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PERIODIC AND ALMOST PERIODIC FUNCTIONS

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Dedicated to Professor Ştefan Cobzaş at his 60^{th} *anniversary*

Abstract. In this paper we present some synthesis results about almost periodic functions. Some of these results were discussed in ([3], [10], [11], [12]). A diagram which represents the function sets mentioned in the work is discussed.

1. Preliminaries

The periodic functions play a central role in mathematics. Unfortunately this class of functions is not linear since the sum of periodic functions which not have a non-zero period in common gives a non-periodic function.

A larger class is the class of almost periodic functions which is a linear space. This class was introduced by Harald Bohr ([7], [8]). Bohr's theory of almost periodic functions was studied in connection with differential equations an other theories. For example Riesz and Nagy present some applications for compact operators and Banach algebras ([17]).

Salomon Bochner presents some generalizations of Bohr's definition ([6]) for functions with values in abstract spaces which are useful in the study of differential equations, Fourier series and Fourier transforms ([14], [15]). Laurent Schwartz has presented a definition for almost periodic distributions ([18]).

Our work aims presenting three definitions and discussing some examples of periodic and almost periodic functions.

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We introduce the following sets of functions (The domain of the functions is $\mathbb{R}$ and the range is a subset of $\mathbb{C}$)

F: the set of functions $f : \mathbb{R} \to \mathbb{C}$

C: the set of continuous functions

B: the set of bounded functions

P: the set of periodic functions

UC: the set of uniform continuous functions

AP: the set of almost periodic functions

TP: the set of trigonometric polynomials

IAP: the set of almost periodic functions with an almost periodic primitive

$$PC = P \cap C, \quad BP = B \cap P, \quad BC = B \cap C.$$

2. Some results about periodic functions

The function

$$f(x) = \cos x + \cos \sqrt{2}x$$

([10], [16]) is clearly not periodic. On the other hand it is the sum of two periodic functions: $\cos \sqrt{2}x$ and $\cos x$, the function $f(x)$ does not attains its infimum -2 but attain its supremum 2.

The function

$$g(x) = \sin x + \sin \sqrt{2}x$$

([10], [13]) is also non-periodic and does not attain the infimum and the supremum.

In each case there exist a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \to +\infty$ and $f(x_n) \to -2$ and in the second example there is a sequence $(y_n)_{n\in\mathbb{N}}$ such that $y_n \to +\infty$ and $f(y_n) \to 2$.

For the class PC we have the following results.

Theorem 2.1. ([2], [10]) *If $f \in CP$ then f attains its infimum and supremum.*

Theorem 2.2. ([2], [10]) *Let $f \in CP$ with period T, $T \in \mathbb{R} \setminus \mathbb{Q}$ then the set:*

$$A = \{f(n) : \ n \in \mathbb{N}\}$$

is dense in $[m, M]$*, where* m *denotes the minimum and* M *the maximum of the function.*

If the function in the previous theorem has rational period, the theorem is not true for example for the function $f(x) = \sin \pi x$, in this case the set A is finite, $A = \{0\}$.

Theorem 2.3. ([2]) *Let* $f \in CP$ *with rational period, then the set:*

$$A_\theta = \{f(n\theta) : \ n \in \mathbb{N}\}$$

is dense in $[m, M]$*,* $\forall \ \theta \in \mathbb{R} \setminus \mathbb{Q}$.

Example 2.1. Consider the function

$$f(x) = p \cos ax + q \cos bx + r$$

with $a, b, p, q, r \in \mathbb{R}$.

(1) If $pq \neq 0$ and $a/b \in \mathbb{R} \setminus \mathbb{Q}$ then $f \in C \setminus P$.

(2) If $pq \neq 0$ and $a/b \in \mathbb{Q}$ with $a \neq l\pi$ and $b \neq s\pi$, $\forall \ l, s \in \mathbb{Q}$ then $f \in CP$ (see example 3.1) and the set

$$\{f(n) \mid n \in \mathbb{N}\}$$

is dense in $[m, M]$.

(3) If $pq \neq 0$ and $a = l\pi$, $b = s\pi$ for some $l, s \in \mathbb{Q} \setminus \{0\}$ then $f \in CP$ and the set

$$A_\theta = \{f(n\theta) : \ n \in \mathbb{N}\}$$

is dense in $[m, M]$.

For a discussion and examples about the maxima and minima for periodic and almost periodic functions see [19].

Example 2.2. The function $f(x) = e^{iax} + b$, $a \in \mathbb{R} \setminus \{0\}$, $b \in \mathbb{C}$ is periodic with period $2\pi/a$.

If a is a non rational multiple of π then the set:

$$\{f(n) : \ n \in \mathbb{N}\}$$

is dense in

$$\mathbb{T}_b = \{z \in \mathbb{C} : \ |z - b| = 1\}.$$

If a is a rational multiple of π then the set:

$$\{f(n\theta):\ n \in \mathbb{N}\}$$

is dense in $\mathbb{T}_b$, $\forall\ \theta \in \mathbb{R} \setminus \mathbb{Q}$.

3. **Almost periodic functions ([1], [2], [4], [6], [9], [11])**

Let $f \in C$, we call $f \in AP$ if it has one of the following mutually equivalent properties:

$(AP1)$ (Corduneanu, Besicovitch, Bohr, Bochner) $\forall\ \varepsilon > 0$ there is a trigonometric polynomial

$$T_\varepsilon(x) = \sum_{k=1}^{n} C_k e^{i\lambda_k x} \quad \text{(depends of } \varepsilon\text{)}$$

$$C_k \in \mathbb{C},\ \lambda_k \in \mathbb{R},\ k = 1, \ldots, n,\ \text{depending on } \varepsilon$$

such that $|f(x) - T_\varepsilon(x)| < \varepsilon,\ \forall\ x \in \mathbb{R}$.

$(AP2)$ (Bochner, Besicovitch) $\forall\ \varepsilon > 0$ there is $l > 0$ such that $\forall\ a \in \mathbb{R}$ there exists $\tau \in [a, a + l]$ such that:

$$|f(x + \tau) - f(x)| < \varepsilon,\ \forall\ x \in \mathbb{R}$$

$(AP3)$ (Bohr, Fink) For every sequence $(\alpha'_n)_{n\in\mathbb{N}}$ one can extract a subsequence $(\alpha_n)_{n\in\mathbb{N}}$ such that $\lim\limits_{n\to+\infty} f(- + \alpha_n)$ exists uniform on $\mathbb{R}$.

The three definitions are equivalent and useful in applications ([1], [5], [14], [15]).

For generalizations of these definitions see: ([1], [8], [6], [12], [18]).

Some properties of the almost periodic functions have been studied in ([1], [10], [14], [15], [16]).

Example 3.1. Any trigonometric polynomial $(AP1)$ is an almost periodic function. We have:

$$TP \subset AP$$

In particular the function of the example 2.1 is almost periodic.

We are interesting into study of the primitive of an almost periodic function.

Theorem 3.1. ([14], [15]) *Let $f \in AP$. Then a primitive F of f is almost periodic if and only F if is bounded on $\mathbb{R}$.*

Example 3.2. Let f be the function of the example 2.1. Then $f \in IAP$ if and only if $r = 0$.

Theorem 3.2. ([4]) *If $f \in CP$ is nonconstant and F is a primitive of f, then:*

$$F(x) = Ax + g(x),$$

where $T > 0$ is the period of f,

$$A = \frac{1}{T}\int_0^T f(t)dt$$

and g is a CP function.

In the paper ([4]) the preceding result has been proved for a function $f : \mathbb{R} \to \mathbb{R}$, $f \in CP$, for a function $f : \mathbb{R} \to \mathbb{C}$ were writing $f = f_1 + if_2$, f_1 the real part of f and f_2 the imaginary part of f and the theorem 3.2 follows.

The following theorems collects various results about the Fourier series theory for almost periodic functions.

Theorem 3.3. ([8], [5], [13], [14], [15]) *Let $f \in AP$ then:*

(1) $$a(f, \lambda) := \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t)e^{-i\lambda t}dt$$

exists and is equal to zero excepting a countable set Λ.

(2) $$\lim_{T\to\infty} \frac{1}{T}\int_a^{a+T} f(t)e^{-i\lambda t}dt$$

exists uniformly for $a \in \mathbb{R}$.

For $\lambda = 0$ we denote its value by $M(f)$, and call it the mean of f.

(3) (Parseval's equality) The Parseval's equality holds:

$$M(|f|^2) = \sum_{\lambda\in\Lambda} |a(f, \lambda)|^2$$

where Λ is the set mentioned in (1).

(4) The series $\sum_{n=1}^{\infty} a(f,\lambda_n)e^{i\lambda_n x}$ is called the Fourier series of f and we write:

$$f \sim \sum_{n=1}^{\infty} a(f,\lambda_n)e^{i\lambda_n x}$$

If the precedent series converges uniform then:

$$f(x) = \sum_{n=1}^{\infty} a(f,\lambda_n)e^{i\lambda_n x}, \quad x \in \mathbb{R}$$

(5) If the derivate (primitive) of f is an almost periodic function then its Fourier series is obtained by formal derivation (integration) of the Fourier series of f.

Theorem 3.4. ([14], [15], [16]) *Let $f \in AP$.*

(1) *If a primitive of f is almost periodic then $M(f) = 0$.*

(2) *If the series $\sum_{n=1}^{\infty} \left|\frac{a(f,\lambda_n)}{\lambda_n}\right| < +\infty$ then $f \in IAP$ and:*

$$\int_0^x f(t)dt = a_0 + \sum_{n=1}^{\infty} \frac{a(f,\lambda_n)}{\lambda_n} e^{i\lambda_n x}$$

(3) *If the exponents in the Fourier series of f have the property:*

$$|\lambda_n| \geq \alpha > 0,\ \forall\, n \in \mathbb{N} \text{ then } f \in IAP.$$

(4) *If $\int_0^x f(t)dt = Ax^\lambda + g(x)$, $x \in \mathbb{R}_+$, $\lambda \geq 0$ and $g \in BC$ then $A = M(f)$ and $\lambda = 1$.*

Example 3.3. We consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} e^{i2^n x}, \quad x \in \mathbb{R}$$

We see that $f(0) = \frac{\pi^2}{6}$ and $f(x) \neq \frac{\pi^2}{6}$, $\forall\, x \neq 0$ then $f \in C \setminus P$. On the other hand $f \in AP$ and $f \notin TP$. We have

$$\int_0^x f(t)dt = a_0 + \sum_{n=1}^{\infty} \frac{1}{i2^n n^2} e^{i2^n x}$$

and $f \in IAP$. We conclude

$$f \in IAP \setminus (CP \cup TP).$$

Example 3.4. ([14], [15]) We consider the function:

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} e^{ix/n^2}, \quad x \in \mathbb{R}.$$

The function is almost periodic, non periodic. We observe that $\lambda_n = 1/n^2$, $n \in \mathbb{N}$ and $\lambda_n \to 0$ also $M(f) = 0$.

If a primitive of f would be almost periodic then:

$$\int_0^x f(t)dt \sim a_0 + \sum_{n=1}^{\infty} e^{ix/n^2}.$$

But this is not possible since the last series is violating the Parseval's equality.

The function f is an example of an almost periodic function which is not a trigonometric polynomial, has mean zero, $\lambda_n \to 0$ and the primitives are not almost periodic. We conclude

$$f \in AP \setminus (CP \cup IAP \cup TP).$$

4. **Diagram of functions**

The relations:

$$AP \subset BC, \quad AP \subset UC, \quad CP \subset AP$$

have been studied ([8], [5], [10], [13], [14], [16]).

The following examples shows the inclusions between the considered sets of functions are strict.

Examples

4.1. $f \in F \setminus (P \cup B \cup C)$

$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

4.2. $f \in C \setminus (UC \cup P \cup B)$

$$f(x) = x^2$$

4.3. $f \in BC \setminus (UC \cup P)$

$$f(x) = e^{ix^2} \quad ([10])$$

4.4. $f \in (UC \cap B) \setminus (P \cup AP)$

$$f(x) = \text{artan} x \quad ([10])$$

4.5. $f \in B \setminus (C \cup P)$

$$f(x) = \begin{cases} \sin 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

4.6. $f \in AP \setminus (CP \cup IAP \cup TP)$

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} e^{\frac{ix}{n^2}} \quad \text{(Example 3.4)}$$

4.7. $f \in UC \setminus (B \cup P)$

$$f(x) = x$$

4.8. $f \in (TP \cap IAP) \setminus P$ (Examples 2.1 and 3.2)

4.9. $f \in P \setminus (B \cup C)$

$$f(x) = \begin{cases} \tan x, & x \neq (2n+1)\frac{\pi}{2}, \quad n \in \mathbb{Z} \\ 0, & x = (2n+1)\frac{\pi}{2}, \quad n \in \mathbb{Z} \end{cases}$$

4.10. $f \in BP \setminus C$

$$f(x) = x - [x]$$

4.11. $f \in CP \setminus TP$

First we consider the function $\psi : [0,2] \to [0,1]$

$$\psi(x) = \begin{cases} x, & x \in [0,1] \\ 2 - x, & x \in]1,2] \end{cases}$$

The function f is the function:

$$f(x) = \psi(x - 2n), \quad x \in [2n, 2(n+1)], \quad n \in \mathbb{Z}.$$

4.12. $f \in TP \cap CP \cap IAP$

(Example 2.1 and 3.2)

4.13. $f \in IAP \setminus (CP \cup TP)$

(Example 3.3)

4.14. $g \in (CP \cap IAP) \setminus TP$

$$g(x) = f(x) - M(f)$$

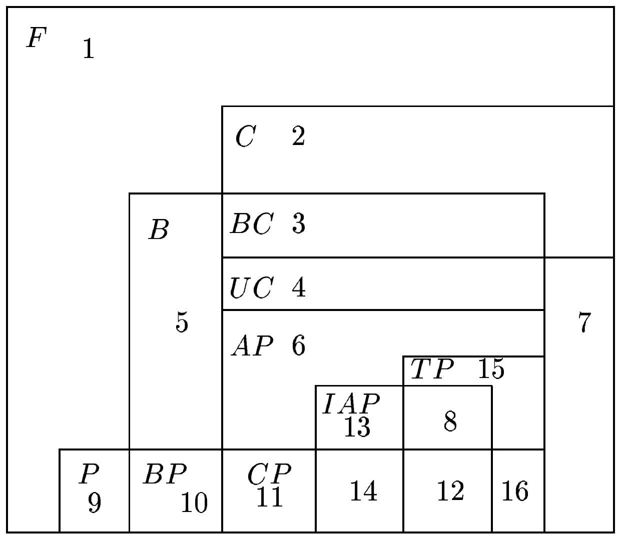
f is the function of the example 4.11

4.15. $f \in TP \setminus (IAP \cup P)$

(Example 2.1 and 3.2)

4.16. $f \in (TP \cap CP) \setminus IAP$

(Examples 2.1 and 3.2)



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