SOME INTEGRAL OPERATORS DEFINED ON p-VALENT FUNCTION BY USING HYPERGEOMETRIC FUNCTIONS

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Abstract. In the present paper we introduce some integral operators and verify the effect of these operators on p-valent functions and find radii of starlikeness and convexity for these operators, finally we introduce the concept of neighborhood.

1. Introduction and Definitions

Let $\mathcal{A}$ denote the family of functions analytic in unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ with positive coefficient and let $\mathcal{A}_p$ be subclass of a consisting functions $f(z)$ of the form

$$f(z) = mz^p + \sum_{n=p-1}^{2p-1} t_{n-p+1} z^{n-p+1} - {}_2F_1(a,b;c;z), \ |z| < 1 \tag{1.1}$$

$$\text{where } {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)n!} z^n$$

$$(a,n) = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1,n-1), \ c > b > 0, c > a+b, m > 0$$

$$\text{and } t_{n-p+1} = \frac{(a,n-p+1)(b,n-p+1)}{(c,n-p+1)(n-p+1)!}.$$

These functions are analytic in the punctured unit disk. For more details on hypergeometric functions ${}_2F_1(a,b;c;z)$ see [4] and [7].

Let $f \in \mathcal{A}$, then we denote by UCV^p the class of uniformly convex p-valent function in Δ and $\alpha - ST$ the class of α - starlike functions also denote by $\alpha - UCV^p$

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the class of α-uniformly convex p-valent function in Δ which are introduced and investigated by Kanas, Wiśniwoska [6] and Silverman [10].

Definition 1. Let $f \in \mathcal{A}_P$ and $0 \leq \alpha < \infty$. Then $f \in \alpha - UCV^p$ if and only if

$$Re\left\{p + \frac{zf''}{f'}\right\} > \alpha\left|\frac{zf''}{f'}\right| \quad z \in \Delta.$$

Definition 2. Let $f \in \mathcal{A}_p$. The class α - uniformly starlike functions $\alpha - UST^p$ is defined as

$$\alpha - UST^p = \left\{f \in \mathcal{A} : Re\left(\frac{zf'}{f}\right) > \alpha\left|\frac{zf'}{f} - p\right|,\ \alpha \geq 0,\ z \in \Delta\right\}$$

Definition 3. (cf. [7]; see also [11] and [12]). Let the function f be of the form $f(z) = z^p - \sum_{n=2}^{\infty} a_n z^n$ and be analytic in Δ. The fractional derivative of f of order δ is defined by

$$D_z^\delta f(z) = \frac{1}{\Gamma(1-\delta)}\frac{d}{dz}\int_0^z \frac{f(\xi)}{(z-\xi)^\delta}d\xi \ (0 \leq \delta < 1) \tag{1.2}$$

where the multiplicity of $(z-\xi)^\delta$ is removed by requiring $\log(z-\xi)$ to be real when $z - \xi > 0$ and so we have

$$D_z^\delta f(z) = \frac{m}{\Gamma(2-\delta)}z^{p-\delta} - \sum_{n=2}^{\infty}\frac{\Gamma(n+p)}{\Gamma(n+p-\delta)}\ a_n z^{n-\delta}. \tag{1.3}$$

Making use of (1.2) and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [11] introduced the operator

$$\Omega_z^\delta f(z) := \Gamma(2-\delta)z^\delta D_z^\delta f(z),\ 0 \leq \delta < 1 \tag{1.4}$$

and for $\delta = 0$ we have $\Omega_z^0 f(z) = f(z)$.

Definition 4. Let $f(z) \in \mathcal{A}_p$ is said to be a member of the $\alpha - UCV_\delta^p(\eta, \phi)$ if $f(z)$ satisfies the inequality

$$Re\left(\frac{z(\Omega_z^\delta f(z))' + \eta z^2(\Omega_z^\delta f(z))''}{(1-\eta)(\Omega_z^\delta f) + \eta z(\Omega_z^\delta f(z))'}\right)$$
$$\geq \alpha\left|\frac{z(\Omega_z^\delta f(z))' + z^2(\Omega_z^\delta f)''}{(1-\eta)\Omega_z^\delta f(z) + \eta z(\Omega_z^\delta f(z))'} - 1\right| + \tan\phi \tag{1.5}$$

where $0 \leq \eta \leq 1, 0 \leq \tan\phi < p, p \in \mathbb{N}, \alpha \geq 0$ and $0 \leq \delta < 1$.

We note that by specializing the parameters $\alpha, \phi, \eta, \delta$ we obtain the following subclasses studied by various authors (by putting $\tan\phi = \beta$).

(I) If $\alpha = 0, \delta = 0$ and $p = 1 \Rightarrow \alpha - UCV(\eta, 0) \equiv p_1(1, \lambda, \beta)$ was studied by Altintas [1].
(II) If $\eta = 1, \delta = 0, \alpha = 0, p = 1 \Rightarrow \alpha - UCV(1, \phi) \equiv C(\beta)$ was studied by Silverman [10].
(III) If $\eta = 0, \delta =, 0, p = 1 \Rightarrow \alpha - UCV(0, \phi) \equiv UCT(k, \beta)$ was studied by R. Bharati, R. Parvatham and A. Swaminathan [5].
(IV) If $p = 1, \eta = 0$ and $\beta = 0$ and $\delta = 0$, that is $k - ST$ introduced by Kanas and Wiśniowsak [6].

2. **Main Results**

In the first theorem we will obtain coefficient bounds, before it we need the following lemmas.

Lemma 1. *Let $w = u + iv$ then*

$$Re(w) \geq \alpha \Leftrightarrow |w - (1 + \alpha)| \leq |w + (1 - \alpha)|.$$

Lemma 2. *Let $w = u + iv$ and α, β be real numbers. Then*

$$Re(w) > \alpha|w - 1| + \beta \Leftrightarrow Re\{w(1 + \alpha e^{i\theta}) - \alpha e^{i\theta}\} > \beta.$$

Theorem 1. *The function $f(z)$ defined by (1.1) is in the class $\alpha - UCV_\delta^p(\eta, \phi)$ if and only if*

$$\sum_{n=p+1}^{\infty} \gamma^p(n, \delta)[(1 - \eta + n\eta)(n(1 + \alpha) - (\alpha + \tan \phi))]k_n \leq m(1 + \eta p - \eta)(p - \tan \phi + \alpha(p - 1)) \quad (2.1)$$

where $\gamma^p(n, \delta) = \frac{\Gamma(2-\delta)\Gamma(n+p)}{\Gamma(n+p-\delta)}$ and $0 \leq \tan \phi < p, \alpha \geq 0, 0 \leq \eta \leq 1, p \in \mathbb{N}$ and $0 \leq \delta < 1$.

Proof. The function $f(z)$ in $\mathcal{A}_p$ can be expressed in the form

$$f(z) = mz^p - \sum_{n=p+1}^{\infty} k_n z^n, \; p \in \mathbb{N} \quad (2.2)$$

such that $k_n = \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+n)\Gamma(n+1)}$ $n \geq p+1$. Also

$$\begin{aligned} \Omega_z^\delta f(z) &= \Gamma(2-\delta) z^\delta D_z^\delta f(z) = mz^p - \sum_{n=p+1}^{\infty} \frac{\Gamma(n+p)}{\Gamma(n+p-\delta)} k_n z^n \\ &= mz^p - \sum_{n=p+1}^{\infty} \gamma^p(n,\delta) k_n z^n \end{aligned} \tag{2.3}$$

Now, let $f(z) \in \alpha - UCV_\delta^p(\eta, \phi)$ that is

$$\begin{aligned} &Re\left\{ \frac{z(\Omega_z^\delta f(z))' + \eta z^2 (\Omega_z^\delta f(z))''}{(1-\eta)(\Omega_z^\delta f(z)) + \eta z (\Omega_z^\delta f(z))'} \right\} \\ &\geq \alpha \left| \frac{z(\Omega_z^\delta f(z))' + \eta z^2 (\Omega_z^\delta f(z))''}{(1-\eta)(\Omega_z^\delta f(z)) + \eta z (\Omega_z^\delta f(z))'} - 1 \right| + \tan\phi \end{aligned}$$

Using Lemma 2 we have

$$\begin{aligned} &Re\left\{ \frac{z(\Omega_z^\delta f(z))' + \eta z^2 (\Omega_z^\delta f(z))''}{(1-\eta)(\Omega_z^\delta f(z)) + \eta z (\Omega_z^\delta f(z))'} (1+\alpha e^{i\theta}) - \alpha e^{i\theta} \right\} \\ &\geq \tan\phi, (0 \leq \tan\phi < p) \end{aligned}$$

or equivalently

$$\begin{aligned} &Re\{([z(\Omega_z^\delta f(z))' + \eta z^2 (\Omega_z^2 f(z))''](1+\alpha e^{i\theta}) - (\alpha e^{i\theta} + \tan\phi) \\ &[(1-\eta)\Omega_z^\delta f(z) + \eta z (\Omega_z^\delta f(z))'])/((1-\eta)\Omega_z^\delta f(z) + \eta z (\Omega_z^\delta f(z))')\} \geq 0 \end{aligned}$$

Then, we can write

$$\begin{aligned} &Re\{[m(1+\eta p - \eta)(p - \tan\phi) - \sum_{n=p+1}^{\infty} \gamma^p(n,\delta)((n + n\eta(n-1) \\ &- \tan\phi(1-\eta+n\eta))k_n z^{n-p} - \alpha e^{i\theta}(m(1-\eta+p\eta)(p-1)) \\ &-\alpha e^{i\theta} \sum_{n=p+1}^{\infty} \gamma^p(n,\delta)(n + n\eta(n-1) - (1-\eta+n\eta))k_n z^{n-p}] \\ &/[m(1-\eta+p\eta) - \sum_{n=p+1}^{\infty} \gamma^p(n,\delta)(1-\eta+n\eta)k_n z^{n-p}]\} > 0 \end{aligned}$$

The above inequality must hold for all z in Δ. Letting $z \to 1^-$ yields

$$Re\{[m(1+\eta p-\eta)(p-\tan\phi)-\sum_{p+1}^{\infty}\gamma^p(n,\delta)(1-\eta+\eta n)(n-\tan\phi)k_n$$
$$-\alpha e^{i\theta}(m(1-\eta+p\eta)(p-1))-\alpha e^{i\theta}\sum_{n=p+1}^{\infty}\gamma^p(n,\delta)(1-\eta+n\eta)(n-1)]$$
$$/[m(1-\eta+p\eta)-\sum_{n=p+1}^{\infty}\gamma^p(n,\delta)(1-\eta+n\eta)k_n]\}>0$$

and so by the mean value theorem we have

$$Re\{m(1+\eta p-\eta)(p-\tan\phi)-\sum_{p+1}^{\infty}\gamma^p(n,\delta)(1-\eta+\eta n)(n-\tan\phi)k_n$$
$$+\alpha e^{i\theta}[m(1-\eta+p\eta)(p-1)]-\alpha e^{i\theta}\sum_{p+1}^{\infty}\gamma^p(n,\delta)(1-\eta+n\eta)(n-1)k_n\}>0$$

Therefore we obtain

$$\sum_{n=p+1}^{\infty}\gamma^p(n,\delta)(1-\eta+n\eta)(n-\tan\phi+\alpha(n-1))k_n<m(1+\eta p-\eta)(p-\tan\phi+\alpha(p-1))$$

Conversely, let (2.1) hold true. We will show that (1.5) gets satisfied and then $f(z)\in\alpha-UCV_\delta^p(\eta,\phi)$. Using the Lemma 1 it is enough to show that

$$\begin{aligned}
E &= \left|\frac{z(\Omega_z^\delta f(z))'+\eta z^2(\delta_z^\delta f(z))''}{(1-\eta)(\Omega_z^\delta f(z))+\eta z(\Omega_z^\delta f(z)))'}\right.\\
&- \left.\left(1+\alpha\left|\frac{z(\Omega_z^\delta f(z))'+\eta z^2(\Omega_z^\delta f(z))''}{(1-\eta)(\Omega_z^\delta f(z))+\eta z(\Omega_z^\delta f(z))'}-1\right|+\tan\phi\right)\right|\\
&< \left|\frac{z(\Omega_z^\delta f(z))'+\eta z^2(\Omega_z^\delta f(z))''}{(1-\eta)(\Omega_z^\delta f(z))+\eta z(\Omega_z^\delta f(z))'}\right.\\
&+ \left(1-\alpha\left|\frac{z(\Omega_z^\delta f(z))'+\eta z^2(\Omega_z^\delta f(z))''}{(1-\eta)(\Omega_z^\delta f(z))+\eta z(\Omega_z^\delta f(z))'}-1\right|-\tan\phi\right)=F
\end{aligned}$$

We must show $E < F$ or $F - E > 0$. For letting $e^{i\theta} = \frac{B}{|B|}$ where $B = (1-\eta)(\Omega_z^\delta f(z)) + \eta z(\Omega_z^\delta f(z))'$, we may write

$$\begin{aligned} E &= \frac{1}{|B|}|z(\Omega_z^\delta f(z))' + \eta z^2(\Omega_z^\delta f(z))'' - (1+\tan\phi)[(1-\eta)(\Omega_z^\delta f(z)) \\ &\quad +\eta z(\Omega_z^\delta f(z))']| - \alpha e^{i\theta}|(1-\eta)z(\Omega_z^\delta f(z))' + \eta z^2(\Omega_z^\delta f(z))'' \\ &\quad -(1-\eta)(\Omega_z^\delta f(z))|| \\ &< \frac{|z|^p}{|B|}(m(1+\eta p-\eta)(p-1-\tan\phi+\alpha(p-1)) \\ &\quad + \sum_{n=p+1}^{\infty}\gamma^p(n,\delta)(1+\eta n-\eta)[(n-1-\tan\phi)+\alpha(n+1)]k_n \end{aligned}$$

Also, we have

$$\begin{aligned} F &= \frac{1}{|B|}|z(\Omega_z^\delta f(z))' + \eta z^2(\Omega_z^\delta f(z))'' + (1-\tan\phi)((1-\eta)(\Omega_z^\delta f(z)) \\ &\quad -\eta z(\Omega_z^\delta f(z))') - \alpha e^{i\theta}|(1-\eta)z(\Omega_z^\delta f(z))' + \eta z^2(\Omega_z^\delta f(z))'' - (1-\eta)(\Omega_z^\delta f(z))|| \\ &> \frac{|z|^p}{|B|}(m(1+\eta p-\eta)(p+1-\tan\phi+\alpha(p-1)) \\ &\quad - \sum_{n=p+1}^{\infty}\gamma^p(n,\delta)(1-\eta+\eta n)(n+1-\tan\phi+\alpha(n+1))k_n. \end{aligned}$$

It is easy to verify that $F - E > 0$, if (2.1) holds and so the proof is complete. □

Corollary 1. *If* $f(z) \in \alpha - UCV_\delta^p(\eta,\phi)$, *then*

$$k_n \le \frac{m(\alpha(p-1)+p-\tan\phi)(1-\eta+\eta p)}{\gamma^p(n,\delta)[(1+n\eta-\eta)(n(1+\alpha)-(\alpha+\tan\phi)]},\ n \ge p+1$$

where $0 \le \tan\phi < p, \alpha \ge 0, 0 \le \eta \le 1, p \in \mathbb{N}$ *and* $\gamma^p(n,\delta) = \frac{\Gamma(2-\delta)\Gamma(n+p)}{\Gamma(n+p-\delta)}$.

Corollary 2. $f(z) \in \alpha - UCV_0^1(\eta,\phi)$ *if and only if*

$$\sum_{n=p+1}^{\infty}(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\tan\phi))k_n \le m(1-\tan\phi),\ 0 \le \tan\phi < 1$$

that is a class introduced by E. Aqlan and S. R. Kulkarni [3].

Corollary 3. $f(z) \in 0 - UCV_0^1(\eta,\phi)$ *if and only if*

$$\sum_{n=p+1}^{\infty}(1-\eta+n\eta)(n-\tan\phi)k_n \le m(1-\tan\phi),\ 0 \le \tan\phi < 1$$

that is a class studied by Altintas [1].

Corollary 4. *$f(z) \in \alpha - UCV_0^1(0, \phi)$ if and only if*

$$\sum_{n=p+1}^{\infty} (n(1+\alpha) - (\alpha + \tan\phi))k_n \leq m(1 - \tan\phi),\ 0 \leq \tan\phi < 1.$$

That is class studied by R. Bharati, R. Parvatham and A. Swaminathan [5].

3. Special Functions and Integral Operators on $\alpha - UCV_\delta^p(\eta, \phi)$

Definition 5. Let c be a real number such that $c > -p$. For $f \in \alpha - UCV_\delta^p(\eta, \phi)$, we define F_c by

$$F_c(z) = \frac{c+p}{z^c}\int_0^z s^{c-1} f(s)ds \tag{3.1}$$

Theorem 2. *$F_c(z)$ defined by (3.1) belongs to $\alpha - UCV_\delta^p(\eta, \phi)$.*

Proof. Let $f(z) = mz^p - \sum\limits_{n=p+1}^{\infty} k_n z^n \in \alpha - UCV_\delta^p(\eta, \phi)$ then

$$F_c(z) = \frac{c+p}{z^c}\int_0^z \left(ms^{c-1+p} - \sum_{n=p+1}^{\infty} k_n s^{n+c-1} \right) ds = mz^p - \sum_{n=p+1}^{\infty} \frac{c+p}{n+c} k_n z^n.$$

Since $f(z) \in \alpha - UCV_\delta^p(\eta, \phi)$ and $\frac{c+p}{c+n} < 1, n \geq p+1$ and by Theorem 1, $F_c(z) \in \alpha - UCV_\delta^p(\eta, \phi)$ if

$$\begin{aligned} &\sum_{n=p+1}^{\infty} \gamma^p(n,\delta)[(1-\eta+n\eta)(n(1+\alpha) - (\alpha+\tan\phi))]\frac{c+p}{c+n}k_n \\ &\leq \sum_{n=p+1}^{\infty} \gamma^p(n,\delta)[(1-\eta+n\eta)(n(1+\alpha) - (\alpha+\tan\phi))]k_n \\ &\leq m(\alpha(p-1) + p - \tan\phi)(1-\eta+\eta p) \end{aligned} \tag{3.2}$$

So $F_c(z) \in \alpha - UCV_\delta^p(\eta, \phi)$. ∎

Theorem 3. *The function $F_c(z)$ defined in 3.1 is starlike of order $\lambda(0 \leq \lambda < p)$ in $|z| < r_1(\eta, \phi, \alpha, \delta, n, p, c, \lambda)$ where*

$$\begin{aligned} r_1(\eta, \phi, \alpha, \delta, n, p, c, \lambda) &= \inf_{n \geq p+1} \left\{ \frac{[(1-\eta+, n\eta)(n(1+\alpha) - (\alpha+\tan\phi)]}{m(\alpha(p-1)+p-\tan\phi)(1-\eta+\eta p)} \right. \\ &\left. \left(\frac{c+n}{c+p}\right)\left(\frac{m(p-\lambda)}{2p-n-\lambda}\right)\gamma^p(n,\delta)\right\}^{\frac{1}{n-p}} \end{aligned}$$

The bound for $|z|$ is sharp for each n with extremal function being of the form

$$F_{c,n}(z) = mz^p - \frac{m(\alpha(p-1)+p-\tan\phi)(1-\eta+\eta p)}{\gamma^p(n,\delta)[(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\tan\phi)]}\frac{c+n}{c+p}\ z^n, n \geq p+1.$$

Proof. We must show that

$$\left|\frac{zF_c'(z)}{F_c(z)} - p\right| < p - \lambda \tag{3.3}$$

But we have

$$\left|\frac{zF_c'(z)}{F_c(z)} - p\right| \leq \frac{\sum\limits_{n=p+1}^{\infty} \frac{c+p}{c+n} k_n (p-n)|z|^{n-p}}{m - \sum\limits_{n=p+1}^{\infty} \frac{c+p}{c+n} k_n |z|^{n-p}}.$$

Therefore (3.3) holds if

$$\sum_{n=p+1}^{\infty} \left(\frac{c+p}{c+n}\right)\left(\frac{2p-n-\lambda}{m(p-\lambda)}\right) k_n |z|^{n-p} < 1.$$

Now in view of (3.2) the last inequality holds if

$$|z|^{n-p} < \frac{(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\tan\phi)]}{m(\alpha(p-1)+p-\tan\phi)(1-\eta+\eta p)}\left(\frac{m(p-\lambda)}{2p-n-\lambda}\right)\left(\frac{c+n}{c+p}\right)\gamma^p(n,\delta).$$

This gives the required result.

□

Corollary 5. *The function $F_c(z)$ defined in 3.1 is convex of order $\lambda (0 \leq \lambda < p)$ in $|z| < r_2 = r_2(\eta, \phi, \alpha, \delta, n, p, c, \lambda)$ where*

$$\begin{aligned} r_2(\eta,\phi,\alpha,\delta,n,p,c,\lambda) &= \inf_{n\geq p+1}\left\{\frac{(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\tan\phi))}{m(\alpha(p-1)+p-\tan\phi)(1-\eta+\eta p)}\right. \\ &\qquad \left.\left(\frac{m(p-\lambda)}{2p-n-\lambda}\right)\left(\frac{c+n}{c+p}\right)\gamma^p(n,\delta)\right\}^{\frac{1}{n-p}} \end{aligned}$$

Proof. We must show that $\left|\frac{zF_c''(z)}{F_C'(z)}\right| < p - \lambda$ for $|z| < r_2$ and $c > -p$.
But we have

$$\left|\frac{zF_c''(z)}{F_c'(z)}\right| \leq \frac{mp(p-1) + \sum\limits_{n=p+1}^{\infty} \frac{c+p}{c+n} k_n n(n-1)|z|^{n-p}}{mp - \sum\limits_{n=p+1}^{\infty} \frac{c+p}{c+n} k_n n |z|^{n-p}}.$$

Therefore $\left|\frac{zF_c''(z)}{F_C'(z)}\right| < p-\lambda$ holds if

$$\sum_{n=p+1}^{\infty} \frac{n(n-1+p-\lambda)}{mp(\lambda-1)} \left(\frac{c+p}{c+n}\right) k_n |z|^{n-p} < 1.$$

The last inequality holds if

$$|z|^{n-p} < \frac{(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\tan\phi))}{m(\alpha(p-1)+p-\tan\phi)(1-\eta+\eta p)} \left(\frac{mp(\lambda-1)}{n(n-1+p-\lambda}\right)$$
$$\left(\frac{c+n}{c+p}\right)\gamma^p(n,\delta).$$

This gives the required result. □

Definition 6. Let c be a real number such that $c > -p$ and let $f \in \alpha - UCV_\delta^p(\eta,\phi)$, Komato operator in [8] is defined by

$$G(z) = \int_0^1 \frac{(c+1)^\xi}{\Gamma(\xi)} t^c (\log\frac{1}{t})^{\xi-1} \frac{f(tz)}{t^p} dt, \ c > -1, \xi \geq 0. \tag{3.4}$$

Theorem 4. *$G(z)$ defined in 3.4 belongs to $\alpha - UCV_\delta^p(\eta,\phi)$.*

Proof. Since $\int_0^1 t^c(-\log t)^{\xi-1}dt = \frac{\Gamma(\xi)}{(c+1)^\xi}$ and $\int_0^1 t^{n+c-p}(-\log t)^{\xi-1}dt$
$= \frac{\Gamma(\xi)}{(c+n-p+1)^\xi}$
$n \geq p+1$. Therefore we obtain

$$\begin{aligned} G(z) &= \frac{(c+1)^\xi}{\Gamma(\xi)} \left[\int_0^1 t^c z^p \log(\frac{1}{t})^{\xi-1} dt - \sum_{n=p+1}^{\infty} \int_0^1 \log(\frac{1}{t})^{\xi-1} t^{n-p+c} k_n z^n dt\right] \\ &= mz^p - \sum_{n=p+1}^{\infty} \left(\frac{c+1}{c+n-p+1}\right)^\xi k_n z^n. \end{aligned} \tag{3.5}$$

Therefore and with use of Theorem 1 and $\frac{c+1}{c+1+n-p} < 1$ for $n \geq p+1$ we can write

$$\begin{aligned} &\sum_{n=p+1}^{\infty} \gamma^p(n,\delta)[(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\tan\phi))] \left(\frac{c+1}{c+n-p+1}\right)^\xi k_n \\ &\leq m(\alpha(p-1)+p-\sin\phi)(1-\eta+\eta p) \end{aligned} \tag{3.6}$$

So $G(z) \in \alpha - UCV_\delta^p(\eta,\phi)$. □

Theorem 5. *The function $G(z)$ defined in 3.4 is starlike of order $\lambda(0 \leq \lambda < 1)$ in $|z| < r_1 = r_1(\eta,\phi,\alpha,\delta,n,p,c,\xi,\lambda)$ where*

$$r_1(\eta,\phi,\alpha,\delta,n,p,c,\xi,\lambda) = \inf_{n\geq p+1}\left\{\frac{(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\tan\phi)}{m(\alpha(p-1)+p-\sin\phi)(1-\eta+\eta p)}\right.$$
$$\left.\left(\frac{m(p-\lambda)}{2p-n-\lambda}\right)\left(\frac{c+n-p+1}{c+1}\right)^{\xi}\gamma^p(n,\delta)\right\}^{\frac{1}{n-p}}$$

Proof. We must show that $\left|\frac{zG'(t)}{G(t)}-p\right|<p-\lambda$ or we must show

$$\left|\frac{zG'(t)}{G(t)}-p\right| \leq \frac{\sum_{n=p+1}^{\infty}\left(\frac{c+1}{c+n-p+1}\right)^{\xi}(p-n)k_n|z|^{n-p}}{m-\sum_{n=p+1}^{\infty}\left(\frac{c+1}{c+n-p+1}\right)^{\xi}k_n|z|^{n-p}} < p-\lambda.$$

The last inequality holds if

$$\sum_{n=p+1}^{\infty}\left(\frac{c+1}{c+n-p+1}\right)^{\xi}\frac{(2p-(n+\lambda))}{m(p-\lambda)}k_n|z|^{n-p}<1.$$

Now in view of (3.6), (3.5) the last inequality holds if

$$|z|^{n-p} \leq \frac{\gamma^p(n,\delta)(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\tan\phi))}{m(\alpha(p-1)+p-\tan\phi)(1-\eta+\eta p)}$$
$$\left(\frac{m(p-\lambda)}{(2p-(n+\lambda))}\right)\left(\frac{c+n-p+1}{c+1}\right)^{\xi}$$

This gives the required result. □

Corollary 6. *The function $G(z)$ defined in (3.4) is convex of order $\lambda(0\leq\lambda<p)$ in $|z|<r_2=r_2(\eta,\phi,\alpha,\delta,n,p,c,\xi,\lambda)$ where*

$$r_2(\eta,\phi,\alpha,\delta,n,p,c,\xi,\lambda) = \inf_{n\geq p+1}\left\{\frac{(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\tan\phi)}{m(\alpha(p-1)+p-\sin\phi)(1-\eta+\eta p)}\right.$$
$$\left.\left(\frac{c+n-p+1}{c+1}\right)^{\xi}\left(\frac{p(1-\lambda)}{n(p+n-\lambda-1)}\right)\gamma^p(n,\delta)\right\}^{\frac{1}{n-p}}$$

Proof. We must show that $\left|\frac{zG''(z)}{G'(z)}\right|<p-\lambda,\ |z|<r_2$ or

$$\left|\frac{zG''(z)}{G'(z)}\right| = \left|\frac{mp(p-1)z^{p-1}-\sum_{n=p+1}^{\infty}\left(\frac{c+1}{c+n-p+1}\right)^{\xi}k_n n(n-1)z^{n-1}}{mpz^{p-1}-\sum_{n=p+1}^{\infty}\left(\frac{c+1}{c+n-p+1}\right)_k^{\xi}nz^{n-1}}\right| < p-\lambda$$

Therefore

$$\sum_{n=p+1}^{\infty}\left(\frac{c+1}{c+n-p+1}\right)^{\xi}\left(\frac{n(p-\lambda+n-1)}{mp(1-\lambda)}\right)k_n|z|^{n-p}<1. \quad (3.7)$$

Therefore (3.7) holds if

$$|z|^{n-p}<\frac{\gamma^p(n,\delta)(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\tan\phi))}{m(\alpha(p-1)+p-\tan\phi)(1-\eta+\eta p)}$$
$$\left(\frac{c+n-p+1}{c+1}\right)^{\xi}\left(\frac{mp(1-\lambda)}{n(p+n-\lambda-1)}\right)$$

□

Definition 7. Let $f \in \alpha - UCV_\delta^p(\eta,\phi)$. Function $H_\mu(z)$ defined by

$$H_\mu(z)=(1-\mu)mz^p+\mu p\int_0^z\frac{f(t)}{t}dt \; 0\le\mu<1, z\in\Delta \quad (3.8)$$

Theorem 6. *The function $H_\mu(z)$ defined in (3.8) belongs to $\alpha - UCV_\delta^p(\eta,\phi)$ if $0\le\mu\le1$.*

Proof. Let $f(z)\in\alpha-UCV_\delta^p(\eta,\phi)$ and is of the form (1.1) so

$$H_\mu(z)=(1-\mu)mz^p+\mu p(\int_0^z(mt^{p-1}-\sum_{n=p+1}^{\infty}k_nt^{n-1})dt)=mz^p-\sum_{n=p+1}^{\infty}(\frac{\mu p}{n}k_n)z^n \quad (3.9)$$

By Theorem 1 we must show

$$\begin{aligned}&\sum_{n=p+1}^{\infty}\gamma^p(n,\delta)[(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\tan\phi))]\frac{\mu p}{n}k_n\\ &\le\sum_{n=p+1}^{\infty}\gamma^p(n,\delta)[(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\tan\phi))]\frac{\mu p}{p+1}k_n\\ &\le\sum_{n=p+1}^{\infty}\gamma^p(n,\delta)[(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\tan\phi))]k_n\\ &\le m(\alpha(p-1)+p-\tan\phi)(1-\eta+\eta p)\end{aligned}$$

So $H_\mu(z)\in\alpha-UCV_\delta^p(\eta,\phi)$. □

Theorem 7. *By the similar method which we applied for Theorem 5 and Corollary 6, we obtain the radii of starlikeness and convexity of order* $\lambda (0 \leq \lambda \leq p)$ *for* $H_\mu(z)$ *respectively as following*

$$\begin{aligned} r_1(\eta,\phi,\alpha,\delta,n,p,\mu,\lambda) &= \inf_{n\geq p+1}\left\{\frac{(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\tan\phi))\gamma^p(n,\delta)}{m(\alpha(p-1)+p-\tan\phi)(1-\eta+\eta p)}\right. \\ &\qquad \left.\left(\frac{m(p-\lambda)}{2p-n-\lambda}\right)\left(\frac{n}{\mu p}\right)\right\}^{\frac{1}{n-p}} \\ r_2(\eta,\phi,\alpha,\delta,n,p,\mu,\lambda) &= \inf_{n\geq p+1}\left\{\frac{(1-\eta+n\eta)(n(1+\alpha)-(\alpha+\tan\phi))\gamma^p(n,\delta)}{m(\alpha(p-1)+p-\sin\phi)(1-\eta+\eta p)}\right. \\ &\qquad \left.\left(\frac{mp(1-\lambda)}{\mu(p+n-\lambda-1)}\right)\right\}^{\frac{1}{n-p}} \end{aligned}$$

where $0 \leq \mu \leq 1$.

4. (n, λ) - Neighborhood

Definition 8. ([9], [2]) : Let $\lambda \geq 0$ and $f(z) \in \mathcal{A}_p$ and f defined by (1.1). We define the
(n, λ) - neighborhood of a function $f(z)$ by

$$N_{n,\lambda}(f) = \left\{ g \in \mathcal{A}_p : g(z) = mz^p - \sum_{n=p+1}^{\infty} k'_n z^n \text{ and } \sum_{n=p+1}^{\infty} n|k_n - k'_n| \leq \lambda \right\} \tag{4.1}$$

For the identity function $e(z) = z$, we have

$$N_{n,\lambda}(e) = \left\{ g \in \mathcal{A}_p : g(z) = mz^p - \sum_{n=p+1}^{\infty} k'_n z^n \text{ and } \sum_{n=p+1}^{\infty} n|k'_n| \leq \lambda \right\} \tag{4.2}$$

Theorem 8. *Let*

$$\lambda = \frac{(p+1)m(\alpha(p-1)+p-\tan\phi)(1-\eta+\eta p)}{\gamma^p(p+1,\delta)(1+p\eta)(p(1+\alpha)+1-\tan\phi)}.$$

where $\gamma^p(p+1,\delta) = \dfrac{\Gamma(2-\delta)\Gamma(2p+1)}{\Gamma(2p-\delta)}$. *Then*

$$\alpha - UCV_\delta^p(\eta,\phi) \subset N_{n,\lambda}(e).$$

Proof. For $f \in \alpha - UCV_{\delta}^{p}(\eta, \phi)$ we have from (2.1)

$$(1+p\eta)(p(1+\alpha)+1-\tan\phi)]\gamma^{p}(p+1,\delta)\sum_{n=p+1}^{\infty}k_{n}$$
$$\leq \sum_{n=p+1}^{\infty}[(1-\eta+n\eta)(n(1+\alpha)-\alpha-\tan\phi)]\gamma^{p}(n,\delta)k_{n}$$
$$\leq m(\alpha(p-1)+p-\tan\phi)(1-\eta+\eta p)$$

Therefore

$$\sum_{n=p+1}^{\infty}k_{n} \leq \frac{m(\alpha(p-1)+p-\tan\phi)(1-\eta+\eta p)}{\gamma^{p}(p+1,\delta)(1+p\eta)(p(1+\alpha)+1-\tan\phi)}\ , \tag{4.3}$$

and on the other hand we have for $|z| < r$

$$|f'(z)| \leq mp|z|^{p-1} + |z|^{p}\sum_{n=p+1}^{\infty}nk_{n}$$
$$\leq mpr^{p-1} + r^{p}\sum_{n=p+1}^{\infty}nk_{n}$$

$$\text{(from (4.3))}\ \ \leq pr^{p-1} + r^{p}\frac{(p+1)m(\alpha(p-1)+p-\tan\phi)(1-\eta+\eta p)}{\gamma^{p}(p+1,\delta)(1+p\eta)(p(1+\alpha)+1-\tan\phi)}.$$

From above inequalities we conclude

$$\sum_{n=p+1}^{\infty}nk_{n} \leq \frac{(p+1)m(\alpha(p-1)+p-\tan\phi)(1-\eta+\eta p)}{\gamma^{p}(p+1,\delta)(1+p\eta)(p(1+\alpha)+1-\tan\phi)} = \lambda.$$

□

Definition 9. The function $f(z)$ defined by (1.1) is said to be a member of the class $\alpha - UCV_{\delta}^{p,\xi}(\eta, \phi)$ if there exists a function $g \in \alpha - UCV_{\delta}^{p}(\eta, \phi)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| \leq p - \xi,\ z \in \Delta,\ 0 \leq \xi < p.$$

Theorem 9. *If $g \in \alpha - UCV_{\delta}^{p}(\eta, \phi)$ and*

$$\xi = p - \frac{\lambda}{p+1}\mu(\eta, \phi, \alpha, \delta, p) \tag{4.4}$$

such that

$$\mu(\eta,\phi,\alpha,\delta,p) = [\gamma^p(p+1,\delta)(1+p\eta)(p(\alpha+1)+1-\tan\phi)] /[m\gamma^p(p+1,\delta)(1+p\eta)(p(\alpha+1)+1-\tan\phi) -m(\alpha(p-1)+p-\tan\phi)(1-\eta+\eta p)]$$

then $N_{n,\lambda}(g) \subset \alpha - UCV_\delta^{p,\xi}(\eta,\phi)$.

Proof. Let $f \in N_{n,\lambda}(g)$, then we have from (4.1) that $\sum_{n=p+1}^{\infty} n|k_n - k'_n| \leq \lambda$ which readily implies the coefficient inequality

$$\sum_{n=p+1}^{\infty} |k_n - k'_n| \leq \frac{\lambda}{p+1}.$$

Also since $g \in \alpha - UCV_\delta^p(\eta,\phi)$ we have from (2.1)

$$\sum_{n=p+1}^{\infty} k'_n \leq \frac{m(\alpha(p-1)+p-\tan\phi)(1-\eta+\eta p)}{\gamma^p(p+1,\delta)(1+p\eta)(p(\alpha+1)+1-\tan\phi)}$$

so that

$$\left|\frac{f(z)}{g(z)} - 1\right| < \frac{\sum_{n=p+1}^{\infty}|k_n - k'_n|}{m - \sum_{n=p+1}^{\infty} k'_n} \leq \left(\frac{\lambda}{p+1}\right)$$

$$(\gamma^p(p+1,\delta)(1+p\eta)(p(\alpha+1)+1-\tan\phi) /m\gamma^p(p+1,\delta)(1+p\eta)(p(\alpha+1)+1-\tan\phi) -m(\alpha(p-1)+p-\tan\phi)(1-\eta+\eta p))$$

$$= \left(\frac{\lambda}{p+1}\right)\mu(\eta,\phi,\alpha,\delta,p) = p - \xi$$

Then $\left|\frac{f(z)}{g(z)} - 1\right| < p - \xi$. Thus, by definition 9, $f \in \alpha - UCV_\delta^{p,\xi}(\eta,\phi)$ for ξ given by (4.4). □

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