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EQUICONTINUITY AND SINGULARITIES OF FAMILIES OF MONOMIAL MAPPINGS

WOLFGANG W. BRECKNER and TIBERIU TRIF

Dedicated to Professor Ştefan Cobzaş at his 60^{th} *anniversary*

Abstract. The starting-point for the present paper is the principle of condensation of the singularities of families consisting of continuous linear mappings that act between normed linear spaces. It is proved that this basic functional analytical principle can be generalized for families of continuous monomial mappings of degree n between topological linear spaces. The obtained principle yields a generalization of the principle of uniform boundedness published by I. W. Sandberg [IEEE Trans. Circuits and Systems CAS-32 (1985), 332–336] and recently rediscovered by R. Miculescu [Math. Reports (Bucharest) 5 (55) (2003), 57–59]. Furthermore, by applying the new nonlinear principle there are revealed Baire category properties of certain subsets of the normed linear space $C[a,b]$ involved with Riemann-Stieltjes integrability.

1. Introduction

One of the most important and most useful results in the theory of real or complex normed linear spaces is the following theorem, known as the principle of condensation of the singularities.

Theorem 1.1. *Let X and Y be normed linear spaces, and let $(F_j)_{j\in J}$ be a family of continuous linear mappings from X into Y such that*

$$\sup\{\, \|F_j\| \mid j \in J\,\} = \infty.$$

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Then the set of all $x \in X$ satisfying

$$\sup\{\,\|F_j(x)\| \mid j \in J\,\} = \infty$$

is residual, i.e. its complement is a set of the first category.

This theorem immediately provides the next theorem called the principle of uniform boundedness and considered to represent also a major functional analytical result.

Theorem 1.2. *Let X and Y be normed linear spaces of which X is complete, and let $(F_j)_{j\in J}$ be a family of continuous linear mappings from X into Y. Then*

$$\sup\{\,\|F_j(x)\| \mid j \in J\,\} < \infty \qquad \text{for all } x \in X$$

if and only if

$$\sup\{\,\|F_j\| \mid j \in J\,\} < \infty.$$

Both these theorems have been extensively investigated and have been generalized in several directions. In some papers more general spaces have been considered instead of the normed linear spaces X and Y. For instance, Ş. Cobzaş and I. Muntean [5] dealt with the topological structure of the set of singularities associated with a nonequicontinuous family of continuous linear mappings from a topological linear space into another topological linear space and pointed out cases when this set of singularities is an uncountable infinite G_δ-set. In other papers the linear mappings F_j $(j \in J)$ have been replaced by nonlinear mappings of a certain type. Moreover, W. W. Breckner [1] has proved a very general principle of condensation of the singularities which does not require any algebraic structure of the considered spaces and neither assumptions as to the shape of the mappings that are concerned.

For a detailed information on diverse generalizations of the Theorems 1.1 and 1.2 the reader is referred to the surveys by W. W. Breckner [2, 3] as well as to T. Trif [14, 15].

In the present paper we deal with the equicontinuity of families of monomial mappings. Moreover, by following the general line of proving principles of condensation of the singularities we show that the Theorems 1.1 and 1.2 can be generalized for families of continuous monomial mappings of degree n acting between topological linear spaces. Consequently, these generalizations integrate well into the framework described in [1]. Besides, it should be mentioned that the new principle of uniform boundedness turns out to be a generalization of the principle of uniform boundedness proved by I. W. Sandberg [11] and recently rediscovered by R. Miculescu [9]. The paper ends with an application of the obtained nonlinear principle of condensation of singularities that directly reveals Baire category properties of certain subsets of the normed linear space $C[a,b]$ involved with Riemann-Stieltjes integrability.

2. Monomial mappings

All linear spaces as well as all topological linear spaces that will occur in this paper are over $\mathbf{K}$, where $\mathbf{K}$ is either the field $\mathbf{R}$ of real numbers or the field $\mathbf{C}$ of complex numbers. If X is a linear space, then its zero-element is denoted by o_X. The set of all positive integers is $\mathbf{N}$.

Throughout this section let X and Y be linear spaces. Furthermore, let n be a positive integer. A mapping $F\colon X^n \to Y$ is said to be:

(i) *symmetric* if

$$F(x_1,\ldots,x_n) = F(x_{\sigma(1)},\ldots,x_{\sigma(n)})$$

for each $(x_1,\ldots,x_n) \in X^n$ and each bijection $\sigma\colon \{1,\ldots,n\} \to \{1,\ldots,n\}$;

(ii) *n-additive* if for each $i \in \{1,\ldots,n\}$ the mapping

$$\forall\, x \in X \;\longmapsto\; F(x_1,\ldots,x_{i-1},x,x_{i+1},\ldots,x_n) \in Y$$

is additive whenever $x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n \in X$ are fixed.

If $F\colon X^n \to Y$ is an n-additive mapping, then it can be shown that

$$F(r_1x_1,\ldots,r_nx_n) = r_1\cdots r_n F(x_1,\ldots,x_n)$$

for all $x_1, \ldots, x_n \in X$ and all rational numbers $r_1, \ldots, r_n$. If in addition X and Y are topological linear spaces and F is continuous, then we even have

$$F(a_1x_1, \ldots, a_nx_n) = a_1 \cdots a_n F(x_1, \ldots, x_n)$$

for all $x_1, \ldots, x_n \in X$ and all $a_1, \ldots, a_n \in \mathbf{R}$.

Given a mapping $F\colon X^n \to Y$, the mapping $F^*\colon X \to Y$, defined by

$$F^*(x) := F(\underbrace{x, \ldots, x}_{n \text{ times}}) \qquad \text{for all } x \in X,$$

is said to be the *diagonalization* of F. Any symmetric n-additive mapping can be expressed by means of its diagonalization as the following proposition points out (see A. M. McKiernan [8, Lemma 1] or D. Ž. Djoković [6, Lemma 2]).

Proposition 2.1. *If $F\colon X^n \to Y$ is a symmetric n-additive mapping, then*

$$F(u_1, \ldots, u_n) = \frac{1}{n!} \left(\Delta_{u_1} \cdots \Delta_{u_n} F^*\right)(x)$$

for all $u_1, \ldots, u_n, x \in X$, where $\Delta_u\colon Y^X \to Y^X$ is defined for each $u \in X$ by

$$(\Delta_u f)(x) := f(x+u) - f(x) \qquad \textit{for all } f \in Y^X \textit{ and all } x \in X.$$

A mapping $Q\colon X \to Y$ is said to be a *monomial mapping of degree* n if there exists a symmetric n-additive mapping $F\colon X^n \to Y$ such that $Q = F^*$. In virtue of Proposition 2.1 there exists for each monomial mapping $Q\colon X \to Y$ of degree n a single symmetric n-additive mapping $F\colon X^n \to Y$ such that $Q = F^*$.

A monomial mapping $Q\colon X \to Y$ of degree n has the homogeneity property $Q(rx) = r^n Q(x)$ for every $x \in X$ and every rational number r. If in addition X and Y are topological linear spaces and Q is continuous, this property implies

$$Q(ax) = a^n Q(x) \qquad \text{for every } x \in X \text{ and every } a \in \mathbf{R}.$$

Finally, we mention a useful characterization of the monomial mappings of degree n (see A. M. McKiernan [8, Corollary 3] or D. Ž. Djoković [6, Corollary 3]).

Proposition 2.2. *A mapping $Q\colon X \to Y$ is a monomial mapping of degree n if and only if*

$$\frac{1}{n!}\left(\Delta_u^n Q\right)(x) = Q(u) \qquad \textit{for all } u, x \in X.$$

The monomial mappings of degree 1 coincide with the additive mappings, while the monomial mappings of degree 2 are called *quadratic*.

3. **Equicontinuity of families of monomial mappings**

Let X and Y be topological linear spaces, and let $\mathcal{F} := (F_j)_{j\in J}$ be a family of mappings from X into Y. If x is a point in X, then $\mathcal{F}$ is said to be *equicontinuous at x* if for every neighbourhood V of o_Y there exists a neighbourhood U of o_X such that

$$\{\, F_j(x+u) - F_j(x) \mid j \in J \,\} \subseteq V \qquad \text{for all } u \in U.$$

If $\mathcal{F}$ is equicontinuous at each point of X, then $\mathcal{F}$ is said to be *equicontinuous on X*.

For families of symmetric n-additive mappings the following characterization of the equicontinuity is valid.

Theorem 3.1. *Let n be a positive integer, let X and Y be topological linear spaces, let $\mathcal{F} := (F_j)_{j\in J}$ be a family of symmetric n-additive mappings from X^n into Y, and let $\mathcal{F}^* := (F_j^*)_{j\in J}$. Then the following assertions are equivalent:*

1° *$\mathcal{F}^*$ is equicontinuous on X.*

2° *$\mathcal{F}^*$ is equicontinuous at o_X.*

3° *$\mathcal{F}$ is equicontinuous at o_{X^n}.*

4° *$\mathcal{F}$ is equicontinuous on X^n.*

Proof. Since the implications $1^\circ \Rightarrow 2^\circ$ and $4^\circ \Rightarrow 3^\circ$ are obvious, it remains to prove that $2^\circ \Rightarrow 3^\circ$, $3^\circ \Rightarrow 1^\circ$ and $1^\circ \Rightarrow 4^\circ$.

We start by proving the implication $2^\circ \Rightarrow 3^\circ$. Let V be any neighbourhood of o_Y. Choose a balanced neighbourhood V_0 of o_Y such that

$$\underbrace{V_0 + \cdots + V_0}_{2^n \text{ terms}} \subseteq V. \tag{1}$$

The equicontinuity of $\mathcal{F}^*$ at o_X ensures the existence of a neighbourhood U_0 of o_X such that

$$\{ F_j^*(u) \mid j \in J \} \subseteq V_0 \qquad \text{for all } u \in U_0. \tag{2}$$

Now select a neighbourhood U of o_X such that

$$\underbrace{U + \cdots + U}_{n \text{ terms}} \subseteq U_0. \tag{3}$$

We claim that

$$\{ F_j(u_1, \ldots, u_n) \mid j \in J \} \subseteq V \qquad \text{for all } (u_1, \ldots, u_n) \in U^n. \tag{4}$$

Indeed, let j be any index in J and let $(u_1, \ldots, u_n)$ be any point in U^n. According to Proposition 2.1 we have

$$\begin{aligned} F_j(u_1, \ldots, u_n) &= \frac{1}{n!} \left(\Delta_{u_1} \cdots \Delta_{u_n} F_j^* \right)(o_X) \\ &= \frac{1}{n!} \sum_{(a_1, \ldots, a_n) \in A} (-1)^{n-(a_1+\cdots+a_n)} F_j^*(a_1 u_1 + \cdots + a_n u_n), \end{aligned} \tag{5}$$

where $A := \{0,1\}^n$. Since

$$a_1 u_1 + \cdots + a_n u_n \in \underbrace{U + \cdots + U}_{n \text{ terms}} \subseteq U_0 \qquad \text{for all } (a_1, \ldots, a_n) \in A,$$

we conclude in virtue of (2) that

$$F_j^*(a_1 u_1 + \cdots + a_n u_n) \in V_0 \qquad \text{for all } (a_1, \ldots, a_n) \in A.$$

Taking into account that $\operatorname{card} A = 2^n$ and that V_0 is balanced, we get by (5)

$$F_j(u_1, \ldots, u_n) \in \underbrace{V_0 + \cdots + V_0}_{2^n \text{ terms}} \subseteq V.$$

Consequently, (4) is true. From (4) it follows that $\mathcal{F}$ is equicontinuous at o_{X^n}.

Next we prove that $3^\circ \Rightarrow 1^\circ$. Let x be any point in X, and let V be any neighbourhood of o_Y. Choose a balanced neighbourhood V_0 of o_Y such that

$$\underbrace{V_0 + \cdots + V_0}_{2^n - 1 \text{ terms}} \subseteq V.$$

The equicontinuity of $\mathcal{F}$ at o_{X^n} ensures the existence of a balanced neighbourhood U_0 of o_X such that

$$\{\, F_j(u_1, \dots, u_n) \mid j \in J \,\} \subseteq V_0 \qquad \text{for all } (u_1, \dots, u_n) \in U_0^n. \tag{6}$$

Select a rational number $r \in \left]0, 1\right]$ such that $rx \in U_0$. We assert that

$$\{\, F_j^*(x + r^{n-1}u) - F_j^*(x) \mid j \in J \,\} \subseteq V \qquad \text{for all } u \in U_0. \tag{7}$$

Indeed, let $j \in J$ and $u \in U_0$ be arbitrarily chosen. Then we have

$$\begin{aligned} F_j^*(x+&r^{n-1}u) - F_j^*(x) \\ &= F_j(\underbrace{x + r^{n-1}u, \dots, x + r^{n-1}u}_{n \text{ times}}) - F_j(\underbrace{x, \dots, x}_{n \text{ times}}) \\ &= \sum_{k=1}^{n} \binom{n}{k} F_j(\underbrace{x, \dots, x}_{n-k \text{ times}}, \underbrace{r^{n-1}u, \dots, r^{n-1}u}_{k \text{ times}}) \\ &= \sum_{k=1}^{n} \binom{n}{k} F_j(\underbrace{rx, \dots, rx}_{n-k \text{ times}}, r^{k-1}u, \underbrace{r^{n-1}u, \dots, r^{n-1}u}_{k-1 \text{ times}}). \end{aligned} \tag{8}$$

Since U_0 is balanced and $r \in \left]0, 1\right]$, we see that (6) implies

$$F_j(\underbrace{rx, \dots, rx}_{n-k \text{ times}}, r^{k-1}u, \underbrace{r^{n-1}u, \dots, r^{n-1}u}_{k-1 \text{ times}}) \in V_0$$

for each $k \in \{\, 1, \dots, n \,\}$. Therefore it follows from (8) that

$$F_j^*(x + r^{n-1}u) - F_j^*(x) \in \underbrace{V_0 + \cdots + V_0}_{2^n - 1 \text{ terms}} \subseteq V.$$

Hence (7) is true. If we set $U := r^{n-1}U_0$, then U is a neighbourhood of o_X satisfying

$$\{\, F_j^*(x + u) - F_j^*(x) \mid j \in J \,\} \subseteq V \qquad \text{for all } u \in U.$$

Consequently, $\mathcal{F}^*$ is equicontinuous at x.

Finally, we prove that $1^\circ \Rightarrow 4^\circ$. Let $(x_1, \dots, x_n)$ be any point in X^n, and let V be any neighbourhood of o_Y. Choose a balanced neighbourhood V_0 of o_Y such that (1) holds. Let $A := \{0,1\}^n$. Since $\mathcal{F}^*$ is equicontinuous on X, there exists for each $(a_1,\dots,a_n) \in A$ a neighbourhood U_a of o_X such that

$$\{\, F_j^*(a_1x_1+\cdots+a_nx_n+u) - F_j^*(a_1x_1+\cdots+a_nx_n) \mid j \in J \,\} \subseteq V_0 \tag{9}$$

for all $u \in U_a$. Next we choose a neighbourhood U of o_X such that

$$\underbrace{U+\cdots+U}_{n \text{ terms}} \subseteq \bigcap_{a\in A} U_a.$$

Then it results from (9) that

$$\{\, F_j^*(a_1(x_1+u_1)+\cdots+a_n(x_n+u_n)) - F_j^*(a_1x_1+\cdots+a_nx_n) \mid j \in J \,\} \subseteq V_0 \tag{10}$$

for all $(a_1,\dots,a_n) \in A$ and all $(u_1,\dots,u_n) \in U^n$. But, according to Proposition 2.1 we have

$$\begin{aligned}
&F_j(x_1+u_1,\dots,x_n+u_n) - F_j(x_1,\dots,x_n)\\
&\quad= \frac{1}{n!}\left[\left(\Delta_{x_1+u_1}\cdots\Delta_{x_n+u_n}F_j^*\right)(o_X) - \left(\Delta_{x_1}\cdots\Delta_{x_n}F_j^*\right)(o_X)\right]\\
&\quad= \frac{1}{n!}\sum_{(a_1,\dots,a_n)\in A}(-1)^{n-(a_1+\cdots+a_n)}\left[F_j^*(a_1(x_1+u_1)+\cdots+a_n(x_n+u_n))\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad\left.-F_j^*(a_1x_1+\cdots+a_nx_n)\right]
\end{aligned}$$

for all $j \in J$ and all $(u_1,\dots,u_n) \in U^n$. By (10) it follows that

$$\{\, F_j(x_1+u_1,\dots,x_n+u_n) - F_j(x_1,\dots,x_n) \mid j \in J \,\} \subseteq \underbrace{V_0+\cdots+V_0}_{2^n \text{ terms}} \subseteq V$$

for all $(u_1,\dots,u_n) \in U^n$. Consequently, $\mathcal{F}$ is equicontinuous at $(x_1,\dots,x_n)$. $\square$

Corollary 3.2. *Let n be a positive integer, let X and Y be topological linear spaces, and let $\mathcal{Q} := (Q_j)_{j\in J}$ be a family of monomial mappings of degree n from X into Y. Then $\mathcal{Q}$ is equicontinuous on X if and only if it is equicontinuous at some point of X.*

Proof. Necessity. Obvious.

Sufficiency. Suppose that $x \in X$ is a point at which $\mathcal{Q}$ is equicontinuous. Then $\mathcal{Q}$ is equicontinuous at o_X. Indeed, when $x = o_X$, then this assertion is trivial.

When $x \neq o_X$, then it can be proved as follows. Let V be any neighbourhood of o_Y. Choose a balanced neighbourhood V_0 of o_Y such that

$$\underbrace{V_0 + \cdots + V_0}_{n \text{ terms}} \subseteq V.$$

The equicontinuity of $\mathcal{Q}$ at x ensures the existence of a neighbourhood U_0 of o_X such that

$$\{\, Q_j(x+u) - Q_j(x) \mid j \in J \,\} \subseteq V_0 \qquad \text{for all } u \in U_0. \tag{11}$$

Select a neighbourhood U of o_X that satisfies (3). Taking into account that

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} = 0,$$

the Proposition 2.2 implies

$$\begin{aligned} Q_j(u) &= \frac{1}{n!} \left(\Delta_u^n Q_j\right)(x) \\ &= \frac{1}{n!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} Q_j(x+ku) \\ &= \frac{1}{n!} \sum_{k=1}^{n} (-1)^{n-k} \binom{n}{k} [Q_j(x+ku) - Q_j(x)] \end{aligned} \tag{12}$$

for all $j \in J$ and all $u \in U$. Since

$$ku \in \underbrace{U + \cdots + U}_{k \text{ terms}} \subseteq U_0$$

for all $k \in \{\, 1, \ldots, n \,\}$ and all $u \in U$, it follows from (11) that

$$\{\, Q_j(x+ku) - Q_j(x) \mid j \in J \,\} \subseteq V_0$$

for all $k \in \{\, 1, \ldots, n \,\}$ and all $u \in U$. Since V_0 is balanced, (12) implies that

$$\{\, Q_j(u) \mid j \in J \,\} \subseteq \underbrace{V_0 + \cdots + V_0}_{n \text{ terms}} \subseteq V \qquad \text{for all } u \in U.$$

Consequently, $\mathcal{Q}$ is equicontinuous at o_X.

By applying now the implication $2^\circ \Rightarrow 1^\circ$ stated in Theorem 3.1, it follows that $\mathcal{Q}$ is equicontinuous on X. $\square$

Corollary 3.3. *Let n be a positive integer, let X and Y be topological linear spaces, and let $Q\colon X \to Y$ be a monomial mapping of degree n. Then Q is continuous on X if and only if it is continuous at some point of X.*

In the special case when $n = 1$ this corollary is well-known. When $n = 2$ it generalizes a result stated by S. Kurepa [7, Theorem 2] under the assumption that X is a normed linear space and $Y = \mathbf{R}$. In addition we note that a similar continuity result involving quadratic set-valued mappings was obtained by W. Smajdor [12, Theorem 4.2].

Proposition 3.4. *Let n be a positive integer, let X and Y be normed linear spaces, and let $(F_j)_{j\in J}$ be a family of symmetric n-additive mappings from X^n into Y. Then the following assertions are equivalent:*

1° *$(F_j^*)_{j\in J}$ is equicontinuous at o_X.*

2° $\sup\{\, \|F_j(x_1,\dots,x_n)\| \mid j \in J,\ \|x_1\| \le 1,\ \dots,\ \|x_n\| \le 1 \,\} < \infty$.

3° $\sup\{\, \|F_j^*(x)\| \mid j \in J,\ \|x\| \le 1 \,\} < \infty$.

Proof. 1° $\Rightarrow$ 2° According to the implication 2° $\Rightarrow$ 3° in Theorem 3.1, the family $(F_j)_{j\in J}$ is equicontinuous at o_{X^n}. Therefore there exists a neighbourhood U of o_X such that

$$\|F_j(u_1,\dots,u_n)\| \le 1 \qquad \text{for all } j \in J \text{ and all } (u_1,\dots,u_n) \in U^n.$$

Let r be a positive rational number such that $\{\, x \in X \mid \|x\| \le r \,\} \subseteq U$. Then we have

$$\|F_j(x_1,\dots,x_n)\| = \frac{1}{r^n}\|F_j(rx_1,\dots,rx_n)\| \le \frac{1}{r^n}$$

for all $j \in J$ and all $(x_1,\dots,x_n) \in X^n$ satisfying $\|x_1\| \le 1,\ \dots,\ \|x_n\| \le 1$. Consequently, assertion 2° is true.

2° $\Rightarrow$ 3° Obvious.

3° $\Rightarrow$ 1° Let V be a neighbourhood of o_Y. Choose a positive real number a such that $\{\, y \in Y \mid \|y\| \le a \,\} \subseteq V$. In addition, choose a positive rational number r such that $br^n \le a$, where

$$b := \sup\{\, \|F_j^*(x)\| \mid j \in J,\ \|x\| \le 1 \,\}.$$

Then we have

$$\|F_j^*(u)\| = r^n \left\| F_j^* \left(\frac{1}{r} u \right) \right\| \leq br^n \leq a$$

for all $j \in J$ and all $u \in X$ with $\|u\| \leq r$. Consequently, the neighbourhood $U := \{ x \in X \mid \|x\| \leq r \}$ of o_X satisfies

$$\{ F_j^*(u) \mid j \in J \} \subseteq V \qquad \text{for all } u \in U.$$

Hence $(F_j^*)_{j\in J}$ is equicontinuous at o_X. □

4. **Singularities of families of monomial mappings**

Let X and Y be topological linear spaces, and let $\mathcal{F} := (F_j)_{j\in J}$ be a family of mappings from X into Y. If x is a point in X, then $\mathcal{F}$ is said to be *bounded at* x if the set $\{ F_j(x) \mid j \in J \}$ is bounded, i.e. for each neighbourhood V of o_Y there exists a positive real number a such that $\{ F_j(x) \mid j \in J \} \subseteq aV$. If M is a subset of X and $\mathcal{F}$ is bounded at each point of M, then $\mathcal{F}$ is said to be *pointwise bounded on* M.

Any point in X at which $\mathcal{F}$ is not bounded is said to be a *singularity* of $\mathcal{F}$. The set of all singularities of $\mathcal{F}$ is denoted by $S_{\mathcal{F}}$. Clearly, $\mathcal{F}$ is pointwise bounded on X if and only if $S_{\mathcal{F}} = \emptyset$.

Theorem 4.1. *Let n be a positive integer, let X and Y be topological linear spaces, and let $\mathcal{Q} := (Q_j)_{j\in J}$ be a family of monomial mappings of degree n from X into Y which is equicontinuous at o_X. Then $\mathcal{Q}$ is pointwise bounded on X.*

Proof. Let x be any point in X. We prove that $\mathcal{Q}$ is bounded at x. Let V be a neighbourhood of o_Y. Since $\mathcal{Q}$ is equicontinuous at o_X, there exists a neighbourhood U of o_X such that

$$\{ Q_j(u) \mid j \in J \} \subseteq V \qquad \text{for all } u \in U.$$

Choose a rational number $r \neq 0$ such that $rx \in U$. Then $\{ Q_j(rx) \mid j \in J \} \subseteq V$, whence

$$\{ Q_j(x) \mid j \in J \} \subseteq \frac{1}{r^n} V.$$

Consequently, $\mathcal{Q}$ is bounded at x. □

The converse of Theorem 4.1 is not true. The pointwise boundedness of $\mathcal{Q}$ on X does not imply the equicontinuity of $\mathcal{Q}$ at o_X, not even when $n = 1$. But, taking into consideration the next theorem, which is a principle of condensation of the singularities of families of continuous monomial mappings between topological linear spaces, we will be able to point out cases when the pointwise boundedness of $\mathcal{Q}$ implies the equicontinuity of $\mathcal{Q}$ at o_X (and therefore on the whole space X).

Theorem 4.2. *Let n be a positive integer, let X and Y be topological linear spaces, and let $\mathcal{Q} := (Q_j)_{j\in J}$ be a family of continuous monomial mappings of degree n from X into Y which is not equicontinuous at o_X. Then the following assertions are true:*

1° $S_{\mathcal{Q}}$ is a residual set.

2° If, in addition, X is of the second category, then $S_{\mathcal{Q}}$ is of the second category, dense in X and with $\operatorname{card} S_{\mathcal{Q}} \geq \aleph$.

Proof. 1° Since $\mathcal{Q}$ is not equicontinuous at o_X, there exists a neighbourhood V of o_Y such that for every neighbourhood U of o_X there is a $u \in U$ satisfying

$$\{\, Q_j(u) \mid j \in J \,\} \not\subseteq V.$$

Choose a closed balanced neighbourhood V_0 of o_Y such that

$$\underbrace{V_0 + \cdots + V_0}_{n+1 \text{ terms}} \subseteq V.$$

For each positive integer m put

$$S_m := \bigcap_{j\in J} \{\, x \in X \mid Q_j(x) \in mV_0 \,\}.$$

Since V_0 is closed and all the mappings Q_j $(j \in J)$ are continuous, it follows that all the sets S_m are closed. We claim that all these sets are nowhere dense. Indeed, otherwise there exists a positive integer m such that $\operatorname{int} S_m \neq \emptyset$. Choose any point $x_0 \in \operatorname{int} S_m$. Next select a neighbourhood U_0 of o_X such that $x_0 + U_0 \subseteq S_m$ and after that select a neighbourhood U of o_X satisfying (3). Fix any $j \in J$ and any $u \in U$. In virtue of Proposition 2.2 we have

$$Q_j(u) = \frac{1}{n!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} Q_j(x_0 + ku). \tag{13}$$

Since

$$x_0 + ku \in x_0 + \underbrace{U + \cdots + U}_{k \text{ terms}} \subseteq x_0 + U_0 \subseteq S_m$$

for all $k \in \{ 0, 1, \ldots, n \}$, it follows that

$$Q_j(x_0 + ku) \in mV_0 \qquad \text{for all } k \in \{ 0, 1, \ldots, n \}.$$

Taking into account that V_0 is balanced, we obtain from (13) that

$$Q_j(u) \in m(\underbrace{V_0 + \cdots + V_0}_{n+1 \text{ terms}}) \subseteq mV,$$

whence

$$Q_j\left(\frac{1}{m}\, u\right) = \frac{1}{m^n}\, Q_j(u) \in \frac{1}{m^{n-1}}\, V \subseteq V.$$

Since $j \in J$ and $u \in U$ were arbitrarily chosen, we have

$$\left\{ Q_j\left(\frac{1}{m}\, u\right) \,\middle|\, j \in J \right\} \subseteq V \qquad \text{for all } u \in U,$$

which contradicts the choice of V. Consequently, all the sets S_m are nowhere dense, as claimed.

It is immediately seen that

$$X \setminus S_{\mathcal{Q}} \subseteq \bigcup_{m=1}^{\infty} S_m.$$

Therefore $X \setminus S_{\mathcal{Q}}$ is a set of the first category, i.e. $S_{\mathcal{Q}}$ is a residual set.

2° Since X is of the second category, it follows in virtue of a well-known result in the theory of topological linear spaces that X is a Baire space. Consequently, the residual set $S_{\mathcal{Q}}$ is of the second category and dense. Therefore $S_{\mathcal{Q}}$ is not empty. Let x be any point in $S_{\mathcal{Q}}$. Since $\mathcal{Q}$ is bounded at o_X, we have $x \neq o_X$. Besides we have

$$\{ Q_j(ax) \mid j \in J \} = a^n \{ Q_j(x) \mid j \in J \} \qquad \text{for all } a \in \mathbf{R}.$$

Since the set $\{ Q_j(x) \mid j \in J \}$ is not bounded, it follows that

$$\{ ax \mid a \in \mathbf{R} \setminus \{0\} \} \subseteq S_{\mathcal{Q}},$$

whence $\operatorname{card} S_{\mathcal{Q}} \geq \aleph$. $\square$

Together the Theorems 4.1 and 4.2 yield the following theorem revealing cases when the equicontinuity at o_X of a family $\mathcal{Q}$ of continuous monomial mappings of degree n from a topological linear space X into a topological linear space Y is equivalent to the pointwise boundedness of $\mathcal{Q}$ on X.

Theorem 4.3. *Let n be a positive integer, let X and Y be topological linear spaces, and let $\mathcal{Q}$ be a family of continuous monomial mappings of degree n from X into Y. Then the following assertions are equivalent:*

1° *X is of the second category and $\mathcal{Q}$ is pointwise bounded on X.*

2° *There exists a subset $M \subseteq X$ of the second category on which $\mathcal{Q}$ is pointwise bounded.*

3° *X is of the second category and $\mathcal{Q}$ is equicontinuous at o_X.*

Proof. $1^\circ \Rightarrow 2^\circ$ Obvious.

$2^\circ \Rightarrow 3^\circ$ Since $M \subseteq X$, it follows that X is of the second category. Analogously, it follows from $M \subseteq X \setminus S_{\mathcal{Q}}$ that $X \setminus S_{\mathcal{Q}}$ is of the second category. In other words, $S_{\mathcal{Q}}$ is not a residual set. According to assertion 1° in Theorem 4.2 the family $\mathcal{Q}$ must be equicontinuous at o_X.

$3^\circ \Rightarrow 1^\circ$ Results by Theorem 4.1. $\square$

Corollary 4.4. *Let n be a positive integer, let X and Y be normed linear spaces, and let $(F_j)_{j\in J}$ be a family of continuous symmetric n-additive mappings from X^n into Y. Then the following assertions are equivalent:*

1° *X is of the second category and*

$$\sup\{\, \|F_j^*(x)\| \mid j \in J\,\} < \infty \qquad \textit{for all } x \in X.$$

2° *There exists a subset $M \subseteq X$ of the second category such that*

$$\sup\{\, \|F_j^*(x)\| \mid j \in J\,\} < \infty \qquad \textit{for all } x \in M. \tag{14}$$

3° *X is of the second category and*

$$\sup\{\, \|F_j(x_1,\dots,x_n)\| \mid j \in J,\ \|x_1\| \leq 1,\ \dots,\ \|x_n\| \leq 1\,\} < \infty. \tag{15}$$

Proof. $1° \Rightarrow 2°$ Obvious.

$2° \Rightarrow 3°$ The inequality (14) expresses that the family $(F_j^*)_{j\in J}$ is pointwise bounded on M. By applying the implication $2° \Rightarrow 3°$ from Theorem 4.3 it follows that X is of the second category and that $(F_j^*)_{j\in J}$ is equicontinuous at o_X. Therefore, by the implication $1° \Rightarrow 2°$ in Proposition 3.4, the inequality (15) is true.

$3° \Rightarrow 1°$ Let $x \in X$ be arbitrarily chosen. When $x = o_X$, then

$$\sup\{\, \|F_j^*(x)\| \mid j \in J \,\} = 0.$$

When $x \neq o_X$, then the number $a := 1/\|x\|$ satisfies

$$\|F_j^*(x)\| = \frac{1}{a^n}\|F_j(\underbrace{ax, \ldots, ax}_{n \text{ times}})\| \leq \frac{b}{a^n}$$

for all $j \in J$, where

$$b := \sup\{\, \|F_j(x_1, \ldots, x_n)\| \mid j \in J,\ \|x_1\| \leq 1,\ \ldots,\ \|x_n\| \leq 1 \,\}.$$

Consequently, assertion $1°$ is true. $\square$

It should be remarked that Theorem 4.2 is a generalization of Theorem 1.1, while Theorem 4.3 and Corollary 4.4 are generalizations of Theorem 1.2. Besides, Corollary 4.4 is also a generalization of the principle of uniform boundedness proved by I. W. Sandberg [11, Theorem 2], which recently was rediscovered by R. Miculescu [9, Theorem 2].

5. An application to the theory of the Riemann-Stieltjes integral

Throughout this section a and b are real numbers satisfying the inequality $a < b$. Any finite sequence $(x_0, x_1, \ldots, x_n)$ of points of the interval $[a, b]$ such that $a = x_0 < x_1 < \cdots < x_n = b$ is called a *subdivision* of $[a, b]$.

If $\Delta := (x_0, x_1, \ldots, x_n)$ is a subdivision of $[a, b]$, then the number

$$\mu(\Delta) := \max\{\, x_1 - x_0, \ldots, x_n - x_{n-1} \,\}$$

is called the *mesh* of Δ and any finite sequence $(c_1, \ldots, c_n)$ such that $c_j \in [x_{j-1}, x_j]$ for all $j \in \{1, \ldots, n\}$ is called a *selection* assigned to Δ. The set of all selections assigned to Δ will be denoted by $\mathcal{S}_\Delta$.

A. Pelczynski and S. Rolewicz [10, Corollary] proved that a function $f : [a, b] \to \mathbf{R}$ is Riemann-Stieltjes integrable with respect to itself over $[a, b]$ if and only if for each $\varepsilon \in]0, \infty[$ there exists a $\delta \in]0, \infty[$ such that for any subdivision $\Delta := (x_0, x_1, \ldots, x_n)$ of $[a, b]$ with $\mu(\Delta) < \delta$ the inequality

$$\sum_{j=1}^{n} [f(x_j) - f(x_{j-1})]^2 < \varepsilon$$

holds. According to this result each function $f : [a, b] \to \mathbf{R}$, which is Riemann-Stieltjes integrable with respect to itself over $[a, b]$, has to be continuous. On the other hand, the main result established by A. Pelczynski and S. Rolewicz [10, Theorem 3], concerning the Riemann-Stieltjes integrals of the form

$$\int_a^b \Phi(f(x)) df(x),$$

reveals that not every continuous function $f : [a, b] \to \mathbf{R}$ is Riemann-Stieltjes integrable with respect to itself over $[a, b]$. Actually, the set consisting of all continuous functions $f : [a, b] \to \mathbf{R}$ having the property that f is not Riemann-Stieltjes integrable with respect to itself over $[a, b]$ is very large. More precisely, the following theorem holds.

Theorem 5.1. *Let $C[a, b]$ be the linear space of all real-valued continuous functions defined on $[a, b]$ endowed with the usual uniform norm*

$$\|f\| = \max \{ |f(x)| \mid x \in [a, b] \} \qquad (f \in C[a, b]),$$

and let $\widetilde{RS}[a, b]$ be the set of all functions $f \colon [a, b] \to \mathbf{R}$ having the property that f is Riemann-Stieltjes integrable with respect to itself over $[a, b]$. Then the following assertions are true:

1° *The set $C[a, b] \setminus \widetilde{RS}[a, b]$ is residual, whence of the second category, dense in $C[a, b]$ and with* $\operatorname{card}(C[a, b] \setminus \widetilde{RS}[a, b]) \geq \aleph$.

2° *The set $\widetilde{RS}[a, b]$ is of the first category and dense in $C[a, b]$.*

Proof. Let $\varphi\colon [0,1] \to [a,b]$ be defined by $\varphi(t) := a + t(b-a)$. Taking into account that the mapping

$$\forall\ f \in C[a,b] \ \longmapsto\ f \circ \varphi \in C[0,1]$$

is an isometric isomorphism as well as that a function $f\colon [a,b] \to \mathbf{R}$ is Riemann-Stieltjes integrable with respect to itself over $[a,b]$ if and only if $f \circ \varphi$ is Riemann-Stieltjes integrable with respect to itself over $[0,1]$, it suffices to prove the theorem in the special case when $a = 0$ and $b = 1$.

Let $\mathcal{D}$ be the set consisting of all subdivisions of $[0,1]$. Given a subdivision $\Delta := (x_0, x_1, \ldots, x_n) \in \mathcal{D}$ and a selection $\xi := (c_1, \ldots, c_n) \in \mathcal{S}_\Delta$, let $Q_{\Delta,\xi}\colon C[0,1] \to \mathbf{R}$ be the mapping defined by

$$Q_{\Delta,\xi}(f) := \sum_{j=1}^{n} f(c_j)[f(x_j) - f(x_{j-1})] \qquad \text{for all } f \in C[0,1].$$

It is easily seen that $Q_{\Delta,\xi}$ is continuous. Besides, we notice that $Q_{\Delta,\xi}$ is a quadratic mapping, because it is the diagonalization of the symmetric bilinear mapping $F_{\Delta,\xi}\colon C[0,1] \times C[0,1] \to \mathbf{R}$, defined by

$$F_{\Delta,\xi}(f,g) := \frac{1}{2}\sum_{j=1}^{n} f(c_j)[g(x_j) - g(x_{j-1})] + \frac{1}{2}\sum_{j=1}^{n} g(c_j)[f(x_j) - f(x_{j-1})]$$

for all $f, g \in C[0,1]$.

1° Passing to the proof of the first assertion of the theorem, we consider for every positive integer n the family

$$\mathcal{Q}_n := \{\, Q_{\Delta,\xi} \mid \Delta \in \mathcal{D},\ \xi \in \mathcal{S}_\Delta,\ \mu(\Delta) \le 1/n \,\}.$$

We claim that for every positive integer n the family $\mathcal{Q}_n$ is not equicontinuous at the zero-element of $C[0,1]$.

Indeed, let n be any positive integer. Define the function $f\colon [0,1] \to \mathbf{R}$ by

$$f(x) := \begin{cases} 0 & \text{if} \quad x = 0 \\ \sqrt{x}\,\left|\cos\frac{\pi}{x}\right| & \text{if} \quad 0 < x \le \frac{1}{n} \\ \frac{1}{\sqrt{n}} & \text{if} \quad \frac{1}{n} < x \le 1. \end{cases}$$

For each positive integer p set

$$\Delta_p := \Bigg(0, \frac{1}{n+p}, \frac{2}{2(n+p)-1}, \frac{1}{n+p-1}, \frac{2}{2(n+p)-3}, \frac{1}{n+p-2}, \dots, \frac{1}{n+1}, \frac{2}{2n+1}, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots, \frac{n-1}{n}, 1\Bigg)$$

and

$$\xi_p := \Bigg(0, \frac{2}{2(n+p)-1}, \frac{1}{n+p-1}, \frac{2}{2(n+p)-3}, \frac{1}{n+p-2}, \dots, \frac{2}{2n+1}, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\Bigg).$$

Obviously, Δ_p is a subdivision of $[0,1]$ with $\mu(\Delta_p) \le 1/n$ and ξ_p is a selection assigned to Δ_p. Since

$$\begin{aligned} Q_{\Delta_p,\xi_p}(f) &= \sum_{j=n}^{n+p-1} f\left(\frac{1}{j}\right)\left[f\left(\frac{1}{j}\right) - f\left(\frac{2}{2j+1}\right)\right] \\ &= \frac{1}{n} + \frac{1}{n+1} + \dots + \frac{1}{n+p-1} \end{aligned}$$

for every $p \in \mathbf{N}$, it follows that

$$\sup\{\, Q_{\Delta_p,\xi_p}(f) \mid p \in \mathbf{N}\,\} = \infty.$$

Consequently, f is a singularity of $\mathcal{Q}_n$. By applying Theorem 4.1 we conclude that $\mathcal{Q}_n$ is not equicontinuous at the zero-element of $C[0,1]$, as claimed.

By virtue of Theorem 4.2 we deduce that all the sets $S_{\mathcal{Q}_n}$ $(n \in \mathbf{N})$ are residual, hence the set

$$S := \bigcap_{n=1}^{\infty} S_{\mathcal{Q}_n}$$

is residual, too.

Since $S \subseteq C[0,1] \setminus \widetilde{RS}[0,1]$, it follows that the set $C[0,1] \setminus \widetilde{RS}[0,1]$ is residual, whence of the second category, dense in $C[0,1]$ and with

$$\operatorname{card}(C[0,1] \setminus \widetilde{RS}[0,1]) \ge \aleph.$$

2° The fact that $\widetilde{RS}[0,1]$ is of the first category follows from assertion 1°. On the other hand, since $\widetilde{RS}[0,1]$ contains the restrictions to $[0,1]$ of all polynomials, it follows that $\widetilde{RS}[0,1]$ is dense in $C[0,1]$. □

Remark. There is also another way to prove Theorem 5.1. The characterization of the functions that are Riemann-Stieltjes integrable with respect to themselves over $[a,b]$, recalled at the beginning of this section, yields that $\widetilde{RS}[a,b] \subseteq CBV_2[a,b]$, where $CBV_2[a,b]$ denotes the set of all real-valued functions defined on $[a,b]$ that are continuous and of bounded variation of order 2. Taking into consideration that $CBV_2[a,b]$ is a set of the first category in $C[a,b]$ (see, for instance, [4, Corollary 2.8]), it follows that $\widetilde{RS}[a,b]$ is also of the first category in $C[a,b]$. Apparently this proof avoids the condensation of singularities, but in reality the property of $CBV_2[a,b]$ to be of the first category in $C[a,b]$ is a consequence of a principle of condensation of the singularities of a family of nonnegative functions as shown in [4].

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Babeş-Bolyai University,
Faculty of Mathematics and Computer Science,
Str. Kogălniceanu Nr. 1, RO-400084 Cluj-Napoca, Romania
E-mail address: `breckner@math.ubbcluj.ro`

Babeş-Bolyai University,
Faculty of Mathematics and Computer Science,
Str. Kogălniceanu Nr. 1, RO-400084 Cluj-Napoca, Romania
E-mail address: `ttrif@math.ubbcluj.ro`