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A NONSMOOTH EXTENSION FOR THE BERNSTEIN-STANCU OPERATORS AND AN APPLICATION

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Abstract. D.D. Stancu defined in [8] a class of approximation operators which are more general than the well-known Bernstein operators. We define here a new type of approximation operators which extend the Bernstein-Stancu operators. These new operators have the advantage that the points where the given function $f : [0,1] \to \mathbf{R}$ is calculated can be independently chosen in each interval of the equidistant division of the interval $[0,1]$. Moreover, all possible such choices of intermediary points cover the whole interval $[0,1]$. Finally, we consider a particular case as an application.

1. **Introduction**

The Bernstein approximations $B_m f$, $m \in \mathbb{N}$ associated to a given continuous function $f : [0,1] \to \mathbb{R}$ is the polynomial

$$(B_m f)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k}{m}\right),$$

where

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k}$$

are called the Bernstein fundamental polynomials of $m-$th degree (see [2]). Bernstein used this approximation to give the first constructive proof of the Weierstrass theorem. One of the many remarkable properties of Bernstein approximation is that

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each derivative of the polynomial function $B_m f$ of any order converges to the corresponding derivative of f ([6]). Other important properties are shape-preservation and variation-diminuation [4]. These many properties can be viewed as compensation for the slow convergence of $B_m f$ to f.

In 1968, D.D. Stancu defined in [8] a linear positive operator depending on two non-negative parameters α and β satisfying the condition $0 \le \alpha \le \beta$. Those operators defined for any non-negative integer m, associate to every function $f \in C([0,1])$ the polynomial $P_m^{(\alpha,\beta)} f$,

$$f \in C([0,1]) \longmapsto P_m^{(\alpha,\beta)} f,$$

in the following way:

$$\left(P_m^{(\alpha,\beta)} f\right)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k+\alpha}{m+\beta}\right).$$

Note that for $\alpha = \beta = 0$ the Bernstein-Stancu operators become the classical Bernstein operators B_m. It is known that the Bernstein-Stancu operators verify the following relations (e.g. [1]):

Lemma 1.1 *For Bernstein-Stancu operators* $P_m^{(\alpha,\beta)}$, $m \in \mathbb{N}$, *the following relations hold true:*

1) $\left(P_m^{(\alpha,\beta)} e_0\right)(x) = 1$

2) $\left(P_m^{(\alpha,\beta)} e_1\right)(x) = x + \dfrac{\alpha - \beta x}{m+\beta}$

3) $\left(P_m^{(\alpha,\beta)} e_2\right)(x) = x^2 + \dfrac{mx(1-x) + (\alpha - \beta x)(2mx + \beta x + \alpha)}{(m+\beta)^2}$,

where

$$e_j(x) = x^j \quad , \quad j = 0, 1, 2$$

are test functions.

For proofs and other comments see [1].

2. **The Results**

In order to define the new class of operators, for all non-negative integers m and $k = 0, 1, \ldots, m$ consider the non-negative reals α_{mk}, β_{mk} so that

$$\alpha_{mk} \le \beta_{mk}.$$

Further, let us denote by A, respective B, the infinite dimensional lower triangular matrices

$$A = \begin{pmatrix} \alpha_{00} & 0 & 0 & 0 & 0 & ... \\ \alpha_{10} & \alpha_{11} & 0 & 0 & 0 & \ldots \\ \alpha_{20} & \alpha_{21} & \alpha_{22} & 0 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & ... & \ldots \end{pmatrix}$$

and

$$B = \begin{pmatrix} \beta_{00} & 0 & 0 & 0 & 0 & ... \\ \beta_{10} & \beta_{11} & 0 & 0 & 0 & \ldots \\ \beta_{20} & \beta_{21} & \beta_{22} & 0 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & ... & \ldots \end{pmatrix}.$$

Under these assumptions, we define an approximation operator denoted by

$$P_m^{(A,B)} : C([0,1]) \to C([0,1]),$$

with the formula

$$\left(P_m^{(A,B)} f\right)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k+\alpha_{mk}}{m+\beta_{mk}}\right) \quad , \quad f \in C([0,1]).$$

Remark that the Bernstein-Stancu operators is a particular type of operators $P_m^{(A,B)}$, in case when the matrices A and B are of the form

$$A = \begin{pmatrix} \alpha & 0 & 0 & 0 & 0 & ... \\ \alpha & \alpha & 0 & 0 & 0 & \ldots \\ \alpha & \alpha & \alpha & 0 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & ... & \ldots \end{pmatrix}$$

and

$$B = \begin{pmatrix} \beta & 0 & 0 & 0 & 0 & ... \\ \beta & \beta & 0 & 0 & 0 & \ldots \\ \beta & \beta & \beta & 0 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & ... & \ldots \end{pmatrix},$$

or equivalent,

$$\alpha_{mk} = \alpha \quad , \quad \beta_{mk} = \beta,$$

for all non-negative integers m and $k \leq m$.

The operators $P_m^{(A,B)}$ have some advantages in comparison with Bernstein-Stancu operators. The approximations of a given continuous function f are calculated using some known values of f. In case of Bernstein operators, the $m-$th approximation is given in function of the values of f at points

$$0 < \frac{1}{m} < \frac{2}{m} < ... < \frac{m-1}{m} < 1,$$

while in case of Stancu operators, the $m-$th approximation is given in terms of the values at points

$$\frac{\alpha}{m+\beta} < \frac{\alpha+1}{m+\beta} < \frac{\alpha+2}{m+\beta} < ... < \frac{\alpha+m}{m+\beta}.$$

This choice of the intermediary points are in some sense strictly, because they depend each other. In the first case, the intermediary points are in arithmetic progression and in the second case, the intermediary points are also under some restrictions. The success of the approximation method appears only if we know the values of the function f at that particular points. The $P_m^{(A,B)}$ operators defined here allow a great liberty for choice of the intermediary points. Indeed, we can independently choose intermediary points in each interval

$$\left[\frac{k}{m}, \frac{k+1}{m}\right] \quad , \quad 0 \leq k \leq m-1.$$

It is sufficient to have $\beta_{mk} \leq 1$, to imply

$$\frac{k+\alpha_{mk}}{m+\beta_{mk}} \leq \frac{k+\beta_{mk}}{m+\beta_{mk}} \leq \frac{k+1}{m+\beta_{mk}} \leq \frac{k+1}{m}.$$

Thus under similar weak assumptions, we can have

$$\frac{k+\alpha_{mk}}{m+\beta_{mk}} \in \left[\frac{k}{m}, \frac{k+1}{m}\right] \quad , \quad 0 \leq k \leq m-1,$$

so the possible values of the intermediary points cover the whole interval $[0,1]$. The operators $P_m^{(A,B)}$ are linear, in the sense that

$$P_m^{(A,B)}(\mu f + \lambda g) = \mu P_m^{(A,B)} f + \lambda P_m^{(A,B)} g,$$

for all real numbers λ, μ and $f, g \in C([0,1])$ and positive defined, *i.e.*

$$P_m^{(A,B)} f \geq 0 \quad , \quad if\ f \geq 0.$$

We will use the following result due to H. Bohman and P.P. Korovkin.

Theorem 2.1 *Let $L_m : C([a,b]) \to C([a,b])$, $m \in \mathbb{N}$ be a sequence of linear, positive operators such that*

$$(L_m e_0)(x) = 1 + u_m(x)$$

$$(L_m e_1)(x) = x + v_m(x)$$

$$(L_m e_2)(x) = x^2 + w_m(x)$$

with

$$\lim_{m\to\infty} u_m(x) = \lim_{m\to\infty} v_m(x) = \lim_{m\to\infty} w_m(x) = 0,$$

uniformly on $[0,1]$. *Then for every continuous function* $f \in C([0,1])$, *we have*

$$\lim_{m\to\infty} (L_m f)(x) = f(x),$$

uniformly on $[0,1]$.

For proofs and other results see [3], [5]. In order to prove that the operators $P_m^{(A,B)}$ are approximation operators, we give the following main result:

Theorem 2.2 *Given the infinite dimensional lower triangular matrices*

$$A = \begin{pmatrix} \alpha_{00} & 0 & 0 & \dots & ... & ... \\ \alpha_{10} & \alpha_{11} & 0 & 0 & ... & \dots \\ \alpha_{20} & \alpha_{21} & \alpha_{22} & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & ... & \dots \end{pmatrix}$$

and

$$B = \begin{pmatrix} \beta_{00} & 0 & 0 & \dots & ... & ... \\ \beta_{10} & \beta_{11} & 0 & 0 & ... & \dots \\ \beta_{20} & \beta_{21} & \beta_{22} & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & ... & \dots \end{pmatrix}$$

with the following properties:

a) $0 \leq \alpha_{mk} \leq \beta_{mk}$, *for every non-negative integers* m *and* $k \leq m$

b) $\alpha_{mk} \in [a,b]$, $\beta_{mk} \in [c,d]$ *for every non-negative integers* m *and* $k \le m$ *and for some non-negative real numbers* $0 \le a < b$ *and* $0 \le c < d$.
Then for every continuous function $f \in C([0,1])$, *we have*

$$\lim_{m\to\infty} P_m^{(A,B)} f = f \quad , \quad uniformly\ on\ [0,1].$$

Proof. Let us compute the values of the operators $P_m^{(A,B)}$ on test functions e_j, $j = 0,1,2$. We have

$$\left(P_m^{(A,B)} e_0\right)(x) = \sum_{k=0}^{m} p_{m,k}(x) e_0\left(\frac{k+\alpha_{mk}}{m+\beta_{mk}}\right) = \sum_{k=0}^{m} p_{m,k}(x) = 1,$$

$$\left(P_m^{(A,B)} e_1\right)(x) = \sum_{k=0}^{m} p_{m,k}(x) e_1\left(\frac{k+\alpha_{mk}}{m+\beta_{mk}}\right) = \sum_{k=0}^{m} p_{m,k}(x) \frac{k+\alpha_{mk}}{m+\beta_{mk}},$$

respective

$$\left(P_m^{(A,B)} e_2\right)(x) = \sum_{k=0}^{m} p_{m,k}(x) e_2\left(\frac{k+\alpha_{mk}}{m+\beta_{mk}}\right) = \sum_{k=0}^{m} p_{m,k}(x) \left(\frac{k+\alpha_{mk}}{m+\beta_{mk}}\right)^2,$$

Now, from the inequalities

$$a \le \alpha_{mk} \le b \quad , \quad c \le \beta_{mk} \le d,$$

we obtain the estimations

$$\frac{k+a}{m+d} \le \frac{k+\alpha_{mk}}{m+\beta_{mk}} \le \frac{k+b}{m+c},$$

for all non-negative integers $k \le m$. By multiplying each member of the inequality by $p_{m,k}(x)$ and taking the sum with respect to k it follows that

$$\sum_{k=0}^{m} p_{m,k}(x) \frac{k+a}{m+d} \le \sum_{k=0}^{m} p_{m,k}(x) \frac{k+\alpha_{mk}}{m+\beta_{mk}} \le \sum_{k=0}^{m} p_{m,k}(x) \frac{k+b}{m+c}$$

or

$$\left(P_m^{(a,d)} e_1\right)(x) \le \left(P_m^{(A,B)} e_1\right)(x) \le \left(P_m^{(b,c)} e_1\right)(x).$$

Now, by replacing $\left(P_m^{(a,d)} e_1\right)(x)$ and $\left(P_m^{(b,c)} e_1\right)(x)$ with their expressions from Lemma 1.1, we obtain the estimations

$$x + \frac{a-dx}{m+d} \le \left(P_m^{(A,B)} e_1\right)(x) \le x + \frac{b-cx}{m+c}.$$

Hence, for all $x \in [0,1]$, we have

$$\left|\left(P_m^{(A,B)}e_1\right)(x) - x\right| \le \max\left\{\left|\frac{a-dx}{m+d}\right|, \left|\frac{b-cx}{m+c}\right|\right\}.$$

But, for all $x \in [0,1]$, we also have

$$\left|\frac{a-dx}{m+d}\right| \le \frac{|a|+|d|}{m+d} \le \frac{|a|+|d|}{m}$$

and

$$\left|\frac{b-cx}{m+c}\right| \le \frac{|b|+|c|}{m+c} \le \frac{|b|+|c|}{m}.$$

Now, with the notation

$$q = \max\{|a|+|d|, |b|+|c|\},$$

we obtain

$$\left|\left(P_m^{(A,B)}e_1\right)(x) - x\right| \le \frac{q}{m} \to 0 \quad , \quad as\ m \to \infty,$$

for all $x \in [0,1]$, so

$$\lim_{m\to\infty}\left(P_m^{(A,B)}e_1\right)(x) = x \quad , \quad uniformly\ on\ [0,1].$$

Moreover, from the inequality

$$\left(\frac{k+a}{m+d}\right)^2 \le \left(\frac{k+\alpha_{mk}}{m+\beta_{mk}}\right)^2 \le \left(\frac{k+b}{m+c}\right)^2$$

we obtain

$$\sum_{k=0}^{m} p_{m,k}(x)\left(\frac{k+a}{m+d}\right)^2 \le \sum_{k=0}^{m} p_{m,k}(x)\left(\frac{k+\alpha_{mk}}{m+\beta_{mk}}\right)^2 \le \sum_{k=0}^{m} p_{m,k}(x)\left(\frac{k+b}{m+c}\right)^2$$

or

$$\left(P_m^{(a,d)}e_2\right)(x) \le \left(P_m^{(A,B)}e_2\right)(x) \le \left(P_m^{(b,c)}e_2\right)(x).$$

Now, by replacing $\left(P_m^{(a,d)}e_2\right)(x)$ and $\left(P_m^{(b,c)}e_2\right)(x)$ with their expressions from Lemma 1.1, we obtain the estimations

$$x^2 + \frac{mx(1-x)+(a-dx)(2mx+dx+a)}{(m+d)^2} \le \left(P_m^{(A,B)}e_2\right)(x) \le$$

$$\le x^2 + \frac{mx(1-x)+(b-cx)(2mx+cx+b)}{(m+c)^2}.$$

Hence

$$\left|\left(P_m^{(A,B)}e_2\right)(x) - x^2\right| \le$$

$$\leq \max\left\{\left|\tfrac{mx(1-x)+(a-dx)(2mx+dx+a)}{(m+d)^2}\right|, \left|\tfrac{mx(1-x)+(b-cx)(2mx+cx+b)}{(m+c)^2}\right|\right\}.$$

Using that $x \in [0,1]$, we obtain

$$\left|\frac{mx(1-x)+(a-dx)(2mx+dx+a)}{(m+d)^2}\right| \leq \frac{m+(|a|+|d|)(2m+|d|+|a|)}{m^2}$$

and

$$\left|\frac{mx(1-x)+(b-cx)(2mx+cx+b)}{(m+c)^2}\right| \leq \frac{m+(|b|+|c|)(2m+|c|+|b|)}{m^2}.$$

Now, with the notation

$$w = \max\left\{(|a|+|d|)(2m+|d|+|a|), (|b|+|c|)(2m+|c|+|b|)\right\},$$

we obtain that

$$\left|\left(P_m^{(A,B)}e_2\right)(x) - x^2\right| \leq \frac{m+w}{m^2} \to 0 \ , \quad as\ m \to \infty.$$

According with Bohman-Korovkin theorem,

$$\lim_{m\to\infty}\left(P_m^{(A,B)}e_2\right)(x) = x^2 \quad , \quad uniformly\ on\ [0,1].$$

Corollary 2.1 *Assume that* $0 \leq \alpha_{mk} \leq \beta_{mk}$, *for all non-negative integers* $k \leq m$. *For each integer* $k \geq 0$, *assume that the sequences* $(\alpha_{mk})_{m\in\mathbb{N}}$ *and* $(\beta_{mk})_{m\in\mathbb{N}}$ *are convergent to* α_k *and* β_k, *respectively, such that the sequences* $(\alpha_k)_{k\in\mathbb{N}}$ *and* $(\beta_k)_{k\in\mathbb{N}}$ *are bounded. Then for every continuous function* $f \in C([0,1])$,

$$\lim_{m\to\infty}\left(P_m^{(A,B)}f\right)(x) = f(x) \quad , \quad uniformly\ in\ [0,1].$$

Proof. We are in the hypotheses of the Theorem 2.1. From the fact that the sequences $(\alpha_{mk})_{m\in\mathbb{N}}$ and $(\beta_{mk})_{m\in\mathbb{N}}$ are convergent to α_k and β_k, respectively, we can find a positive integer m_1 for which

$$|\alpha_{mk} - \alpha_k| < 1,$$

for all integers $m \geq m_1$ and we can find m_2 for which

$$|\beta_{mk} - \beta_k| < 1,$$

for all integers $m \geq m_2$. Now, for every integer $m \geq m_1 + m_2$, we have

$$\alpha_{mk} < 1 + \alpha_k \leq 1 + M$$

and

$$\beta_{mk} < 1 + \beta_k \leq 1 + M,$$

where

$$M = \max \left\{ \sup_{k \in \mathbb{N}} \alpha_k, \sup_{k \in \mathbb{N}} \beta_k \right\}.$$

3. **A particular case**

An interesting case is when the matrix B has all nonzero entries equal to a positive constant β,

$$B = \begin{pmatrix} \beta & 0 & 0 & \dots & ... & ... \\ \beta & \beta & 0 & 0 & ... & \dots \\ \beta & \beta & \beta & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & ... & \dots \end{pmatrix}.$$

We also impose the conditions

$$\alpha_{mk} \leq \beta,$$

for all non-negative integers $k \leq m$. Under these assumptions, we define an approximation operator denoted by

$$P_m^{(A,\beta)} : C[0,1] \to C([0,1],$$

with the formula

$$\left(P_m^{(A,\beta)} f \right)(x) = \sum_{k=0}^{m} p_{m,k}(x) f\left(\frac{k + \alpha_{mk}}{m + \beta} \right) \quad , \qquad f \in C\,[0,1]\,.$$

Lemma 3.1 *For every continuous function $f \in C[0,1]$, the following relations hold true:*

a) $\left(P_m^{(A,\beta)} e_0 \right)(x) = 1$

b) $\left(P_m^{(A,\beta)} e_1 \right)(x) = x - \dfrac{\beta x}{m + \beta} + \dfrac{1}{m + \beta} \cdot \displaystyle\sum_{k=0}^{m} \alpha_{mk} p_{m,k}(x).$

Proof. a) We have

$$\left(P_m^{(A,\beta)}e_0\right)(x)=\sum_{k=0}^{m}p_{m,k}(x)e_0\left(\frac{k+\alpha_{mk}}{m+\beta}\right)=\sum_{k=0}^{m}p_{m,k}(x)=1.$$

b) We have

$$\left(P_m^{(A,\beta)}e_1\right)(x)=\sum_{k=0}^{m}p_{m,k}(x)e_1\left(\frac{k+\alpha_{mk}}{m+\beta}\right)=\sum_{k=0}^{m}p_{m,k}(x)\frac{k+\alpha_{mk}}{m+\beta}=$$

$$=\sum_{k=0}^{m}\frac{k}{m+\beta}\cdot p_{m,k}(x)+\frac{1}{m+\beta}\cdot\sum_{k=0}^{m}\alpha_{mk}p_{m,k}(x)=$$

$$=\left(P_m^{(0,\beta)}e_1\right)(x)+\frac{1}{m+\beta}\cdot\sum_{k=0}^{m}\alpha_{mk}p_{m,k}(x)=$$

$$=x-\frac{\beta x}{m+\beta}+\frac{1}{m+\beta}\cdot\sum_{k=0}^{m}\alpha_{mk}p_{m,k}(x).$$

Theorem 3.1 *Given the infinite dimensional lower triangular matrix*

$$A=\begin{pmatrix}\alpha_{00} & 0 & 0 & \dots & ... & ...\\ \alpha_{10} & \alpha_{11} & 0 & 0 & ... & \dots\\ \alpha_{20} & \alpha_{21} & \alpha_{22} & 0 & 0 & \dots\\ \dots & \dots & \dots & \dots & ... & \dots\end{pmatrix}$$

and a positive real number β with the following properties:

a) $\alpha_{mk}\leq\beta$, for every non-negative integers m and $k\leq m$

b) $\alpha_{mk}\in[a,b]$, for every non-negative integers m and $k\leq m$ and for some non-negative real numbers $a<b$.

Then for every continuous function $f\in C([0,1])$, we have

$$\lim_{m\to\infty}P_m^{(A,\beta)}f=f\quad,\quad uniformly\ on\ [0,1].$$

Proof. Let us compute the values of the operators $P_m^{(A,\beta)}$ on test functions e_j, $j=0,1,2$. We have

$$\left(P_m^{(A,\beta)}e_0\right)(x)=\sum_{k=0}^{m}p_{m,k}(x)e_0\left(\frac{k+\alpha_{mk}}{m+\beta}\right)=\sum_{k=0}^{m}p_{m,k}(x)=1,$$

$$\left(P_m^{(A,\beta)}e_1\right)(x)=\sum_{k=0}^{m}p_{m,k}(x)e_1\left(\frac{k+\alpha_{mk}}{m+\beta}\right)=\sum_{k=0}^{m}p_{m,k}(x)\frac{k+\alpha_{mk}}{m+\beta},$$

respective

$$\left(P_m^{(A,\beta)}e_2\right)(x) = \sum_{k=0}^{m} p_{m,k}(x)e_2\left(\frac{k+\alpha_{mk}}{m+\beta}\right) = \sum_{k=0}^{m} p_{m,k}(x)\left(\frac{k+\alpha_{mk}}{m+\beta}\right)^2,$$

Now, from the inequalities

$$a \le \alpha_{mk} \le b,$$

we obtain the estimations

$$\frac{k+a}{m+\beta} \le \frac{k+\alpha_{mk}}{m+\beta} \le \frac{k+b}{m+\beta},$$

for all non-negative integers $k \le m$. By multiplying each member of the inequality by $p_{m,k}(x)$ and taking the sum with respect to k it follows that

$$\sum_{k=0}^{m} p_{m,k}(x)\frac{k+a}{m+\beta} \le \sum_{k=0}^{m} p_{m,k}(x)\frac{k+\alpha_{mk}}{m+\beta} \le \sum_{k=0}^{m} p_{m,k}(x)\frac{k+b}{m+\beta}$$

or

$$\left(P_m^{(a,\beta)}e_1\right)(x) \le \left(P_m^{(A,\beta)}e_1\right)(x) \le \left(P_m^{(b,\beta)}e_1\right)(x)$$

and using the expressions of $\left(P_m^{(a,\beta)}e_1\right)(x)$ we obtain

$$x + \frac{a-\beta x}{m+\beta} \le \left(P_m^{(A,B)}e_1\right)(x) \le x + \frac{b-\beta x}{m+\beta}.$$

Hence, for all $x \in [0,1]$, we have

$$\left|\left(P_m^{(A,B)}e_1\right)(x) - x\right| \le \max\left\{\left|\frac{a-\beta x}{m+\beta}\right|, \left|\frac{b-\beta x}{m+\beta}\right|\right\}.$$

Further, for all $x \in [0,1]$, we obtain

$$\left|\frac{a-\beta x}{m+\beta}\right| \le \frac{|a|+|\beta|}{m+\beta} \le \frac{|a|+|\beta|}{m}$$

and

$$\left|\frac{b-\beta x}{m+\beta}\right| \le \frac{|b|+|\beta|}{m+\beta} \le \frac{|b|+|\beta|}{m}.$$

Now, with the notation

$$q = \max\{|a|+|\beta|, |b|+|\beta|\},$$

we obtain

$$\left|\left(P_m^{(A,\beta)}e_1\right)(x) - x\right| \le \frac{q}{m},$$

for all $x \in [0,1]$, so

$$\lim_{m\to\infty}\left(P_m^{(A,\beta)}e_1\right)(x) = x$$

uniformly on $[0,1]$.

Moreover, from the inequality

$$\left(\frac{k+a}{m+\beta}\right)^2 \le \left(\frac{k+\alpha_{mk}}{m+\beta}\right)^2 \le \left(\frac{k+b}{m+\beta}\right)^2$$

we obtain

$$\sum_{k=0}^{m} p_{m,k}(x)\left(\frac{k+a}{m+\beta}\right)^2 \le \sum_{k=0}^{m} p_{m,k}(x)\left(\frac{k+\alpha_{mk}}{m+\beta}\right)^2 \le \sum_{k=0}^{m} p_{m,k}(x)\left(\frac{k+b}{m+\beta}\right)^2$$

or

$$\left(P_m^{(a,\beta)}e_2\right)(x) \le \left(P_m^{(A,\beta)}e_2\right)(x) \le \left(P_m^{(b,\beta)}e_2\right)(x).$$

Now, by replacing $\left(P_m^{(a,\beta)}e_2\right)(x)$ with its expression we obtain the estimations

$$x^2 + \frac{mx(1-x)+(a-\beta x)(2mx+\beta x+a)}{(m+\beta)^2} \le \left(P_m^{(A,\beta)}e_2\right)(x) \le$$

$$\le x^2 + \frac{mx(1-x)+(b-\beta x)(2mx+\beta x+b)}{(m+\beta)^2}.$$

Hence

$$\left|\left(P_m^{(A,\beta)}e_2\right)(x) - x^2\right| \le$$

$$\le \max\left\{\left|\tfrac{mx(1-x)+(a-\beta x)(2mx+\beta x+a)}{(m+\beta)^2}\right|, \left|\tfrac{mx(1-x)+(b-\beta x)(2mx+\beta x+b)}{(m+\beta)^2}\right|\right\}.$$

Using that $x \in [0,1]$, we obtain

$$\left|\frac{mx(1-x)+(a-\beta x)(2mx+\beta x+a)}{(m+\beta)^2}\right| \le \frac{m+(|a|+|\beta|)(2m+|\beta|+|a|)}{m^2}$$

and

$$\left|\frac{mx(1-x)+(b-\beta x)(2mx+\beta x+b)}{(m+\beta)^2}\right| \le \frac{m+(|b|+|\beta|)(2m+|\beta|+|b|)}{m^2}.$$

Now, with the notation

$$w = \max\left\{(|a|+|\beta|)(2m+|\beta|+|a|), (|b|+|\beta|)(2m+|\beta|+|b|)\right\},$$

we obtain that

$$\left|\left(P_m^{(A,\beta)}e_2\right)(x) - x^2\right| \le \frac{m+w}{m^2} \to 0\ , \quad as\ m \to \infty.$$

According with the Bohman-Korovkin theorem, we have

$$\lim_{m\to\infty}\left(P_m^{(A,\beta)}e_2\right)(x)=x^2,$$

uniformly on $[0,1]$.

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