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ON A CLASS OF LINEAR POSITIVE BIVARIATE OPERATORS OF KING TYPE

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Dedicated to Professor Gheorghe Coman at his 70^{th} *anniversary*

Abstract. The concern of this note is to introduce a general class of linear positive operators of discrete type acting on the space of real valued functions defined on a plane domain. These operators preserve some test functions of Bohman-Korovkin theorem. Following our technique, as a particular class, a modified variant of the bivariate Bernstein-Chlodovsky operators is presented.

1. **Introduction**

Let $(L_n)_{n\geq 1}$ be a sequence of positive linear operators defined on the Banach space $C([a,b])$. A classical theorem of Bohman-Korovkin asserts: if $(L_n e_k)_{n\geq 1}$ converges to e_k uniformly on $[a,b]$, $k \in \{0,1,2\}$, for the test functions $e_0(x) = 1$, $e_1(x) = x$, $e_2(x) = x^2$, then $(L_n f)_{n\geq 1}$ converges to f uniformly on $[a,b]$, for each $f \in C([a,b])$.

J.P. King [8] has presented an example of linear and positive operators $V_n : C([0,1]) \to C([0,1])$, given as follows

$$(V_n f)(x) = \sum_{k=0}^{n} \binom{n}{k} (r_n^*(x))^k (1 - r_n^*(x))^{n-k} f\left(\frac{k}{n}\right), \; f \in C([0,1]), \; x \in [0,1], \quad (1)$$

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where $r_n^* : [0,1] \to [0,1]$,

$$r_n^*(x) = \begin{cases} x^2, & n = 1, \\ -\dfrac{1}{2(n-1)} + \sqrt{\dfrac{n}{n-1}x^2 + \dfrac{1}{4(n-1)^2}}, & n = 2,3,\ldots \end{cases} \tag{2}$$

This sequence preserves two test functions e_0, e_2 and $(V_n e_1)(x) = r_n^*(x)$ holds. Based on Bohman-Korovkin criterion, we get $\lim\limits_{n\to\infty} (V_n f)(x) = f(x)$ for each f belonging to $C([0,1])$, $x \in [0,1]$.

Further results regarding V_n operator have been recently obtained by Gonska and Piţul [5]. Also, by using A-statistical convergence, an analog of King's result has been proved by O. Duman and C. Orhan [4].

In [1] we indicated a general technique to construct sequences of univariate operators of discrete type with the same property as in King's example, i.e., their degree of exactness is null, but they reproduce the third test function of the celebrated criterion.

The central issue of this paper is to present a sequence of bivariate operators with similar properties: to reproduce certain monomials of second degree and to form an approximation process.

2. Preliminaries

Following our announced aim, in this section we recall results regarding the univariate case. Also, basic results concerning the uniform approximation of functions by bivariate operators are delivered.

We set $\mathbb{R}_+ := [0,\infty)$ and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. Following [1], we consider a sequence $(L_n)_{n\geq 1}$ of linear positive operators of discrete type acting on a subspace of $C(\mathbb{R}_+)$ and defined by

$$(L_n f)(x) = \sum_{k=0}^{\infty} u_{n,k}(x) f(x_{n,k}),\ x \geq 0,\ f \in \mathcal{F} \cap E_\alpha, \tag{3}$$

where $u_{n,k} : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous $(n \in \mathbb{N},\ k \in \mathbb{N}_0)$, $(x_{n,k})_{k\geq 0} := \Delta_n$ is a net on $\mathbb{R}_+$ and

$$\mathcal{F} := \{f : \mathbb{R}_+ \to \mathbb{R}_+ :\ \text{the series in (3) is convergent}\},$$

$$E_\alpha := \{f \in C(\mathbb{R}_+) :\ (1 + x^\alpha)^{-1} f(x) \text{ is convergent as } x \to \infty\},$$

$\alpha \geq 2$ being fixed. We mention that the right-hand side of (3) could be a finite sum. We assume that the following identities

$$(L_n e_0)(x) = 1, \quad (L_n e_1)(x) = x, \quad (L_n e_2)(x) = a_n x^2 + b_n x + c_n, \quad x \geq 0, \tag{4}$$

are fulfilled for each $n \in \mathbb{N}$. At this moment, $\{e_0, e_1, e_2\} \subset \mathcal{F} \cap E_\alpha$ holds. Moreover, we assume

$$a_n \neq 0,\ n \in \mathbb{N}, \quad \lim_{n\to\infty} a_n = 1, \quad \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = 0.$$

Based on Bohman-Korovkin theorem these relations guarantee that $(L_n)_{n\geq 1}$ is a positive approximation process, more precisely $\lim\limits_{n\to\infty} (L_n f)(x) = f(x)$ uniformly for every $f \in \mathcal{F} \cap E_\alpha$ and every x belonging to any compact $\mathcal{K} \subset \mathbb{R}_+$.

Since $(Le_1)^2 \leq (Le_0)(Le_2)$ is a common property of any linear positive operator L of summation type, we get

$$(a_n - 1)x^2 + b_n x + c_n \geq 0, \quad x \geq 0,\ n \in \mathbb{N}, \tag{5}$$

which implies

$$c_n \geq 0, \quad a_n \geq 1 \text{ for each } n \in \mathbb{N}, \tag{6}$$

and $\{n \in \mathbb{N} :\ a_n = 1\} \subset \{n \in \mathbb{N} :\ b_n \geq 0\}$. Further on, we are looking for the functions $v_n \in \mathbb{R}_+^{\mathbb{R}_+}$, $n \in \mathbb{N}$, such that $(L_n e_2)(v_n(x)) = x^2$ for each $x \geq 0$ and $n \in \mathbb{N}$, this means

$$a_n v_n^2(x) + b_n v_n(x) + c_n - x^2 = 0, \quad x \geq 0,\ n \in \mathbb{N}. \tag{7}$$

In what follows, throughout the paper, we take

$$c_n = 0, \quad n \in \mathbb{N}, \tag{8}$$

and

$$v_n(x) = \frac{1}{2a_n}\left(\sqrt{b_n^2 + 4a_n x^2} - b_n\right), \quad x \geq 0,\ n \in \mathbb{N}. \tag{9}$$

For each $n \in \mathbb{N}$, $v_n(x)$ is well defined and v_n is a continuous positive function. Also, relation (7) is verified.

Starting from (3) we define the univariate linear positive operators

$$(L_n^* f)(x) = \sum_{k=0}^{\infty} u_{n,k}(v_n(x)) f(x_{n,k}), \quad x \geq 0,\ f \in \mathcal{F} \cap E_\alpha,\ n \in \mathbb{N}, \tag{10}$$

where v_n is given by (9).

The following identities

$$L_n^* e_0 = e_0, \quad L_n^* e_1 = v_n, \quad L_n^* e_2 = e_2 \tag{11}$$

hold. Consequently, one has $\lim_{n\to\infty} L_n^* f = f$ uniformly on compact intervals of $\mathbb{R}_+$ for every $f \in \mathcal{F} \cap E_\alpha$. This result follows from (11) and Korovkin criterion. For each n with the property $b_n \geq 0$ we get $v_n(0) = 0$ and, consequently, one has $(L_n^* f)(0) = (L_n f)(0)$.

Setting $e_{i,j}(x,y) = x^i y^j$, $i \in \mathbb{N}_0$, $j \in \mathbb{N}_0$, $i + j \leq 2$, the test functions corresponding to the bidimensional case, we need a result due to Volkov [10].

Theorem 1. *Let I and J compact intervals of the real line. Let L_{m_1,m_2}, $(m_1, m_2) \in \mathbb{N} \times \mathbb{N}$, be linear positive operators applying the space $C(I \times J)$ into itself. If*

$$\lim_{m_1,m_2} L_{m_1,m_2} e_{i,j} = e_{i,j}, \quad (i,j) \in \{(0,0), (1,0), (0,1)\},$$

$$\lim_{m_1,m_2} L_{m_1,m_2}(e_{2,0} + e_{0,2}) = e_{2,0} + e_{0,2},$$

uniformly on $I \times J$, then the sequence $(L_{m_1,m_2} f)$ converges to f uniformly on $I \times J$ for any $f \in C(I \times J)$.

In a more general frame, Volkov's theorem says: if X is a compact subset of the Euclidean space $\mathbb{R}^p$, then $\left\{\mathbf{1}, pr_1, \ldots, pr_p, \sum_{j=1}^{p} pr_j^2\right\}$ is a Korovkin subset in $C(X)$. Here $\mathbf{1}$ stands for the constant function on X of constant value 1 and $pr_1, \ldots, pr_p$ represent the canonical projections on X, this means $pr_j(x) := x_j$ for every $x = (x_i)_{1 \leq i \leq p} \in X$, where $1 \leq j \leq p$. For a thorough documentation the monograph of Altomare and Campiti [2; *page 245*] can be consulted.

3. **A class of bivariate operators**

Now we are going to present the tensor product extension of L_n^* to the bidimensional case.

Starting from the specified Δ_n net on $\mathbb{R}_+$, we consider $\Delta_{m_1} \times \Delta_{m_2}$, the corresponding net on $\mathbb{R}_+ \times \mathbb{R}_+$. Thus, $(x_{m_1,i}, x_{m_2,j})$, $(i,j) \in \mathbb{N}_0 \times \mathbb{N}_0$, are its knots.

Having in mind the notations of the previous section we introduce the bivariate linear positive operators acting on $\mathcal{D}$ and defined as follows

$$(L^*_{m_1,m_2} f)(x,y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} u_{m_1,i}(v_{m_1}(x))u_{m_2,j}(v_{m_2}(y))f(x_{m_1,i}, x_{m_2,j}), \tag{12}$$

$(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+$. For each index $m \in \mathbb{N}$, the functions $u_{m,k}$, $k \in \mathbb{N}_0$, enjoy the properties implied by (4) and v_m is given by (9). In the above $\mathcal{D}$ consists of all continuous functions $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ with the properties: the series in (12) is convergent and $(1 + x^{\alpha_1})^{-1} f(x,y)$, $(1 + y^{\alpha_2})^{-1} f(x,y)$ are convergent as $x \to \infty$, $y \to \infty$ respectively, where $\alpha_1 \geq 2$, $\alpha_2 \geq 2$ are fixed. Clearly, $e_{i,j} \in \mathcal{D}$ for each $(i,j) \in \mathbb{N}_0 \times \mathbb{N}_0$ with $i + j \leq 2$.

Theorem 2. *Let $L^*_{m_1,m_2}$ be defined by (12).*

(i) The following identities

$$\begin{aligned} &L^*_{m_1,m_2} e_{0,0} = e_{0,0},\ L^*_{m_1,m_2} e_{2,0} = e_{2,0},\ L^*_{m_1,m_2} e_{0,2} = e_{0,2}, \\ &(L^*_{m_1,m_2} e_{1,0})(x,y) = v_{m_1}(x),\ (L^*_{m_1,m_2} e_{0,1})(x,y) = v_{m_2}(y),\ (x,y) \in \mathbb{R}_+ \times \mathbb{R}_+, \end{aligned} \tag{13}$$

hold.

(ii) One has $\lim\limits_{m_1,m_2} L^*_{m_1,m_2} f = f$ uniformly on compact subsets of $\mathbb{R}^2_+$ for every $f \in \mathcal{D}$.

Proof. (i) Taking into account (11) and (12), by a straightforward calculation the stated identities follow.

(ii) Based on (13), the result is implied by (9) and Theorem 1. □

We can explore the rate of convergence of $L^*_{m_1,m_2}$ operators in terms of the first order modulus of smoothness ω_f of the bivariate function f. It is known that for any real valued bounded function f, $f \in B(I \times J)$, where I and J are compact intervals of the real line, the associated mapping ω_f is defined as follows:

$$\omega_f(\delta_1,\delta_2)=\sup\{|f(x_1,y_1)-f(x_2,y_2)|:\ (x_1,y_1),(x_2,y_2)\in I\times J,$$
$$|x_1-y_1|\le\delta_1,\ |x_2-y_2|\le\delta_2\},\ (\delta_1,\delta_2)\in\mathbb{R}_+\times\mathbb{R}_+. \tag{14}$$

Among the properties of ω_f investigated by A.F. Ipatov [7] we recall

$$\omega_f(\lambda_1\delta_1,\lambda_2\delta_2)\le(1+\lambda_1+\lambda_2)\omega_f(\delta_1,\delta_2),\quad \lambda_1>0,\ \lambda_2>0. \tag{15}$$

Let $\mathcal{K}\subset\mathbb{R}_+^2$ be a compact and let $\delta_1>0$, $\delta_2>0$ be fixed. Based on (15) and knowing that $L^*_{m_1,m_2}e_{0,0}=1$, for each $(x,y)\in\mathcal{K}$ we can write

$$|(L^*_{m_1,m_2}f)(x,y)-f(x,y)|$$
$$\le\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}u_{m_1,i}(v_{m_1}(x))u_{m_2,j}(v_{m_2}(y))|f(x_{m_1,i},x_{m_2,j})-f(x,y)|$$
$$\le\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}u_{m_1,i}(v_{m_1}(x))u_{m_{2,j}}(v_{m_2}(y))\omega_f\left(\frac{1}{\delta_1}|x_{m_1,i}-x|,\frac{1}{\delta_2}|x_{m_2,j}-y|\right)$$
$$\le\left(1+\frac{1}{\delta_1}\sum_{i=0}^{\infty}u_{m_1,i}(v_{m_1}(x))|x_{m_1,i}-x|+\frac{1}{\delta_2}\sum_{j=0}^{\infty}u_{m_2,j}(v_{m_2}(y))|x_{m_2,j}-y|\right)\omega_f(\delta_1,\delta_2).$$

On the other hand, Cauchy's inequality and the identities given by (13) imply

$$\sum_{i=0}^{\infty}u_{m_1,i}(v_{m_1}(x))|x_{m_1,i}-x|$$
$$\le\left(\sum_{i=0}^{\infty}u_{m_1,i}(v_{m_1}(x))\right)^{1/2}\left(\sum_{i=0}^{\infty}u_{m_1,i}(v_{m_1}(x))(x_{m_1,i}-x)^2\right)^{1/2}$$
$$=(2x^2-2xv_{m_1}(x))^{1/2},$$

and respectively

$$\sum_{j=0}^{\infty}u_{m_2,j}(v_{m_2}(y))|x_{m_2,j}-y|\le(2y^2-2yv_{m_2}(y))^{1/2}.$$

The above relations enable us to state the following estimate for the pointwise approximation.

Theorem 3. *Let $\mathcal{K}$ be a compact subset of $\mathbb{R}^2_+$. The operators L_{m_1,m_2}, $(m_1, m_2) \in \mathbb{N} \times \mathbb{N}$, defined by (12) verify*

$$|(L_{m_1,m_2}f)(x,y) - f(x,y)| \le \left(1 + \frac{1}{\delta_1}\widetilde{v}_{m_1}(x) + \frac{1}{\delta_2}\widetilde{v}_{m_2}(y)\right)\omega_f(\delta_1,\delta_2), \tag{16}$$

for every $f \in \mathcal{D}$, $(x,y) \in \mathcal{K}$, $\delta_1 > 0$, $\delta_2 > 0$, where

$$\widetilde{v}_m(t) = \sqrt{2t^2 - 2tv_m(t)}, \quad m \in \mathbb{N},\ t \ge 0, \tag{17}$$

and v_m is given at (9).

Remarks. 1° Based on Cauchy's inequality $(L^*e_1)^2 \le (L^*e_0)(L^*e_2)$ and relation (11) as well, we get $t \ge v_m(t)$, for each $t \ge 0$. Consequently, in (17) $\widetilde{v}_m$ is well defined.

2° Endowing $\mathbb{R} \times \mathbb{R}$ with the metric ρ, $\rho(z_1,z_2) = |x_1 - x_2| + |y_1 - y_2|$ for $z_k = (x_k, y_k)$, $k = 1, 2$, we could have estimated the rate of convergence using another type of modulus of smoothness given by

$$\omega_1(f;\delta) = \sup\{|f(z_1) - f(z_2)| :\ z_1 \in \mathcal{K},\ z_2 \in \mathcal{K},\ \rho(z_1,z_2) \le \delta\},$$

for every $f \in B(\mathcal{K})$ and $\delta > 0$. Clearly, (14) implies $\omega_f(\delta_1,\delta_2) \le \omega_1(f;\delta_1 + \delta_2)$. An overview on moduli of smoothness as well as some of their extensions can be found, e.g., in the monograph [2; *Section 5.1*].

3° Examining the construction of v_m we easily deduce $v_m(0) \le v_m(x) \le x$, for each $x \in \mathbb{R}_+$. Moreover, the mapping $x \mapsto x - v_m(x)$ is increasing one. For a compact $I = [\alpha, \beta] \subset \mathbb{R}_+$, we can write

$$\widetilde{v}_m(t) \le \sqrt{2\beta}\left(\max_{t\in I}(t - v_m(t))\right)^{1/2} = \sqrt{2\beta}\sqrt{\beta - v_m(\beta)}.$$

Consequently, if $\mathcal{K} := I \times J = [\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \subset \mathbb{R}^2_+$ then, by choosing in (16) $\delta_j := \sqrt{\beta_j - v_{m_j}(\beta_j)}$, $j \in \{1, 2\}$, we obtain the following global estimate on the compact $\mathcal{K}$

$$\|L_{m_1,m_2}f - f\|_{C(\mathcal{K})} \le (1 + \sqrt{2\beta_1} + \sqrt{2\beta_2})\omega_f\left(\sqrt{\beta_1 - v_{m_1}(\beta_1)}, \sqrt{\beta_2 - v_{m_2}(\beta_2)}\right).$$

Here $\|\cdot\|_{C(\mathcal{K})}$ stands for the usual sup-norm of the space $C(\mathcal{K})$.

4. **Example**

In order to obtain an approximation process of $L^*_{m_1,m_2}$-type, we focus our attention on Bernstein-Chlodovsky operators. Let $(h_n)_{n\geq 1}$ be a sequence of strictly positive real numbers verifying

$$\lim_{n\to\infty} h_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{h_n}{n} = 0.$$

The nth Bernstein-Chlodovsky operator [3], $L_n : C(\mathbb{R}_+) \to C(\mathbb{R}_+)$ is defined by

$$(L_n f)(x) = \begin{cases} \displaystyle\sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{h_n}\right)^k \left(1 - \frac{x}{h_n}\right)^{n-k} f\left(\frac{h_n k}{n}\right), & \text{if} \quad 0 \leq x \leq h_n, \\ f(x), & \text{if} \quad x > h_n. \end{cases} \tag{18}$$

It is known that identities (4) are fulfilled and we get

$$\begin{cases} a_n = 1 - \dfrac{1}{n},\ b_n = \dfrac{h_n}{n},\ c_n = 0, & \text{if} \quad x \in [0, h_n], \\ a_n = 1,\ b_n = c_n = 0, & \text{if} \quad x > h_n. \end{cases} \tag{19}$$

Following (9) we obtain: for $n = 1$, $v_1(x) = x^2$, $x \geq 0$; for $n \geq 2$,

$$v_n(x) = \begin{cases} \dfrac{1}{2(n-1)} \left(\sqrt{h_n^2 + 4n(n-1)x^2} - h_n\right), & \text{if} \quad x \in [0, h_n], \\ x, & \text{if} \quad x > h_n. \end{cases} \tag{20}$$

Returning to (10) via (18), we obtain the modified univariate Bernstein-Chlodovsky operators L^*_n. Accordingly, based on (12), the bivariate extension for each $(x, y) \in [0, h_{m_1}] \times [0, h_{m_2}]$ and $f \in \mathcal{D}$ is defined by

$$(L^*_{m_1,m_2} f)(x, y) = \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} c_{m_1,m_2}(i, j) v^i_{m_1}(x) v^j_{m_2}(y) (h_{m_1} - v_{m_1}(x))^i (h_{m_2} - v_{m_2}(y))^j$$

$$\times f\left(\frac{i}{m_1} h_{m_1}, \frac{j}{m_2} h_{m_2}\right), \tag{21}$$

where $c_{m_1,m_2}(i, j) = \binom{m_1}{i}\binom{m_2}{j} h^{-m_1}_{m_1} h^{-m_2}_{m_2}$ and v_m is described by (20).

We notice the following aspect. From (19) we get $a_1 = 0$ and this should be in contradiction with (6). In fact nothing is wrong because, this time, relation (5) must hold only for $x \in [0, h_n]$, not for each $x \in \mathbb{R}_+$. Consequently, condition $a_n \geq 1$ in (6) is not necessary to take place.

Particular case. If we choose $h_n = 1$ in (18), then L_n becomes the classical nth Bernstein polynomial for each $n \in \mathbb{N}$. In this case relations (19) and (20) imply $v_n(x) = r_n^*(x)$, $x \in [0, 1]$, see (2). The King's operators (1) are reobtained.

Remarks. a) If we choose in (3) $u_{n,k}(x) := e^{-nx}\dfrac{(nx)^k}{k!}$ and $x_{k,n} := k/n$, the well-known Szász-Mirakyan-Favard operator is obtained. A variant of this operator in two dimensions was defined by Totik [9; *p.292*] as follows

$$(S_{n,m}f)(x,y) = e^{-nx-my}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(nx)^i}{i!}\frac{(my)^j}{j!}f\left(\frac{i}{n},\frac{j}{m}\right).$$

b) If we choose in (3) $u_{n,k}(x) := \binom{n-1+k}{k}x^k(1+x)^{-n-k}$ and $x_{k,n} := k/n$, the classical Baskakov operator is obtained. In [6] the authors have considered the Baskakov operator for functions of two variables given by

$$(A_{n,m}f)(x,y) = \frac{1}{(1+x)^n(1+y)^m}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}c_{i,j}(n,m)\left(\frac{x}{1+x}\right)^i\left(\frac{y}{1+y}\right)^j f\left(\frac{i}{n},\frac{j}{m}\right),$$

where $c_{i,j}(n,m) := \binom{n-1+i}{i}\binom{m-1+j}{j}$.

Following our technique, in the same manner we can obtain the modified variants given by relation (12) of the above two classes. By a short computation, relation (9) becomes

$$\text{a) } v_n(x) = \frac{\sqrt{1+4n^2x^2}-1}{2n}, \qquad \text{b) } v_n(x) = \frac{\sqrt{1+4n(n+1)x^2}-1}{2(n+1)},$$

respectively.

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