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COMPACT OPERATORS ON SPACES WITH ASYMMETRIC NORM

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Dedicated to Professor Gheorghe Coman at his 70^{th} *anniversary*

Abstract. The aim of the present paper is to define compact operators on asymmetric normed spaces and to study some of their properties. The dual of a bounded linear operator is defined and a Schauder type theorem is proved within this framework. The paper contains also a short discussion on various completeness notions for quasi-metric and for quasi-uniform spaces.

1. **Introduction**

An asymmetric norm on a real vector space X is a functional $p : X \to [0, \infty)$ satisfying the conditions

$$\text{(AN1) } p(x) = p(-x) = 0 \Rightarrow x = 0; \quad \text{(AN2) } p(\alpha x) = \alpha p(x);$$

$$\text{(AN3) } p(x + y) \le p(x) + p(y),$$

for all $x, y \in X$ and $\alpha \ge 0$. A quasi-metric on a set X is a mapping $\rho : X \times X \to [0, \infty)$ satisfying the conditions

$$\text{(QM1) } \rho(x, y) = \rho(y, x) = 0 \iff x = y; \quad \text{(QM2) } \rho(x, z) \le \rho(x, y) + \rho(y, z),$$

for all $x, y, z \in X$. If the mapping ρ satisfies only the conditions $\rho(x, x) = 0$, $x \in X$, and (QM2), then it is called a *quasi-pseudometric*. If p is an asymmetric norm on a vector space X, then the pair (X, p) is called an asymmetric normed space. Similarly,

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(X,ρ) is called a quasi-metric space. If p is an asymmetric norm on a vector space X, then $\rho(x,y)=p(y-x),\ x,y\in X$, is a quasi-metric on X. A closed, respectively open, ball in a quasi-metric space is defined by

$$B_\rho(x,r)=\{y\in X:\rho(x,y)\le r\},\quad B'_\rho(x,r)=\{y\in X:\rho(x,y)<r\},$$

for $x\in X$ and $r>0$. In the case of an asymmetric norm p one denotes by $B_p(x,r),B'_p(x,r)$ the corresponding balls and by $B_p=B_p(0,1),B'_p=B'_p(0,1)$, the unit balls. In this case the following equalities hold

$$B_p(x,r)=x+rB_p\quad\text{and}\quad B'_p(x,r)=x+rB'_p.$$

The family of sets $B'_\rho(x,r)$, $r>0$, is a base of neighborhoods of the point $x\in X$ for the topology τ_ρ on X generated by the quasi-metric ρ. The family $B_\rho(x,r)$, $r>0$, of closed balls is also a neighborhood base at x for τ_ρ.

A quasi-uniformity on a set X is a filter $\mathcal{U}$ such that

(QU1) $\quad\Delta(X)\subset U,\ \forall U\in\mathcal{U}$;

(QU1) $\quad\forall U\in\mathcal{U},\ \exists V\in\mathcal{U},$ such that $V\circ V\subset U$,

where $\Delta(X)=\{(x,x):x\in X\}$ denotes the diagonal of X and, for $M,N\subset X\times X$,

$$M\circ N=\{(x,z)\in X\times X:\exists y\in X,\ (x,y)\in M\text{ and }(y,z)\in N\}.$$

If the filter $\mathcal{U}$ satisfies also the condition

$$\text{(U3)}\qquad\forall U,\ U\in\mathcal{U}\ \Rightarrow\ U^{-1}\in\mathcal{U},$$

where

$$U^{-1}=\{(y,x)\in X\times X:(x,y)\in U\},$$

then $\mathcal{U}$ is called a uniformity on X. The sets in $\mathcal{U}$ are called *entourages* (or *vicinities*).

For $U\in\mathcal{U}$, $x\in X$ and $Z\subset X$ put

$$U(x)=\{y\in X:(x,y)\in U\}\quad\text{and}\quad U[Z]=\cup\{U(z):z\in Z\}.$$

A quasi-uniformity $\mathcal{U}$ generates a topology $\tau(\mathcal{U})$ on X for which the family of sets

$$\{U(x):U\in\mathcal{U}\}$$

is a base of neighborhoods of the point $x \in X$. A mapping f between two quasi-uniform spaces $(X, \mathcal{U})$, $(Y, \mathcal{W})$ is called *quasi-uniformly continuous* if for every $W \in \mathcal{W}$ there exists $U \in \mathcal{U}$ such that $(f(x), f(y)) \in W$ for all $(x, y) \in U$. By the definition of the topology generated by a quasi-uniformity, it is clear that a quasi-uniformly continuous mapping is continuous with respect to the topologies $\tau(\mathcal{U})$, $\tau(\mathcal{W})$.

If (X, ρ) is a quasi-metric space, then

$$B'_\epsilon = \{(x, y) \in X \times X : \rho(x, y) < \epsilon\},\ \epsilon > 0,$$

is a basis for a quasi-uniformity $\mathcal{U}_\rho$ on X. The family

$$B_\epsilon = \{(x, y) \in X \times X : \rho(x, y) \le \epsilon\},\ \epsilon > 0,$$

generates the same quasi-uniformity. The topologies generated by the quasi-metric ρ and by the quasi-uniformity $\mathcal{U}_\rho$ agree, i.e., $\tau_\rho = \tau(\mathcal{U}_\rho)$.

The lack of the symmetry, i.e., the omission of the axiom (U3), makes the theory of quasi-uniform spaces to differ drastically from that of uniform spaces. An account of the theory up to 1982 is given in the book by Fletcher and Lindgren [21]. The survey papers by Künzi [32, 33, 34, 35] are good guides for subsequent developments. Another book on quasi-uniform spaces is [38].

On the other hand, the theory of asymmetric normed spaces has been developed in a series of papers [6], [8], [22], [23], [24], [25], [25], [26], following ideas from the theory of (symmetric) normed spaces and emphasizing similarities as well as differences between the symmetric and the asymmetric case.

Let (X, p) be an asymmetric normed space. The functional $\bar{p}(x) = p(-x)$, $x \in X$, is also an asymmetric norm on X, called the conjugate of p, $p_s(x) = \max\{p(x), \bar{p}(x)\}$, $x \in X$, is a (symmetric) norm on X and the following inequalities hold

$$|p(x) - p(y)| \le p_s(x - y) \quad \text{and} \quad |\bar{p}(x) - \bar{p}(y)| \le p_s(x - y),\ \forall x, y \in X.$$

For a quasi-metric space one defines similarly the conjugate of ρ by $\bar{\rho}(x,y) = \rho(y,x)$ and the associated (symmetric) metric by $\rho_s(x,y) = \max\{\rho(x,y), \rho(y,x)\}$, for $x, y \in X$.

Let (X,p), (Y,q) be two asymmetric normed space. A linear mapping $A : X \to Y$ is called *bounded*, ((p,q)-bounded if more precision is needed), or *semi-Lipschitz*, if there exists a number $\beta \geq 0$ such that

$$q(Ax) \leq \beta p(x), \tag{1.1}$$

for all $x \in X$. The number β is called a semi-Lipschitz constant for A. For properties of semi-Lipschitz functions and of spaces of semi-Lipschitz functions see [39, 40, 44, 45].

The operator A is continuous with respect to the topologies τ_p, τ_q ((τ_p, τ_q)-continuous) if and only if it is bounded and if and only if it is quasi-uniformly continuous with respect to the quasi-uniformities $\mathcal{U}_p$ and $\mathcal{U}_q$ (see [20] and [24]). Denote by $(X,Y)^\flat_{p,q}$, or simply by $(X,Y)^\flat$ when there is no danger of confusion, the set of all (p,q)-bounded linear operators. The set $(X,Y)^\flat$ need not be a linear subspace but merely a convex cone in the space $(X,Y)^\#$ of all linear operators from X to Y, i.e., $A+B \in (X,Y)^\flat$ and $\alpha A \in (X,Y)^\flat$, for any $A, B \in (X,Y)^\flat$ and $\alpha \geq 0$. Following [24], we shall call $(X,Y)^\flat$ a *semilinear space*. The functional

$$\|A| = \|A|_{p,q} = \sup\{q(Ax) : x \in B_p\} \tag{1.2}$$

is an asymmetric norm on the semilinear space $(X,Y)^\flat$, and $\|A|$ is the smallest semi-Lipschitz constant for A, i.e., the smallest number for which the inequality (1.1) holds.

Denote by $(X,Y)^*_s$ the space of all continuous linear operators from (X,p_s) to (Y,q_s), normed by

$$\|A\| = \|A\|_{p_s,q_s} = \sup\{q_s(Ax) : x \in X,\ p_s(x) \leq 1\}, \quad A \in (X,Y)^*_s. \tag{1.3}$$

It was shown in [24] that $(X,Y)^\flat_{p,q} \subset (X,Y)^*_s$, and $\|A| \leq \|A\|$ for any $A \in (X,Y)^\flat$.

Consider on $\mathbb{R}$ the asymmetric norm $u(\alpha) = \max\{\alpha, 0\}$, $\alpha \in \mathbb{R}$. Its conjugate is $\bar{u}(\alpha) = \max\{-\alpha, 0\}$ and $u_s(\alpha) = |\alpha|$ is the absolute value norm on $\mathbb{R}$. The topology

τ_u on $\mathbb{R}$ generated by u, called the upper topology of $\mathbb{R}$, has as neighborhood basis of a point $\alpha \in \mathbb{R}$ the family of intervals $(-\infty, \alpha+\epsilon)$, $\epsilon > 0$.

The space of all linear bounded functionals from an asymmetric normed space (X, p) to $(\mathbb{R}, u)$ is denoted by $X_p^\flat$. Notice that, due to the fact that p is non-negative, we have

$$\forall x \in X,\ u(\varphi(x)) \leq \beta p(x) \iff \varphi(x) \leq \beta p(x),$$

for any linear functional $\varphi : X \to \mathbb{R}$, so the asymmetric norm of a functional $\varphi \in X_p^\flat$ is given by

$$\|\varphi| = \|\varphi|_p = \sup\{\varphi(x) : x \in X,\ p(x) \leq 1\}.$$

Also, the continuity of φ from (X, τ_p) to $(\mathbb{R}, \tau_u)$ is equivalent to its upper semi-continuity from (X, τ_p) to $(\mathbb{R}, |\,|)$, (see [1, 2, 20]).

In [24] it was defined the analog of the w^*-topology on the space $X_p^\flat$, which we denote by $w^\flat$, having as a base of $w^\flat$-neighborhoods of an element $\varphi_0 \in X_p^\flat$ the sets

$$V_{x_1,...,x_n;\,\epsilon}(\varphi_0) = \{\varphi \in X_p^\flat : \varphi(x_i) - \varphi_0(x_i) \leq \epsilon,\ i = 1, ..., n\}, \tag{1.4}$$

for $n \in \mathbb{N}$, $x_1, ..., x_n \in X$, and $\epsilon > 0$.

Since

$$V_{x;\,\epsilon}(\varphi_0) \cap V_{-x;\,\epsilon}(\varphi_0) = \{\varphi \in X_p^\flat : |\varphi(x) - \varphi_0(x)| \leq \epsilon\},$$

it follows that the topology $w^\flat$ is the restriction to $X^\flat$ of the w^*-topology of $X_s^* = (X, p_s)^*$.

Some results on $w^\flat$-topology were proved in [24] as, for instance, the analog of the Alaoglu-Bourbaki theorem: the polar

$$B_p^\flat = \{\varphi \in X^\flat : \varphi(x) \leq 1,\ \forall x \in B_p\} \tag{1.5}$$

of the unit ball B_p of (X, p) is $w^\flat$-compact. Other results on asymmetric normed spaces, including separation of convex sets by closed hyperplanes and a Krein-Milman type theorem, were obtained in [6]. Asymmetric locally convex spaces were considered in [7]. Best approximation problems in asymmetric normed spaces were studied in [6] and [8].

The topology $w^\flat$ is derived from a quasi-uniformity $\mathcal{W}_p^\flat$ on $X_p^\flat$ with a basis formed of the sets

$$V_{x_1,...,x_n;\,\epsilon} = \{(\varphi_1,\varphi_2) \in X_p^\flat \times X_p^\flat : \varphi_2(x_i) - \varphi_1(x_i) \le \epsilon,\ i = 1,...,n\}, \tag{1.6}$$

for $n \in \mathbb{N}$, $x_1,...,x_n \in X$ and $\epsilon > 0$. Note that, for fixed $\varphi_1 = \varphi_0$, one obtains the neighborhoods from (1.4).

On the space $(X,Y)_s^*$ we shall consider several quasi-uniformities. Namely, for $\mu \in \{p, \bar{p}, p_s\}$ and $\nu \in \{q, \bar{q}, q_s\}$ let $\mathcal{U}_{\mu,\nu}$ be the quasi-uniformity generated by the basis

$$U_{\mu,\nu;\,\epsilon} = \{(A,B); A,B \in (X,Y)_s^*,\ \nu(Bx - Ax) \le \epsilon,\ \forall x \in B_\mu,\},\ \ \epsilon > 0, \tag{1.7}$$

where $B_\mu = \{x \in X : \mu(x) \le 1\}$ denotes the unit ball of (X,μ). The induced quasi-uniformity on the semilinear subspace $(X,Y)_{\mu,\nu}^\flat$ of $(X,Y)_s^*$ is denoted also by $\mathcal{U}_{\mu,\nu}$ and the corresponding topologies by $\tau(\mu,\nu)$. The uniformity $\mathcal{U}_{p_s,q_s}$ and the topology $\tau(p_s,q_s)$ are those corresponding to the norm (1.3) on the space $(X,Y)_s^*$.

In the case of the dual space $X_\mu^\flat$ we shall use the notation $\mathcal{U}_\mu^\flat$ for the quasi-uniformity $\mathcal{U}_{\mu,u}$.

2. Completeness and compactness in quasi-metric and in quasi-uniform spaces

The lack of symmetry in the definition of quasi-metric and quasi-uniform spaces causes a lot of troubles, mainly concerning completeness, compactness and total boundedness in such spaces. There are a lot of completeness notions in quasi-metric and in quasi-uniform spaces, all agreeing with the usual notion of completeness in the case of metric or uniform spaces, each of them having its advantages and weaknesses.

We shall describe briefly some of these notions along with some of their properties.

The first one is that of bicompleteness. A quasi-metric space (X,ρ) is called *bicomplete* if the associated symmetric metric space (X,ρ_s) is complete. A bicomplete asymmetric normed space (X,p) is called also a *biBanach space.* The existence of a bicompletion of an asymmetric normed space was proved in [22]. The notion can be considered also for an *extended* (i.e. taking values in $[0,\infty]$) quasi-metric, or for an extended asymmetric norm on a semilinear space.

In [24] it was defined an extended asymmetric norm on $(X,Y)^*_s$ by

$$\|A|^*_{p,q} = \sup\{q(Ax) : x \in B_p\}, \quad A \in (X,Y)^*_s. \tag{2.1}$$

The identity mapping $\mathrm{id}_{\mathbb{R}}$ is continuous from $(\mathbb{R},u)$ to $(\mathbb{R},u)$, but for $-\mathrm{id}_{\mathbb{R}}$ we have

$$\|-\mathrm{id}_{\mathbb{R}}|^*_{u,u} = \sup\{-\alpha : u(\alpha) \le 1\} \ge \sup\{-\alpha : \alpha \le 0\} = +\infty,$$

because $u(\alpha) = 0 \le 1$ for $\alpha \le 0$. It follows that $\|A|^*_{p,q}$ can take effectively the value $+\infty$.

If the asymmetric normed space (Y,p) is bicomplete, then the space $(X,Y)^*_s$ is complete with respect to the symmetric extended norm $(\|\,|^*_{p,q})_s$ and $(X,Y)^\flat_{p,q}$ is a $(\|\,|^*_{p,q})_s$-closed semilinear subspace of $(X,Y)^*_s$, so it is $\|\,|_{p,q}$-bicomplete (see [24]).

In the case of a quasi-metric space (X,ρ) there are also other completeness notions. We present them following [42], starting with the definitions of Cauchy sequences.

A sequence (x_n) in (X,ρ) is called

(a) *left (right) ρ-Cauchy* if for every $\epsilon > 0$ there exist $x \in X$ and $n_0 \in \mathbb{N}$ such that

$\forall n \ge n_0,\ \rho(x,x_n) < \epsilon$ (respectively $\rho(x_n,x) < \epsilon$);

(b) *ρ-Cauchy* if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$\forall n,k \ge n_0,\ \rho(x_n,x_k) < \epsilon$;

(c) *left (right)-K-Cauchy* if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$\forall n,k,\ n \ge k \ge n_0 \ \Rightarrow\ \rho(x_k,x_n) < \epsilon$ (respectively $\rho(x_n,x_k) < \epsilon$);

(d) *weakly left(right) K-Cauchy* if for every $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$\forall n \ge n_0,\ \rho(x_{n_0},x_n) < \epsilon$ (respectively $\rho(x_n,x_{n_0}) < \epsilon$).

These notions are related in the following way:

left(right) K-Cauchy $\Rightarrow$ weakly left(right) K-Cauchy $\Rightarrow$ left(right) ρ-Cauchy,

and no one of the above implications is reversible (see [42]).

Furthermore, each ρ-convergent sequence is ρ-Cauchy, but for each of the other notions there are examples of ρ-convergent sequences that are not Cauchy, which is a major inconvenience of the theory. Another one is that closed subspaces of a complete (in some sense) quasi-metric spaces need not be complete. If each convergent sequence in a regular quasi-metric space (X, ρ) admits a left K-Cauchy subsequence, then X is metrizable ([36]. This result shows that putting too many conditions on a quasi-metric, or on a quasi-uniform space, in order to obtain results similar to those in the symmetric case, there is the danger to force the quasi-metric to be a metric and the quasi-uniformity a uniformity. In fact, this is a general problem when dealing with generalizations.

For each of these notions of Cauchy sequence one obtains a notion of sequential completeness, by asking that each corresponding Cauchy sequence be convergent in (X, ρ). These notions of completeness are related in the following way:

left (right) ρ-sequentially complete $\Rightarrow$ weakly left (right) K-sequentially complete $\Rightarrow$
$\Rightarrow$ ρ-sequentially complete.

In spite of the obvious fact that left ρ-Cauchy is equivalent to right $\bar{\rho}$-Cauchy, left ρ- and right $\bar{\rho}$-completeness do not agree, due to the fact that right $\bar{\rho}$-completeness means that every left ρ-Cauchy sequence converges in $(X, \bar{\rho})$, while left ρ-completeness means the convergence of such sequences in the space (X, ρ). For concrete examples, see [42].

A subset Y of a quasi-metric space (X, ρ) is called *precompact* if for every $\epsilon > 0$ there exists a finite subset Z of X such that

$$Y \subset \cup\{B_\rho(z, \epsilon) : z \in Z\}.$$

The set Y is called *totally bounded* if for every $\epsilon > 0$, Y can be covered by a finite family of sets of diameter less that ϵ, where the diameter of a subset A of X is defined by

$$\operatorname{diam}(A) = \sup\{\rho(x, y) : x, y \in A\}.$$

As it is known, in metric spaces the precompactness and the total boundedness are equivalent notions, a result that is not longer true in quasi-metric spaces, where precompactness is strictly weaker than total boundedness, see [37] or [38].

In spite of these peculiarities there are some positive results concerning Baire theorem and compactness. For instance, any compact quasi-metric space is left K-sequentially complete and precompact. If (X, ρ) is precompact and left ρ-sequentially complete, then it is sequentially compact (see [19, 42]). Hicks [28] proved some fixed point theorems in quasi-metric spaces (see also [5, 29])

Notice also that in quasi-metric spaces compactness, countable compactness and sequential compactness are different notions (see [18] and [31]).

The considered completeness notions can be extended to quasi-uniform spaces by replacing sequences by filters or nets (for nets, see [52, 53]). Let $(X,\mathcal{U})$ be a quasi-uniform space, $\mathcal{U}^{-1} = \{U^{-1} : U \in \mathcal{U}\}$ the conjugate quasi-uniformity on X, and $\mathcal{U}_s = \mathcal{U} \vee \mathcal{U}^{-1}$ the coarsest uniformity finer than $\mathcal{U}$ and $\mathcal{U}^{-1}$. The quasi-uniform space $(X,\mathcal{U})$ is called *bicomplete* if $(X,\mathcal{U}_s)$ is a complete uniform space. This notion is useful and easy to handle, because one can appeal to results from the theory of uniform spaces which is satisfactorily accomplished.

A subset Y of a quasi-uniform space $(X,\mathcal{U})$ is called *precompact* if for every $U \in \mathcal{U}$ there exists a finite subset Z of X such that $Y \subset U[Z]$. The set Y is called *totally bounded* if for every U there exists a finite family $A_1, ..., A_n$ of subsets of X such that $A_i \times A_i \subset U$, $i = 1, ..., n$, and $Y \subset \cup_{i=1}^n A_i$. In uniform spaces total boundedness and precompactness agree, and a set is compact if and only if it is totally bounded and complete. A subset Y of quasi-uniform space $(X,\mathcal{U})$ is totally bounded if and only if it is totally bounded as a subset of the uniform space $(X,\mathcal{U}_s)$.

Another notion of completeness is that considered by Sieber and Pervin [49]. A filter $\mathcal{F}$ in a quasi-uniform space $(X,\mathcal{U})$ is called *$\mathcal{U}$-Cauchy* if for every $U \in \mathcal{U}$ there exists $x \in X$ such that $U(x) \in \mathcal{F}$. In terms of nets, a net $(x_\alpha, \alpha \in D)$ is called *$\mathcal{U}$-Cauchy* if for every $U \in \mathcal{U}$ there exists $x \in X$ and $\alpha_0 \in D$ such that $(x, x_\alpha) \in U$ for all $\alpha \geq \alpha_0$. The quasi-uniform space $(X,\mathcal{U})$ is called *$\mathcal{U}$-complete* if every $\mathcal{U}$-Cauchy filter (equivalently, every $\mathcal{U}$-Cauchy net) has a cluster point. If every such filter (net) is convergent, then the quasi-uniform space $(X,\mathcal{U})$ is called *$\mathcal{U}$-convergence complete*. Obviously that convergence complete implies complete, but the converse is not true. It is clear that this notion corresponds to that of ρ-completeness of a quasi-metric space. It is worth to notify that the $\mathcal{U}_\rho$-completeness of the associated quasi-uniform space $(X,\mathcal{U}_\rho)$ implies the ρ-sequential completeness of the quasi-metric space (X,ρ), but the converse is not true (see [36]). The equivalence holds for the notion of left K-completeness (which will be defined immediately): a quasi-metric space is left K-sequentially complete if and only if its induced quasi-uniformity $\mathcal{U}_\rho$ is left K-complete ([43]).

A filter $\mathcal{F}$ in a quasi-uniform space $(X,\mathcal{U})$ is called *left K-Cauchy* provided for every $U \in \mathcal{U}$ there exists $F \in \mathcal{F}$ such that $U(x) \in F$ for all $x \in F$. A net $(x_\alpha, \alpha \in D)$ in X is called *left K-Cauchy* provided for every $U \in \mathcal{U}$ there exists $\alpha_0 \in D$ such that $(x_\alpha, x_\beta) \in U$ for all $\beta \geq \alpha \geq \alpha_0$. The quasi-uniform space $(X,\mathcal{U})$ is called *left K-complete* if every left K-Cauchy filter (equivalently, every left K-Cauchy net) converges. If every left K-Cauchy filter converges with respect to the uniformity $\mathcal{U}_s$, then the quasi-uniform space $(X,\mathcal{U})$ is called *Smyth complete* (see [33] and [51]). This notion of completeness has applications to computer science, see [50]. In fact, there are a lot of applications of quasi-metric spaces, asymmetric normed spaces and quasi-uniform spaces to computer science, abstract languages, complexity, see, for instance, [23, 27, 41, 46, 47, 48].

Künzi et al [36] proved that a quasi-metric space is compact if and only if it is precompact and left K-sequentially complete, and studied the relations between completeness, compactness, precompactness, total boundedness and other related notions in quasi-uniform spaces.

Another useful notion of completeness was considered by Doitchinov [13, 14, 15, 16, 17]. A filter $\mathcal{F}$ in a quasi-uniform space $(X,\mathcal{U})$ is called *D-Cauchy* provided there exists a co-filter $\mathcal{G}$ in X such that for every $U \in \mathcal{U}$ there are $G \in \mathcal{G}$ and $F \in \mathcal{F}$ such that $F \times G \subset U$. The quasi-uniform space $(X,\mathcal{U})$ is called *D-complete* provided every D-Cauchy filter converges. A related notion of completeness was considered by Andrikopoulos [3]. For a comparative study of the completeness notions defined by pairs of filters see [10] and [4].

Notice also that these notions of completeness can be considered within the framework of bitopological spaces in the sense of Kelly [30], since a quasi-metric space is a bitopological space with respect to the topologies $\tau(\rho)$ and $\tau(\bar{\rho})$. For this approach see the papers by Deak [11, 12]. It seems that the letter K in the definition of left K-completeness comes from Kelly (see [9]).

3. **Compact operators**

Recall that a subset Z of an asymmetric normed space (X,p) is called *p-precompact* if for every $\epsilon > 0$ there exist $z_1, ..., z_n \in Z$ such that

$$\forall z \in Z,\ \exists i \in \{1,...,n\}, \quad p(z - z_i) \le \epsilon, \tag{3.1}$$

or, equivalently,

$$Z \subset U_\epsilon[\{z_1, ..., z_n\}],$$

where U_ϵ is the entourage

$$U_\epsilon = \{(x,x') \in X \times X : p(x' - x) \le \epsilon\}$$

in the quasi-uniformity $\mathcal{U}_p$.

One obtains an equivalent notion taking the points z_i in X or/and $< \epsilon$ in (3.1).

Let $(X,p),(Y,q)$ be asymmetric normed spaces and, as before, let

$$\mu \in \{p, \bar{p}, p_s\} \ \text{ and } \ \nu \in \{q, \bar{q}, q_s\}. \tag{3.2}$$

A linear operator $A : X \to Y$ is called (μ,ν)-*compact* if the set $A(B_\mu)$ is ν-precompact in Y.

Some properties of compact operators are collected in the following proposition. We shall denote by $(X,Y)^k_{\mu,\nu}$ the set of all linear (μ,ν)-compact operators from X to Y. Notice that, for $\mu = p_s$ and $\nu = q_s$, the space $(X,Y)^\flat_{p_s,q_s}$ agrees with $(X,Y)^*_s$, the (p_s,q_s)-compact operators are the usual linear compact operators between the normed spaces (X,p_s) and (Y,q_s), so the proposition contains some well known results for compact operators on normed spaces.

Proposition 3.1. *Let $(X,p),(Y,q)$ be asymmetric normed spaces. The following assertions hold.*

1. *$(X,Y)^k_{\mu,\nu}$ is a semilinear subspace of $(X,Y)^\flat_{\mu,\nu}$.*
2. *$(X,Y)^k_{p,q}$ is $\tau(p,\bar{q})$-closed in $(X,Y)^\flat_{p,q}$.*

Proof. (1) We give the proof in the case $\mu = p$ and $\nu = q$. The other cases can be treated similarly.

If $A : X \to Y$ is (p,q)-compact, then there exists $x_1,...,x_n \in B_p$ such that

$$\forall x \in B_p,\ \exists i \in \{1,...,n\}, \quad q(Ax - Ax_i) \le 1. \tag{3.3}$$

If for $x \in B_p,\ i \in \{1,...,n\}$ is chosen according to (3.3), then

$$q(Ax) \le q(Ax - Ax_i) + q(Ax_i) \le 1 + \max\{q(Ax_j) : 1 \le j \le n\},$$

showing that the operator A is (p,q)-bounded.

Suppose that $A_1, A_2 : X \to Y$ are (p,q)-compact and let $\epsilon > 0$. By the (p,q)-compactness of the operators A_1, A_2, there exist $x_1,...,x_m$ and $y_1,...,y_n$ in B_p such that

$$\forall x \in B_p,\ \exists i \in \{1,...,m\},\ \exists j \in \{1,...,n\},\ \ q(A_1x - A_1x_i) \le \epsilon \ \text{ and } \ q(A_2x - A_2x_j) \le \epsilon.$$

It follows that for every $x \in B_p$ there exists a pair (i,j) with $1 \le i \le m$ and $1 \le j \le n$ such that

$$q(A_1x + A_2x - A_1x_i - A_2y_j) \le q(A_1x - A_1x_i) + q(A_2x - A_2y_j) \le 2\epsilon,$$

showing that $\{A_1x_i + A_2y_j : 1 \le i \le m,\ 1 \le j \le n\}$ is a finite 2ϵ-net for $(A_1+A_2)(B_p)$.

The proof of the compactness of αA, for $\alpha > 0$ and A compact, is immediate and we omit it.

(2) *The $\tau(p,\bar{q})$-closedness of $(X,Y)^k_{p.q}$.*

Let (A_n) be a sequence in $(X,Y)^k_{p,q}$ which is $\tau(p,\bar{q})$-convergent to $A \in (X,Y)^\flat_{p,q}$.

For $\epsilon > 0$ choose $n_0 \in \mathbb{N}$ such that

$$\forall n \ge n_0,\ \forall x \in B_p,\ \ \bar{q}(A_nx - Ax) \le \epsilon\ \ (\iff q(Ax - A_nx) \le \epsilon). \tag{3.4}$$

Let $x_1, ..., x_m \in B_p$ such that $A_{n_0}x_i,\ 1 \le i \le m,$ is an ϵ-net for $A_{n_0}(B_p)$. Then for every $x \in B_p$ there exists $i \in \{1, ..., m\}$ such that

$$q(A_{n_0}x - A_{n_0}x_i) \le \epsilon,$$

so that, by (3.4),

$$q(Ax - Ax_i) \le q(Ax - A_{n_0}x) + q(A_{n_0}x - A_{n_0}x_i) + q(A_{n_0}x_i - Ax_i) \le 3\epsilon.$$

Consequently, $Ax_i,\ 1 \le i \le m,$ is a 3ϵ-net for $A(B_p)$, showing that $A \in (X,Y)^k_{p,q}$. □

Remark 3.2. The assertion (2) of Proposition 3.1 holds for other types of compactness too, i.e. for the spaces $(X,Y)^k_{\mu,\nu}$ with μ, ν as in (3.2), with similar proofs.

4. **The dual of a bounded linear operator**

Let (X,p), (Y,q) be asymmetric normed spaces and μ, ν as in (3.2). For $A \in (X,Y)^\flat_{\mu,\nu}$ define $A^\flat : Y^\flat_\nu \to X^\flat_\mu$ by

$$A^\flat\psi = \psi \circ A,\ \ \psi \in Y^\flat_s. \tag{4.1}$$

Obviously that $A^\flat$ is properly defined, additive and positively homogeneous. Concerning the continuity we have.

Proposition 4.1. *1. The operator $A^\flat$ is quasi-uniformly continuous with respect to the quasi-uniformities $\mathcal{U}_\nu^\flat$ and $\mathcal{U}_\mu^\flat$ on $Y_\nu^\flat$ and $X_\mu^\flat$, respectively.*

2. The operator $A^\flat$ is also quasi-uniformly continuous with respect to the $w^\flat$-quasi-uniformities on $Y_\nu^\flat$ and $X_\mu^\flat$.

Proof. (1) Take again $\mu = p$ and $\nu = q$. For $\epsilon > 0$ let

$$U_\epsilon = \{(\varphi_1, \varphi_2) \in X_p^\flat \times X_p^\flat : \varphi_2(x) - \varphi_1(x) \le \epsilon,\ \forall x \in B_p\}.$$

If $\|A|_{p,q} = 0$, then $A = 0$, so we can suppose $\|A| = \|A|_{p,q} > 0$. Let

$$V_\epsilon = \{(\psi_1, \psi_2) \in Y_q^\flat \times Y_q^\flat : \psi_2(x) - \psi_1(x) \le \epsilon/\|A|,\ \forall x \in B_q\}.$$

Taking into account that

$$\forall x \in B_p,\ \varphi_2(x) - \varphi_1(x) \le \epsilon/r \iff \forall x' \in rB_p,\ \varphi_2(x') - \varphi_1(x') \le \epsilon,$$

and

$$\forall x \in B_p, \quad q(Ax) \le \|A|p(x) \le \|A|,$$

it follows

$$A^\flat\psi_2(x) - A^\flat\psi_1(x) = \psi_2(Ax) - \psi_1(Ax) \le \epsilon,$$

for all $x \in B_p$, proving the quasi-uniform continuity of A.

(2) For $x_1, ..., x_n \in X$ and $\epsilon > 0$ let

$$V = \{(\varphi_1, \varphi_2) \in X_p^\flat \times X_p^\flat : \varphi_2(x_i) - \varphi_1(x_i) \le \epsilon,\ i = 1, ..., n\}$$

be a $w^\flat$-entourage in $X_p^\flat$. Then

$$U = \{(\psi_1, \psi_2) \in Y_q^\flat \times Y_q^\flat : \psi_2(Ax_i) - \psi_1(Ax_i) \le \epsilon,\ i = 1, ..., n\},$$

is a $w^\flat$-entourage in $Y_q^\flat$ and $(A^\flat\psi_1, A^\flat\psi_2) \in V$ for every $(\psi_1, \psi_2) \in U$, proving the quasi-uniform continuity of $A^\flat$ with respect to the $w^\flat$-quasi-uniformities on $Y_q^\flat$ and $X_p^\flat$. □

Now we can prove the analog of the Schauder theorem for the asymmetric dual.

Theorem 4.2. *Let $(X,p),(Y,q)$ be asymmetric normed spaces. If the linear operator $A : X \to Y$ is (p,q)-compact, then $A^\flat(B_q^\flat)$ is precompact with respect to the quasi-uniformity $\mathcal{U}_p^\flat$ on $X_p^\flat$.*

Proof. For $\epsilon > 0$ let

$$U_\epsilon = \{(\varphi_1,\varphi_2) \in X_p^\flat \times X_p^\flat : \varphi_2(x) - \varphi_1(x) \le \epsilon,\ \forall x \in B_p\},$$

be an entourage in $X_p^\flat$ for the quasi-uniformity $\mathcal{U}_p^\flat$.

Since A is (p,q)-compact, there exist $x_1,...,x_n \in B_p$ such that

$$\forall x \in B_p\,,\ \exists i \in \{1,...,n\}, \quad q(Ax - Ax_i) \le \epsilon. \tag{4.2}$$

By the Alaoglu-Bourbaki theorem, [24, Theorem 4] the set $B_q^\flat$ is $w^\flat$-compact, so by the $(w^\flat,w^\flat)$-continuity of the operator $A^\flat$ (Proposition 4.1), the set $A^\flat(B_q^\flat)$ is $w^\flat$-compact in $X_p^\flat$. Consequently, the $w^\flat$-open cover

$$V_\psi = \{\varphi \in X_p^\flat : \varphi(x_i) - A^\flat\psi(x_i) < \epsilon,\ i = 1,...,n\},\ \psi \in B_q^\flat,$$

contains a finite subcover V_{ψ_k}, $1 \le k \le m$, i.e,

$$A^\flat(B_q^\flat) \subset \bigcup\{V_{\psi_k} : 1 \le k \le m\}. \tag{4.3}$$

Now let $\psi \in B_q^\flat$. By (4.3) there exists $k \in \{1,...,m\}$ such that

$$A^\flat\psi(x_i) - A^\flat\psi_k(x_i) < \epsilon,\ i = 1,...,n.$$

If $x \in B_p$, then, by (4.2), there exists $i \in \{1,...,n\}$, such that

$$q(Ax - Ax_i) \le \epsilon.$$

It follows

$$\begin{aligned}
&\psi(Ax) - \psi_k(Ax) = \\
&= \psi(Ax) - \psi(Ax_i) + \psi(Ax_i) - \psi_k(Ax_i) + \psi_k(Ax_i) - \psi(Ax_i) \\
&\le 2q(Ax - Ax_i) + \epsilon \le 3\epsilon.
\end{aligned}$$

Consequently,

$$\forall x \in B_p\,, \quad (A^\flat \psi - A^\flat \psi_k)(x) \le 3\epsilon,$$

proving that

$$A^\flat(B_q^\flat) \subset U_{3\epsilon}[\{A^\flat \psi_1, ..., A^\flat \psi_m\}].$$

$\square$

Comments. As a measure of precaution, we have defined the compactness of an operator A in terms of the precompactness of the image of the unit ball B_p by A, rather than by the relative compactness of $A(B_p)$, as in the case of compact operators on usual normed spaces. As can be seen from Section 2, the relations between precompactness, total boundedness and completeness are considerably more complicated in the asymmetric case than in the symmetric one. To obtain some compactness properties of the set $A(B_p)$, one needs a study of the completeness of the space $(X, Y)^\flat_{\mu,\nu}$ with respect to various quasi-uniformities and various notions of completeness, which could be the topic of a further investigation.

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