fg|_c^d - \int_c^d f(\nabla^\mu g) d\mu$$

Proof. By definition of ∇^μ and Fubini theorem it follows that

$$\begin{aligned}
& \int_c^d (\nabla^\mu f)(x) g(x) d\mu(x) \\
= \quad & \int_c^d (\nabla^\mu f)(x) \left[g(c) + \int_c^x (\nabla^\mu g)(y) d\mu(y) \right] d\mu(x) \\
= \quad & g(c)\left[f(d) - f(c)\right] + \int_c^d (\nabla^\mu g)(y) \left[\int_y^d (\nabla^\mu f)(x) d\mu(x) \right] d\mu(y) \\
= \quad & g(c)\left[f(d) - f(c)\right] + \int_c^d (\nabla^\mu g)(y) \left[f(d) - f(y)\right] d\mu(y) \\
= \quad & g(c)\left[f(d) - f(c)\right] + f(d)[g(d) - g(c)] - \int_c^d f(y)(\nabla^\mu g)(y) d\mu(y) \\
= \quad & f(d)g(d) - f(c)g(c) - \int_c^d f(y)(\nabla^\mu g)(y) d\mu(y). \qquad (3.27)
\end{aligned}$$

□

In the similar way we can prove the following proposition:

Proposition 3.3. *For any $c, d \in K$ with $c \leq d$ and $f, g \in \mathcal{D}_2^\mu$ we have*

$$\int_c^d (\Delta^\mu f) g d\mu = (\nabla^\mu f) g|_c^d - \int_c^d (\nabla^\mu f)(\nabla^\mu g) d\mu$$

$$\int_c^d \left[(\Delta^\mu f) g - f(\Delta^\mu g)\right] d\mu = (\nabla^\mu f) g - f(\nabla^\mu g)|_c^d$$

These are analogues of the classical Gauss Green formulae.

Now we introduce the Dirichlet boundary condition

$$\mathcal{D}_{2,D}^\mu := \{f \in \mathcal{D}_2^\mu : \, f(a) = f(b) = 0\}.$$

From the last proposition we obtain the following

Corollary 3.1. *$-\Delta^\mu$ is a positive symmetric operator on $\mathcal{D}_{2,D}^\mu$.*

Theorem 3.4. *Let $u^k \in H_0^1(\cup_{|\sigma|=k}\Omega_\sigma)$. If*

$$a_k := \left[\sum_{|\sigma|=k} r_\sigma^s \int_\Omega [u^k \circ \varphi_\sigma(x)]^2 d\nu(x)\right]^{\frac{1}{2}}$$

and

$$b_k := \left[\sum_{|\sigma|=k} r_\sigma^s \int_\Omega [\nabla u^k \circ \varphi_\sigma(x)]^2 d\nu(x)\right]^{\frac{1}{2}}$$

are bounded then there exist a subsequence of (u^k) and functions $u_0(\cdot), u(\cdot) \in L^2_{\overline{\mu}}(K)$ with $\nabla^{\overline{\mu}} u \in L^2_{\overline{\mu}}(K)$, such that

$$\begin{aligned}\lim_{k\to\infty} \sum_{|\sigma|=k} r_\sigma^s \int_{\Omega_\sigma} (\nabla u^k)(x)\varphi(x)d\nu_\sigma(x) &= \int_K \nabla^{\overline{\mu}} u(x)\Phi(x)d\overline{\mu}(x) = \\ &= -\int_K u_0(x)\nabla\Phi(x)d\overline{\mu}(x),\ \forall \Phi \in H_0^1(O).\end{aligned}$$

Proof. By Theorem 3.2 we can choose a subsequence denoted by u^k too, a function $u_0 \in L^2(K\times\Omega)$ and a function $v \in L^2(K\times\Omega)$ such that $u^k \xrightarrow{2} u_0$ and $\nabla u^k \xrightarrow{2} v$.

First we prove the independence of u_0 of the second variable y. To this end let $\Phi \in C^1(O\times O)$. We have

$$\begin{aligned}&\sum_{|\sigma|=k} r_\sigma^s \int_\Omega (\nabla u^k)(\varphi_\sigma(z))\Phi(\varphi_\sigma(z),z)d\nu(z) = \\ = \quad &-\sum_{|\sigma|=k} r_\sigma^s \int_\Omega u^k(\varphi_\sigma(z))[r_\sigma \nabla_x \Phi(\varphi_\sigma(z),z) + \nabla_y \Phi(\varphi_\sigma(z),z))]d\nu(z)\end{aligned}$$

Since $u^k(\varphi_\sigma(z))\nabla_x\Phi(\varphi_\sigma(z),z)$ is bounded in $L^2(\Omega)$ it follows that

$$\sum_{|\sigma|=k} r_\sigma^{s+1} \int_\Omega u^k(\varphi_\sigma(z))\nabla_x\Phi(\varphi_\sigma(z),z)d\nu(z) \to 0.$$

This implies

$$\sum_{|\sigma|=k} r_\sigma^s \int_\Omega (\nabla u^k)(\varphi_\sigma(z))\Phi(\varphi_\sigma(z),z)d\nu(z) \to -\int_K\int_\Omega u_0(x,y)\nabla_y\Phi(x,y)d\overline{\mu}(x)d\nu(y). \tag{3.28}$$

On the other hand

$$\frac{1}{r_\sigma}\nabla u^k(\varphi_\sigma(z))\Phi(\varphi_\sigma(z),z))$$

is bounded in $L^2(\Omega)$ which implies the convergence to 0. Consequently

$$-\int_K\int_\Omega u_0(x,y)\nabla_y\Phi(x,y)d\overline{\mu}(x)d\nu(y)=0$$

for all Φ. Hence the two scale limit is a.e. independent of y, i.e. $u_0(x,y)=u_0(x)$.

Let us now suppose Φ is independent of y. We compute

$$\begin{aligned}
&\sum_{|\sigma|=k} r_\sigma^s\int_\Omega(\nabla u^k)(\varphi_\sigma(z))\Phi(\varphi_\sigma(z))d\nu(z)=\\
=&\quad -\sum_{|\sigma|=k} r_\sigma^s\int_\Omega u^k(\varphi_\sigma(z))\nabla\Phi(\varphi_\sigma(z))d\nu(z)\to\\
\to&\quad -\int_K\int_\Omega u_0(x)\nabla\Phi(x)d\nu(y)d\overline{\mu}(x)=-\int_K u_0(x)\nabla\Phi(x)d\overline{\mu}(x)
\end{aligned}$$

According to the two-scale convergence of ∇u^k, we have

$$\begin{aligned}
&\sum_{|\sigma|=k} r_\sigma^s\int_\Omega(\nabla u^k)(\varphi_\sigma(z))\Phi(\varphi_\sigma(z))d\nu(z)\to\\
\to&\quad \int_K\int_\Omega v(x,y)\Phi(x)d\nu(y)d\overline{\mu}(x).
\end{aligned}$$

The last two relations implies that

$$-\int_K u_0(x)\nabla\Phi(x)d\overline{\mu}(x)=\int_K\int_\Omega v(x,y)\Phi(x)d\nu(y)d\overline{\mu}(x)\tag{3.29}$$

for all $\Phi\in C^1(\Omega)$. Denote

$$V(x)=\int_\Omega v(x,y)d\nu(y)\quad\text{and } u(x)=\int_a^x V(x)d\overline{\mu}(x).$$

Then

$$V(x)=\nabla^{\overline{\mu}}u(x).$$

By a density argument on Φ the assertion follows. □

References

[1] Allaire, G., *Homogeneization et convergence a deux echelle, application a un probleme de convection diffusion*, C.R. Acad Sci. Paris, **312**(1991), 581-586.

[2] Allaire, G., *Homogenization and two-scale convergence*, SIAM J. Math. Anal. **23**(6) (1992), 1482-1518.

[3] Bensoussan, A., Lions, J.L., Papanicolau, G., *Asymptotic analysis of periodic structures,* North Holland, Amsterdam, 1978.

[4] Cioranescu, D., Donato, P., *An Introduction to Homogenization*, Oxford University Press, New York, 1999.

[5] Evans, L.C., *Partial Differential Equations*, AMS, Providence, Rhode Island, 1998.

[6] Hutchinson, J.E., *Fractals and Self Similarity,* Indiana University Mathematics Journal, **30**(1981), no.5, 713-747.

[7] Kolumbán, J., *Two scale convergence of measures* (to appear).

[8] Nguetseng, G., *A general convergence result for a functional related to the theory of homogenization,* SIAM J. Math. Anal., **20**(3)(1989), 608-623.

[9] Nguetseng, G., *Asymptotic analysis for a still variational problem arising in mechanics*, SIAM J. Math. Anal. **21**(6)(1989), 1394-1414.

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