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# CRITERIA FOR UNIT GROUPS IN COMMUTATIVE GROUP RINGS

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**Abstract**. Suppose $G$ is an arbitrary abelian group and $F$ is a field of $charF = p \neq 0$. In the present paper criteria are found the group of all units $UF[G]$ in the group ring $F[G]$ and its subgroup $VF[G]$ of normed units to belong to some central classes of abelian groups under minimal restrictions on $F$ and $G$. In many instances these necessary and sufficient conditions are in a final form and improve or supersede well-known and documented classical results in this aspect such as due to Karpilovsky (Arch. Math. Basel, 1983). The criteria obtained by us are a natural sequel to our recent results published in Glasgow Math. J. (September, 2001) and are generalizations to those stated and argued by us in Math. Balkanica (June, 2000) as well.

## 1. Introduction

Throughout the body of the text, let $F[G]$ be the group ring with prime characteristic $p$ of the abelian group $G$ over the field $F$ of prime characteristic $p$. As usual, $n \in \mathbb{N}$ is a natural number and $\zeta_n$ is a primitive $n$-th root of unity, that is $\zeta_n^n = 1$ while $\zeta_n^k \neq 1 \ \forall \ k < n$. For an abelian group $G$, written via the multiplicative record as is customary when regarding group rings, $G^*$ is the maximal $p$-divisible subgroup of $G$, $G[n] = \{g \in G | g^n = 1\}$ is the $n$-socle of $G$ and $G_t = \cup_{n<\omega} G[n]$ (in the set-theoretic sense) jointly with $G_p$ are the torsion part and its $p$-component in $G$, respectively. For a field $F$, $F^-$ is the algebraic closure of $F$, $F(\zeta_n)$ is a cyclotomic extension of $F$ by inserting $\zeta_n$, $(F(\zeta_n) : F)$ is the binomial index of $F(\zeta_n)$ in $F, F^*$ is

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the multiplicative group of $F$ and $F_d = F^{p^\omega}$ is the maximal $p$-divisible subfield of $F$. In what follows, $K$ denotes an algebraically closed field with characteristic $p$.

All other notions and notations from abelian group rings theory not explicitly defined herein will follow essentially our recent work [2]. For instance, $SF[G]$ is the normed Sylow $p$-subgroup in $F[G]$, $|M|$ is the cardinality of an arbitrary set $M$, $m_n = |\{g \in G | order(g) = n\}|/(F(\zeta_n) : F)$, etc. Apparently $m_n = 0 \Leftrightarrow G[n] \setminus G[k] = \emptyset \ \forall \ k < n \Leftrightarrow G[n] \setminus \cup_{k<n} G[k] = \emptyset$, and $|m_n| = |G[n] \setminus \cup_{k<n} G[k]| \geq \aleph_0$ for some, hence almost all, $n \in \mathbb{N}$ whenever $|G_t| \geq \aleph_0$ since $(F(\zeta_n) : F) < \aleph_0$ is ever fulfilled.

Concerning various technical terms and the terminology used in the abelian group theory, they are in agreement with the classical books [10-12]. Nevertheless, for the sake of completeness and for the convenience of the readers, we include some more specific details; for example, in all that follows, for any abelian group $A$, the cardinal number $r_0 A$ denotes the torsion-free rank of $A$, and $A^1 = \cap_n A^n = \cap_p \cap_m A^{p^m} = \cap_p A^{p^\omega}$ is the first Ulm subgroup of $A$. For simplicity of the exposition, we use the abbreviations $\Sigma$-cyclic and $\Sigma$-countable for direct sums of cyclic groups, respectively for direct sums of countable groups, with the exception of the definition of a $\Sigma$-group that is an abelian group whose high subgroups are direct sums of cyclics.

The main goal of this manuscript is to establish as applications to the structural theorems in [2] necessary and sufficient conditions for the groups $UF[G]$ and $VF[G]$ of all invertible elements (often called units) and normed invertible elements (often called normalised units), respectively, to possess some important properties and to compute explicitly their determinate numerical invariants. The given criteria and computations expand in some way classical facts in this direction proved in ([3,5]; see [6] too), [13] and [19-22; 23-28].

Conforming with the isomorphic descriptions of $UF[G]$ and $VF[G]$, given in [2], we have obtained in [4] certain additional algebraic properties for these groups, which properties are of some importance. Moreover, we indicate also that, a criterion for $VF[G]$ to be a direct sum of $p$-mixed countable abelian groups was established in [9,7], provided $F$ is perfect.

## 2. Main results

Some of the main attainments presented here were previously announced in [1].

And so, we start with

**Theorem 1.** *$VF[G]$ is $\Sigma$-cyclic if and only if $G$ is $\Sigma$-cyclic and at most one of the following conditions hold:*

*1) $G_t = G_p$*

*or*

*2) $G_t \neq G_p$, $F \neq F^-$ and $F(\zeta_n)^*$ is $\Sigma$-cyclic for each $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ which is an order of an element of $G_t/G_p$.*

**Proof.** Certainly, $VF[G]$ being $\Sigma$-cyclic implies by the classical theorem of L. Kulikov ([10], p.110, Theorem 18.1) that $G \subseteq VF[G]$, being a subgroup, is also $\Sigma$-cyclic.

First of all, suppose $F$ is algebraically closed and $VF[G]$ is $\Sigma$-cyclic. Hence $VF[M_t] \subseteq VF[G]$ is also $\Sigma$-cyclic, where $M$ is a group so that $G = G_p \times M$. But besides $VF[M_t]$ is divisible (see [2], formula (8)), and therefore $VF[M_t] = 1$, i.e. $M_t = 1$. Thus $G_t = G_p$.

Conversely, take $G$ to be $\Sigma$-cyclic and $G_t = G_p$. Hence $G$ splits and so $G = G_p \times M$. Owing to Lemma 2.2 of [2], $VF[G] = VF[M] \times SG[G] \cong G/G_t \times SF[G]$ by using the well-known Higman's result on trivial units documented in [14]. Finally $VF[G]$ is $\Sigma$-cyclic because $G/G_t$ is free and because Theorem 2.1 from [2] ensures that $SF[G]$ is $\Sigma$-cyclic.

Let now $F$ be not algebraically closed, i.e. $F \neq F^-$. Suppose $VF[G]$ is $\Sigma$-cyclic. Hence $1 \neq VF[M_t]$ is as well, where $M$ is such a group that $G = G_p \times M$ and $M_t \neq 1$, whence $G_t \neq G_p$. Consequently by formulas (3) and (4) of [2], $F(\zeta_n)^*$ is $\Sigma$-cyclic. The reverse inclusion follows applying formulas (17), (18) and Theorem 2.1(ii) in [2]. This ends the proof.

**Theorem 2.** *$UF[G]$ is $\Sigma$-cyclic if and only if $G$ is $\Sigma$-cyclic and either*

*1') $G_t = G_p$ and $F^*$ is $\Sigma$-cyclic*

*or*

*2') $G_t \neq G_p$ and $F(\zeta_n)^*$ is $\Sigma$-cyclic for every $n$ that is an order of an element of $G_t/G_p$.*

**Proof.** It is analogous to the last theorem, since $F$ is not algebraically closed provided $UF[G]$ is $\Sigma$-cyclic. Indeed, if $F = F^0$ then $F^*$ is divisible $\Sigma$-cyclic, i.e. $F^* = 1$, and thereby $F = \{0, 1\}$, a contradiction with the infinite cardinality of $F$. Finally, we apply that $UF[G] = VF[G] \times F^*$ is $\Sigma$-cyclic only when so are $VF[G]$ and $F^*$. The proof is completed.

**Example.** The condition on $n$ that $m_n \neq 0$ (i.e. that there is an element in $G$ of order $n$) stated in the previous two theorems is necessary. In fact, inductively, let $F_n$ be the finite field of order $2^{3^n}$, and put $F = \cup_{n<\omega} F_n$; $F_n \subseteq F_{n+1}$ so $F$ is a countable field of characteristic $p = 2$. Let $G$ be the direct sum of $\aleph_0$ copies of a cyclic group of order 7. Thus $G_p = 1$ and $G = G_t \neq 1$ with $G^7 = 1$. In order to obtain that $VF[G]$ is $\Sigma$-cyclic, according to Theorems 1 and 2, $F(\zeta_n)^*$ should be $\Sigma$-cyclic only for $n = 7$ but not for every $n \in \mathbb{N}_0$. This is so since $F(\zeta_3)^*$ has 3-component isomorphic to $Z(3^\infty)$. As for $F(\zeta_3)$, we observe that $(\zeta_3)$ is the field of 4-elements, and in the formula $4^{3^{n+1}} - 1 = (4^{3^n} - 1)(4^{2 \cdot 3^n} + 4^{3^n} + 1)$ the second factor is always divisible by 3.

To justify the example, since $F$ contains a primitive 7-th root of unity, namely $\zeta_7 \in F_1$ since $F_1^*$ is cyclic of power 7, whence $F(\zeta_7) = F$, we detect that $VF[G]$ will be $\Sigma$-cyclic if and only if $F^*$ is $\Sigma$-cyclic. This follows from the formula $2^{3^{n+1}} - 1 = (2^{3^n} - 1)(2^{2 \cdot 3^n} + 2^{3^n} + 1)$ because any prime dividing the first factor cannot divide the second one (note that the prime cannot be 3).

We continue in this way with

**Theorem 3.** *$UF[G]$ is bounded if and only if $G$ is bounded and $F^*$ is bounded.*

**Proof.** It is long-known that $SF[G]$ is bounded if and only if $G_p$ is bounded (see for example [2]). We note that $m_k = 0$ precisely when $G$ is bounded with exponent $\exp(G) < k$. That the statements $G$ and $F^*$ are both bounded, is equivalent to $G$ is bounded and, either $F(\zeta_n)^*$ is bounded for each $n$ dividing $|G_t/G_p| < \aleph_0$, or $\bigcup_{n=0}^{\infty} \times m_n F(\zeta_n)^*$ is bounded when $|G_t/G_p| \geq \aleph_0$, follows now easily, since $F^*$ being bounded implies that $F$ is a finite algebraic extension of a simple (hence a finite) field, whence it is finite as well. Appropriate arguments for this are that $F(\mathbb{Z}_n)^*$ is bounded $\Leftrightarrow$ $F^*$ is bounded $\forall\ n < \omega$ and that $\bigcup_{n=0}^{\infty} \times m_n F(\zeta_n)^*$ reduces to $\bigcup_{n \leq \exp(G)} \times m_n F(\zeta_n)^*$. Therefore we wish only to apply Theorem 2.2 point (e) of [2]. The proof is complete.

**Theorem 4.** *$UF[G]$ is finitely generated if and only if $G_p \neq 1$, $F$ and $G$ are finite; or $G_p = 1$, $G$ and $F$ are finitely generated.*

**Proof.** First assume $G_p \neq 1$. Let $UF[G]$ be finitely generated. Then it is elementary that $1 \neq SF[G]$ is finite. But if $|F| \geq \aleph_0$ or $|G| \geq \aleph_0$, we derive as in [14] that $|SF[G]| = \max(|F|, |G|) \geq \aleph_0$, that is false. Thus obviously $F$ and $G$ are both finite. Conversely, if $F$ and $G$ are finite, then $UF[G]$ is finite, hence finitely generated.

Now let $G_p = 1$. In that case the proof goes by a standard application of Theorem 2.2 in [2] in view of the fact that a subgroup of a finitely generated group has the same property (cf. [10]). The equivalence of the second part half, namely that $G_p = 1$, $G$ and $F$ are finitely generated $\Leftrightarrow$ $G_p = 1$ and $G$ along with $F(\zeta_n)^*$ are finitely generated for every $n$ dividing $|G_t| < \aleph_0$, holds at once since $F^*$ being finitely generated forces that so do both $F(\zeta_n)^*$ $\forall\ n$ and $F = F^* \cup \{0\}$.

The proof is finished in all generality.

**Remark.** A criterion for $UR[G]$ to be finitely generated was also founded by Karpilovsky (see [13, Theorem 3]) when $R$ is a finitely generated commutative unitary ring of arbitrary characteristic and $G$ is an arbitrary abelian group. However, in our situation, $F$ need not be finitely generated a priory, as this fact follows easily from the same property for $UF[G]$.

Generally, does it follow that $UR[G]$ being finitely generated forces the same property for $R$? If yes, the problem of finding the criterion for $UR[G]$ to be finitely generated will be completely resolved. However, this question is quite difficult and its solution seems to be in the distant future.

In the next statement, we will use the simple but useful fact that $G$ being $\Sigma$-countable yields that both $G_t$ and $G_p$ are $\Sigma$-countable groups as well.

**Proposition 5.** *Let $G$ be splitting and $F$ perfect. If $G$ and $F(\zeta_n, \mu_q)^*$ are $\Sigma$-countable groups then the group $UF[G]$ is $\Sigma$-countable when $|G_t/G_p| \geq \aleph_0$ and if $G$ and $F(\zeta_n)^*$ are $\Sigma$-countable groups then the group $UF[G]$ is $\Sigma$-countable when $|G_t/G_p| < \aleph_0$.*

**Proof.** This follows by a standard application of (19), (20) and of Claim 2.1, all from [2]. The proposition is verified.

**Remark.** By the same statements, as in the situation for $\Sigma$-cyclic groups, criteria can be established for $VF[G]$ to be bounded, finitely generated and $\Sigma$-countable. Nevertheless, we omit the reproduction of their explicit form.

The following two group-theoretic observations are well-known and have routine proofs - they shall be used below without further reference: an isotype subgroup of a direct product of a divisible and a bounded group inherits this group property; a pure subgroup of a divisible group is divisible. Moreover, it is not difficult to check that an outer direct sum of equal algebraically compact groups is also an algebraically compact group.

After this, we need one more technicality, which is crucial.

**Lemma 6.** *Suppose $G_t = G_p$. Then $G$ is pure in $VF[G]$.*

**Proof.** We shall use the definition for the property "purity" by differing two basic cases:

**Case 1.** For each natural $n$ so that $p|/n$ we write $n = p_1^{t_1} \dots p_s^{t_s}$ as the canonical form of $n$, where $p_1, \dots, p_s \neq p$ are distinct primes, $s \in \mathbb{N}$, $t_1, \dots, t_s \in \mathbb{N}_0$. Since $VF[G] = GSF[G]$ (see e.g. [21, 22] or [8]), by the usage of the modular law we conclude that $G \cap V^n F[G] = G \cap (GSF[G])^n = G \cap (G^n SF[G]) = G^n(G \cap SF[G]) = G^n G_p = G^n$.

**Case 2.** $p/n$, whence we write $n = p^{k_1} q_2^{k_2} \dots q_m^{k_m}$ to be the canonical form of $n$, where $q_2, \dots, q_m \neq p$ are different primes, $m \in \mathbb{N}$, $k_1, \dots, k_m \in \mathbb{N}_0$. As above we deduce $G \cap V^n F[G] = G \cap (GSF[G])^n = G \cap (G^n SF^{p^{k_1}}[G^{p^{k_1}}]) = G^n(G \cap SF^{p^{k_1}}[G^{p^{k_1}}]) = G^n G_p^{p^{k_1}} = G^n$. The proof is over.

**Remark.** When $G$ is not $p$-mixed, that is $G_t \neq G_p$, $G$ need not be a pure subgroup of $VF[G]$ in general (see the Remark after Corollary 9). Even more $G_t$ is not pure in $V_tF[G] = SF[G]VF[G_t]$ assuming extra that $G_p \neq 1$. Another argumentation is when $G_p = 1$. Henceforth, in this situation, $V_tF[G] = VF[G_t]$ and thus $V_tK[G]$, by point (a') proved below, must be always divisible whereas $G_t$ may not be so.

Now we are ready to attack the following.

**Theorem 7.** *Let $1 \neq G_t$ be $p$-torsion. Then*

*(a) $VF[G]$ is divisible if and only if $G$ is divisible and $F$ is perfect.*

*(b) $VF[G]$ is a direct sum of a divisible group and of a bounded group if and only if $G$ is a direct sum of a divisible group and of a bounded group and, either $G_t$ is not reduced, $G/G_t$ is $p$-divisible and $F$ is perfect, or $G_t$ is reduced.*

*(c) $VF[G]$ is algebraically compact if and only if $G$ is algebraically compact and, either $G_t$ is unbounded algebraically compact, $G/G_t$ is $p$-divisible and $FF$ is perfect, or $G_t$ is bounded.*

*(d) $VF[G]$ is coperiodical if and only if $G$ is coperiodical and, either $G_t$ is unbounded coperiodical, $G/G_t$ is $p$-divisible and $F$ is perfect, or $G_t$ is bounded.*

**Proof.** (a) Choose $VF[G]$ to be divisible. Hence $G_t = G_p$ is divisible as it is a pure subgroup. Thus $G = G_t \times M$ and by formula (6) of [2], $VF[G] = VF[M] \times SF[G] \cong G/G_t \times SF[G]$ using again the classical Higman's result on the trivial units (cf. [14]). Further $G/G_t$ is divisible, i.e. so is $G$, and moreover $SF[G]$ is also divisible. So, $S^pF[G] = SF^p[Gp] = SF[G]$, equivalently $F = F^p$, and $F$ is perfect as asserted.

Conversely, assume $G$ divisible and $F$ perfect. Hence $G_p$ is divisible as it is pure in $G$, and besides $G/G_t$ is also divisible as it is a factor-group. Thus $G \cong G_t \times G/G_t$ and similarly to the above, $VF[G] \cong G/G_t \times SF[G]$. Finally, $SF[G]$ and $VF[G]$ are both divisible groups.

(b) Suppose $VF[G]$ is a direct sum of a divisible group and of a bounded group. Hence $G_p$ as an isotype subgroup is one also. Therefore $G = G_p \times M$ (see [10]) and as above $VF[G] \cong G/G_t \times SF[G]$. Thus $G/G_t$ is a direct sum of a divisible and a bounded group, i.e. the same is $G$. On the other hand $SF[G]$ belongs to this group class, i.e. it is algebraically compact (cf. [10]). But $SF[G] \cong SF[M][Gp]$ (see [0]) because $M_p = 1$, and thus $SF[M] = 1$. That is why, following [0], if $G_p$ is not reduced, then $SF[G]$ algebraically compact yields $FM$ is perfect since $FM$ is without nilpotent elements (notice that $F$ has a trivial nil-radical and $M$ has no $p$-elements). Hence, $F$ is perfect and $G/G_p$ is $p$-divisible.

Oppositely, if the conditions from the text hold, then $G_p$ is algebraically compact as an isotype subgroup in $G$. So, $G = G_p \times M$ (cf. [10]) and by equality (6) from [2], $VF[G] \cong G/G_t \times SF[G] \cong G/G_t\theta SF[M][G_p]$ (see [0]). We only need to apply [0] and the result follows immediately.

(c) If $G$ is $p$-primary, the point follows directly from [0]. So, we may presume that $G \neq G_p$. Referring to Lemma 6 and ([10], p.190, Exercise 3), $VF[G]/G \cong SF[G]/G_p$ is algebraically compact provided that so is $VF[G]$. Therefore $SF[G]/G_p$ is a direct sum of a divisible and a bounded group (cf. [10]). But $(SF[G]/G_p)_d = (SF_d[G^*])G_p/G_p$ via [8], hence the quotient-group $SF[G]/SF_d[G^*]G_p$ is bounded, i.e. there is $k \in \mathbb{N}$ such that $SF^{p^k}[G^{p^k}] \subseteq SF_d[G^*]G_p$. The last reduces to $F^{p^k} = F_d$ and $G^{p^k} = G^*$ when $G_p$ is not reduced. Indeed, consider the element $1 + rg(1 - g_p)$ where $r \in F^{p^k}$, $g \in G^{p^k} \setminus G_p^{p^k}$ and $g_p \in G_p^{p^k} \setminus \{1\}$. Thus $1 + rg(1 - g_p) = (f_1a_1 + \cdots + f_ta_t)c_p$, where $f_i \in F_d$, $a_i \in G^*$, $c_p \in G_p$; $1 \leq i \leq t \in \mathbb{N}$. Henceforth, the canonical forms imply that $r \in F_d$ and $g \in G^*$, $g_p \in (G^*)_p$. Furthermore $(G^{p^k})_p = (G^*)_p$, i.e. $G_p^{p^k}$ is divisible, which is equivalent to $G_p$ being algebraically compact by [10]. But $G^{p^k}/G_p^{p^k} \cong (G/G_p)^{p^k}$ is $p$-divisible, i.e. so is $G/G_p$. Finally, it is a plain exercise to verify that $G^{p^k}$ is $p$-divisible, i.e. $G^{p^k} = G^*$. On the other hand, as we have already seen, $F^{p^k} = F^{p^{k+1}}$ whence $F$ is perfect. Next, if $G_p$ is reduced, we have $(G^*)_p = 1$ hence $SF_d[G^*] = 1$ and so the foregoing inclusion takes the form $SF^{p^k}[G^{p^k}] \subseteq G_p$ or equivalently $G_p^{p^{k+1}} = 1$. So, in both cases, $G_p$, being a pure subgroup, is a direct factor of $SF[G]$, hence $G$ is a direct factor of $VF[G] = GSF[G]$. Then $G$ is algebraically compact exploiting [10]. This verifies the first half.

For the converse implication, we observe that $G_p$ is a direct factor of $G$, i.e. in other words $G$ is $p$-splitting, whence $G/G_t$ is algebraically compact. Thus by what we have shown above, $VF[G] \cong G/G_t \times SF[G] \cong G/G_t \times SF[G/G_p][G_p]$. By making use of [0] and [10], the point is exhausted.

(d) Since $VF[G]$ is coperiodical, we refer to [10] to infer that $VF[G]/G \cong SF[G]/G_p$ is coperiodical too. Therefore, again by using of [10], the proof goes on the same arguments and conclusions as in (c).

This proves the theorem.

After this, we proceed by proving the following.

**Theorem 8.**

*(a') $VK[G]$ is divisible if and only if $G/G_t$ and $G_p$ are divisible.*

*(b') $VK[G]$ is a direct sum of a divisible and a bounded group if and only if $G/G_t$ and $G_p$ are a direct sum of a divisible and a bounded group, and $G/G_p$ is $p$-divisible provided $G_p$ is not reduced.*

*(c') $VK[G]$ is reduced algebraically compact if and only if $G/G_t$ and $G_p$ are reduced algebraically compact.*

*(d') $VK[G]$ is reduced coperiodical if and only if $G/G_t$ and $G_p$ are reduced coperiodical.*

*(e') Let $G$ be $p$-splitting. $VK[G]$ is $\Sigma$-countable if and only if $G/G_t$ and $G_p$ are $\Sigma$-countable.*

**Proof.** (a') $VK[G]$ divisible insures that $G_p$ is divisible as its pure subgroup, whence $G$ is $p$-splitting. Further the proof follows immediately from the description of $VK[G]$ in ([2], section 2, formulas (11)-(12)) and from the group-theoretic facts given in [10]. The reverse implication is similar.

(b') We firstly deal with the necessity. Certainly, the fact that $G_p$ is isotype in $VK[G]$ yields that $G_p$ is a direct sum of a divisible and a bounded group, so $G$ is $p$-splitting. Further, the proof follows directly by virtue of formulae (11)-(12) in [2] and utilizing the criterion in [0] for $SK[G]$ to be algebraically compact combined with some group-theoretic facts obtained in [10]. The sufficiency is analogical.

(c') Foremost, assume that $VK[G]$ is reduced algebraically compact. Evidently $G_p$ is reduced being a subgroup. Assume also that $B$ is an unbounded basic subgroup of $G_p$. Therefore we write $B = \bigcup_{n=1}^{\infty} B_n$, where all subgroups

$B_n$ are homogeneous of order $p^n$. We now construct the infinite sequence $g_n = \prod_{i=1}^{n}(1+b_i^{p^{i-1}}-b_{i+1}p^i)$, where $b_i \in B_i$; $n \in \mathbb{N}$. Clearly $g_n^p = 1$, and for each $k \in \mathbb{N}$ we have $g_{n+l}g_n^{-1} = \prod_{i=n+1}^{n+l}(1+b_i^{p^{i-1}}-b_{i+1}p^i) \in S^{p^k}K[G] \subseteq S^kK[G] \subseteq V^kK[G]$ for every $n \geq k$ and arbitrary positive integer $L$. We note that the first inclusion holds since if $p|/k$ we have $S^kK[G] = SK[G]$, while if $p/k$ we have $k = p^s m$ for some $s, m \in \mathbb{N}$ with $(m,p) = 1$ and so $S^kK[G] = S^{p^s}K[G] \supseteq S^{p^k}K[G]$ by observing that $s < k$. That is why $(g_n)$ is a Cauchy sequence in $VK[G]$ and consequently we can apply the well-known Kaplansky theorem ([10], p.191, Theorem 39.1) which guarantees that $(g_n)$ must be convergent to an element of $VK[G]$ in its $Z$-adic topology. And so, let $g = \sum_{j=1}^{t}\alpha_j g_j \in VK[G]$ be the boundary of $(g_n)$. Furthermore, for all $k \geq 1$ and $n \geq k$, we derive

$$\sum_{j=1}^{t}\alpha_j g_j = \left[\prod_{i=1}^{n}(1+b_i^{p^{i-1}}-b_{i+1}^{p^i})\right](r_{1n}(k)a_{1n}(k)p^k + \cdots + r_{s_n n}(k)a_{s_n n}(k)p^k),$$

where $r_{1n}(k), \ldots, r_{s_n n}(k) \in K$; $a_{1n}(k), \ldots, a_{s_n n}(k) \in G$; $s_n \in \mathbb{N}$. It is easily seen that the left hand-side of the last equality is constant about $n$, while the right hand-side depends on $n$ and contains a number of elements in the canonical form that is $\geq n > t$. In fact, it is easy to see that there is $k \in \mathbb{N}$ so that all products of $b_i p^{i-1}$'s for different various indices $i$ running $\mathbb{N}$ are not in $G_p^k$. If the reverse holds, these products belong to $B \cap G^{p^\omega} = 1$, which is demonstrably false because in that case $b_i^{p^{i-1}} = 1 \;\forall\; 1 \leq i \leq n$ whereas $order(b_i) = p^i$. Moreover, because of the direct decomposition of $B$, these products of $b_i^{p^{i-1}}$'s are independent and their number depend on $n$. By taking $n > t$, the claim really sustained. Finally, we deduce that $(g_n)$ is not a convergent, i.e. it is a divergent, sequence in $VK[G]$ when $B$ is unbounded. Thereby $B$ is bounded, i.e. $G_p$ must be so by referring to ([10, 12]). Henceforth, appealing to [10], $G$ is $p$-splitting and the proof follows by means of formulas (11)-(12) from [2] and the simple observations stated before Lemma 6. The treatment of the converse question is similar.

(d') $VK[G]$ being coperiodical implies that $VK[G]/V^1K[G]$ is algebraically compact (see e.g. [10]), where $V^1K[G]$ is the first Ulm subgroup of $VK[G]$. Now we consider the sequence $(h_n) = (g_nV^1K[G])$ where $(g_n)$ is constructed as in the previous point. Clearly $g_n \notin V^1K[G]$, otherwise $g_n \in V^{p^\omega}K[G] = VK[G^{p^\omega}]$ and so $b_i^{p^{i-1}} \in B \cap G^{p^\omega} = B^{p^\omega} = 1$, a contradiction. Besides, it is a routine technical work to check that $(h_n)$ is a Cauchy sequence since $(g_n)$ is. Further the proof goes by the same arguments as in the preceding statement. The sufficiency is analogous.

(e') Since a direct factor of a $\Sigma$-countable group is $\Sigma$-countable (see [10], a theorem of Kaplansky - C. Walker) and any divisible group is $\Sigma$-countable, then owing to the isomorphism (11) from [2], it is enough to show only that $SK[G]$ is $\Sigma$-countable if and only if $G_p$ is $\Sigma$-countable. In fact, this is precisely Claim 2.1 of [2] and thus we are done. This deduces the theorem.

**Corollary 9.** *Let $G$ be divisible. Then $G$ is a direct factor of $VK[G]$ with divisible complementary factor. Thus $VK[G]$ is divisible.*

**Remark.** We can restate point (a') like this: $VK[G]$ is divisible if and only if $G/G_t$ is divisible and $G$ is $p$-divisible. From this, it follows that if $VK[G]$ is divisible, $G$ need not be so. Consequently, a principal question is whether or not the divisibility of $VF[G]$ does imply that $G$ is splitting. If yes, one can employ formulas (16) (and eventually (19) and (20)) from [2] to find a criterion for $VF[G]$ to be divisible.

If $X$ is an arbitrary abelian group, as emphasized in the introduction, we shall say that $r_0(X)$ is the torsion-free rank of $X$. Mollov [24,25] has calculated the torsion-free rank of $UE[G]$ for semisimple $EG$ whose $G$ is torsion (see also [14]). Later on, Mollov and Nachev [26,27] have computed the torsion-free rank $r_0UE^t[G]$ of the group of units in a commutative semisimple twisted group algebra $E^t[G]$ in terms of $E$ and of $G$, when $G$ is torsion or torsion-free. Specifically, they calculated in a more general aspect this rank for semisimple abelian $E^t[G]$ when $G$ is arbitrary, but in terms of $E, G$ and $E^t[G_t]$. So, the result is incomplete, since a characterization of $E^t[G_t]$ that depends only of $E, G_t$ and the system of factors of $E^t[G]$ was not given here.

Nevertheless, contrasting with their result, we compute $r_0UE[G]$ for a modular or a semisimple group ring $E[G]$ over a splitting or a torsion group $G$ as well as over a $p$-splitting group $G$ but over an algebraically closed field $E$, both in the two cases only in terms of $E$ and $G$. That is, of course, more precise.

Before doing this, we require one more result.

**Theorem 10.** *The group $VF[G]$ is torsion if and only if $G$ is $p$-torsion, or $G$ and $F^*$ are torsion provided $G \neq G_p$.*

**Proof.** If $G = G_p$, it is a simple matter to check that $VF[G]$ is a $p$-group. That is why we deal only with $G \neq G_p$. First assume $VF[G]$ is torsion. Hence $G$ is torsion and so $G = G_p \times M$. Thus $VF[M]$ is torsion, and consequently by [25] or [14] we conclude that $F$ is an algebraic extension of a finite field, i.e. $F^*$ is torsion.

To treat the converse, write $G = G_p \times M$. Therefore, in accordance with Lemma 2.2 of [2], we obtain $VF[G] = VF[M] \times SF[G]$. But $M$ and $F^*$ are both torsion. By virtue of ([25], [14]), $VF[M]$ is torsion, i.e. so does $VF[G]$. This finishes the proof.

Our aims here are the following.

**Theorem 11.** *Let $G$ be torsion. Then $r_0 VF[G] = 0$ if $F$ is an algebraic extension of a finite field or if $G = G_p$, and $r_0 VF[G] = \max(|F|, |G/G_p|)$ otherwise.*

**Proof.** First take $G$ to be $p$-primary or $F$ to be an algebraic extension of a finite field. Consequently Theorem 10 assures that $VF[G]$ is torsion, and so $r_0 VF[G] = 0$. In the remaining cases we write $G \cong G_p \times G/G_p$. Therefore formula (6) in [2] implies $VF[G] \cong VF[G/G_p] \times SF[G]$. Hence, $r_0 VF[G] = r_0 VF[G/G_p]$ (see [10]), whence we use [24,25] to conclude that $r_0 VF[G] = \max(|F|, |G/G_p|)$, as stated. The theorem is proved.

**Theorem 12.** *Suppose $G$ splits and $E$ is a field. Then if $char(E) = 0$,*

$$r_0 UE[G] = \max(|E|, |G_t|, r_0(G)); \tag{1}$$

*and if $char(E) = p \neq 0$;*

$$r_0 UE[G] = \begin{cases} |G_t/G_p| r_0(G), & |G_t/G_p| \geq \aleph_0 \\ \sum_{d/|G_t/G_p|} m_d r_0(G), & |G_t/G_p| < \aleph_0 \end{cases} \tag{2}$$

*provided $E$ is an algebraic extension of a finite field, or*

$$r_0 UE[G] = \max(|E|, |G_t/G_p|, r_0(G)) \tag{3}$$

*otherwise.*

**Proof.** Given $char(E) = 0$. In virtue of the isomorphism (15) from [2] together with [10], we have $r_0UE[G] = r_0UE[G_t]+\Sigma_\alpha r_0(G/G_t)$, where $\alpha$ is computed as in [2]. But $r_0(G/G_t) = r_0(G)$ and thus [24,25] lead us to $r_0UE[G_t] = max(|E|, |G_t|)$, because $E$ is infinite. Consequently by virtue of ([15], p.206, Theorem 7), $r_0UE[G] = max(|E|, G_t|)+(\alpha r_0(G) = max(|E|, |G_t|)+max(|Gt|, r_0(G)) = max(|E|, |G_t|, r_0(G))$.

For $char(E) = p > 0$ and $E$ an algebraic extension of a simple (i.e. of a finite) field we derive via [14] that $UE[G_t/G_p]$ is torsion. In view of formula (16) in [2] and of ([15], p.206, Theorem 7) combined with [10], we deduce that $r_0UE[G] = \sum(|G_t/G_p|r_0(G) = |G_t/G_p|r_0(G)$ for the infinite situation or $r_0UE[G] = \sum_\beta r_0(G) = \beta r_0(G)$ where $\beta = \sum d/|G_t/G_p|m_d$ for the finite one.

In the remaining case, the same formula (16) plus [10], [15] and Theorem 11 are guarantors that $r_0UE[G] = \max(|E|, |G_t/G_p|) + \sum_{|G_t/G_p|} r_0(G) = \max(|E|, |G_t/G_p|) + |G_t/G_p|r_0(G) = max(|E|, |G_t/G_p|, r_0(G))$, as desired. So, the theorem is true.

**Theorem 13.** *Let $E$ be a field. Then if $char(E) = 0$,*

$$r_0UE^-[G] = \max(|E^-|, |G_t|, r_0(G)); \tag{4}$$

*and if $char(E) = p > 0$ and $G$ is $p$-splitting,*

$$r_0UE^-[G] = |G_t/G_p|r_0(G) \tag{5}$$

*provided $E$ is an algebraic extension of a finite field, or*

$$r_0UE^-[G] = max(|E^-|, |G_t/G_p|, r_0(G)) \tag{6}$$

*otherwise.*

**Proof.** The result follows employing formulas (8)-(12) from [2] along with [10] and [15]. The conclusions are similar to these of the foregoing theorem. The proof is finished.

Now, we shall begin with other types of results by arguing the following (a part of the results presented here generalize those obtained by Mollov in [24] and [25]; see [14] and [19] as well).

**Proposition 14.** *Suppose $G$ is a direct sum of finite cyclic groups. Then $VF[G]$ is nontrivial free modulo torsion if and only if $F(\zeta_n)^*$ is free modulo torsion for each $n$ which is an order of an element of $G$.*

**Proof.** Clearly $G$ is torsion and $G = G_p \times M$ for some group $M$. Referring to ratio (6) from [2], we may write $VF[G] = VF[M] \times SF[G]$. Thus $VF[G]/V_tF[G] \cong VF[M]/V_tF[M]$ and the result follows by application of [25] or [14]. The statement is shown.

We can extend the last affirmation to the next claim.

**Proposition 15.** *Let $G$ be $\Sigma$-cyclic. Then $UF[G]$ is nontrivial free modulo torsion if and only if $F(\zeta_n)^*$ is free modulo torsion for every $n$ which is an order of one element of $G$.*

**Proof.** It is not difficult to see by application of formulae (17-18) from [2] that $UF[G]/U_tF[G] \cong (\times_\delta G/G_t) \times (\Pi_n \times_{m_n} F(\zeta n)^*/F(\zeta n)^*_t)$, where $\delta$ is finite or infinite defined in the same manner as in [2]. This proves the result.

**Proposition 16.** *Suppose $G$ $p$-splits. Then $UK[G]$ is nontrivial free modulo torsion if and only if $G$ is free modulo torsion and $K$ is an algebraic extension of a finite field.*

**Proof.** The isomorphism (11) of [2] obviously yields that $UK[G]/U_tK[G] \cong (\times_{|G_t/G_p|} G/G_t) \times (\times_{|G_t/G_p|} K^*/K^*_t)$. Thus $UK[G]$ is free modulo torsion precisely when $G/G_t$ is free and $K^*/K^*_t = 1$ since the latter quotient is divisible. Finally, $K^*$ is torsion, as desired. The affirmation is established.

**Proposition 17.** *Let $G$ be torsion. If $F(\zeta_n)^*$ is divisible modulo torsion for each $n$ which is an order of an element of $G$, then $VF[G]$ is divisible modulo torsion.*

**Proof.** Write $G = G_p \times M$. As we have seen, $VF[G] = VF[M] \times SF[G]$. Hence $VF[G]/V_tF[G] \cong VF[M]/V_tF[M]$. Finally either [25] or [14] gives the claim, thus completing the proof.

**Proposition 18.** *Let $G$ be $\Sigma$-cyclic. Then $UF[G]$ is nontrivial divisible modulo torsion if and only if $G$ is torsion and $F(\zeta_n)^*$ is divisible modulo torsion for every $n$ which is an order of an element of $G$.*

**Proof.** By what we have already shown above, $UF[G]/U_tF[G]$ is divisible only when $G/G_t$ is divisible free and $F(\zeta n)^*/F(\zeta_n)^*_t$ is divisible. Finally $G = G_t$ and $F(\zeta_n)^*$ is divisible modulo torsion, as promised, thus finishing the proof.

**Proposition 19.** *Let $G$ be p-splitting. The group $UK[G]$ is divisible modulo torsion if and only if the group $G$ is divisible modulo torsion.*

**Proof.** By what we have just given above, $UK[G]/U_tK[G]$ is divisible only if the same is valid for $G/G_t$, because $K^*$ is divisible whence divisible modulo torsion. Thus $G$ is really divisible modulo torsion, as expected. The proof is complete.

The following is our crucial tool for the further investigation (see, for instance, cf. [24] and [25]).

**Definition 20.** We recall that the field $F$ belongs to the class $\mathcal{P}$ if $F(\zeta_n)^*$ splits for every primitive $n$-th root of unity $\zeta_n$ in $F^-$. Denote by $\mathcal{PI}$ and $\mathcal{PR}$ the subclasses of $\mathcal{P}$ which contain fields $F$ with the following two corresponding properties: the torsion-free factor of $F(\zeta_n)^*$, that is, the quotient $F(\zeta_n)^*/F(\zeta_n)^*_t$, is free or divisible for each $\zeta_n$.

An example for a field that belongs to $\mathcal{PI}$ is the following (e. g. see May [16] or [17,18]): If $L$ is a field such that the multiplicative group $E^*$ of every finite extension $E$ of $L$ is free modulo torsion, then all extensions $F$ of $L$ generated by the algebraic elements of a bounded degree over $L$ belong to the class $\mathcal{PI}$. Besides, if $K$ is algebraically closed but $K$ is not an absolute algebraic field ($K^* \neq K^*_t$), then $K \in \mathcal{PR}$ ([11], p.298, Theorem 77.1 or [12]).

**Proposition 21.** *Suppose $G$ is a torsion direct sum of cyclic groups such that $G \neq G_p$ and $E$ is a neat transcedental extension of the field $F$. Then if $F \in \mathcal{P}$, the group $VE[G]$ splits; and if $F \in \mathcal{PI}$, the group $VE[G]$ is splitting of torsion-free rank* $\max(|E|, |G/G_p|)$.

**Proof.** Write $G = G_p \times M$, therefore $VE[G] = VE[M] \times SE[G]$. Hence, $VE[G]$ splits if and only if the same holds for $VE[M]$. So, we need only subsequently apply ([25], [14]) and Theorem 11. The proof is completed.

**Proposition 22.** *Suppose $G$ is $\Sigma$-cyclic. Then if $F(\zeta_n)^*$ splits for each $n$ which is an order of an element of $G$ (in particular if $F \in \mathcal{P}$), the group $UF[G]$ is splitting.*

**Proof.** It follows obviously from dependencies (17) and (18) of [2] that $U_tF[G]$ is a direct factor of $UF[G]$, as claimed. This concludes the proof.

**Proposition 23.** *If $G$ splits and, either $F$ is an algebraic extension of a finite field (i.e. it is an absolute algebraic field), or $G_t/G_p$ is $\Sigma$-cyclic and $F \in \mathcal{P}$, then $UF[G]$ splits.*

**Proof.** Consulting with formula (16) of [2], we argue $UF[G] \cong UF[G_t/G_p] \times (\times_\delta G/G_t) \times SF[G]$, where $\delta$ is finite when $|G_t/G_p|$ is finite or is infinite when $|G_t/G_p|$ is infinite. If now $F$ is an absolute algebraic field, then $UF[G_t/G_p]$ is torsion. Thus, in this case, $UF[G]$ splits. In the remaining one, when $G_t/G_p$ is $\Sigma$-cyclic and $F \in \mathcal{P}$, according to Proposition 22 we conclude that $UF[G_t/G_p]$ splits, therefore $UF[G]$ splits as well, as wanted. This is the end of the proof.

**Corollary 24.** *If $G$ is $\Sigma$-cyclic and $F \in \mathcal{PI}$, then $UF[G]$ is splitting.*

We close the study with the following.

**Proposition 25.** *Assume $G$ is $p$-splitting. Then $UK[G]$ splits.*

**Proof.** The group $K^*$ is divisible, hence splitting. Therefore, the statement holds by application of formula (11) from [2]. The proof is deduced.

**Remarks.** The conditions $G \neq G_p$ in Theorems 10, 11 plus the restrictions $m_n \neq 0$ in Theorems 1 and 2 were omitted from [1] involuntarily. Their formulations in [1] are in an equivalent record.

Moreover, the condition $G_q^{p^\omega} \cong 1$ in Theorem 2.2 (f) on p.370 of [2] must be written and read as $G_q^{q^\omega} \neq 1$. The sentence on p.371-line 2(+) of [2], namely: "... $E$ is an algebraic extension of finite field ..." must be assumed as "... $E^-$ is an algebraic extension of a finite field ...", and on line 12(-) of the same page the reference "[36]" must be "[37]", although both the corrections are clear from the context.

Besides, the equality $A = \bigcup_{\alpha<\lambda} B_\alpha$ on p. 223 of [3] should be replaced by $A = \bigcup_{\alpha<\lambda} G_\alpha$. In that aspect, the letter $\bigcup_{\alpha<\lambda} \bigcup_{\mu<\alpha} G_\mu$ on p. 224 of [3] must be replaced by $\prod_{\alpha<\lambda} \bigcup_{\mu<\alpha} G_\mu$.

Also the identity $G = \bigcup_{\beta<\tau} C_\beta$ from [7, p. 258] would be interpreted as $G = \bigcup_{\beta<\tau} G_\beta$.

We terminate this article with problems of some interest and importance, which immediately arise, namely:

## 3. Open questions and conjectures

What are the general criteria for $UF[G]$ to be divisible or algebraically compact or coperiodical or $\Sigma$-countable or Warfield or simply presented or a $\Sigma$-group? The finding of such necessary and sufficient conditions for the classes of all quoted groups will definitely be of some significance. In the present research exploration we have partially settled some of these problems.

On the other hand, the calculation of the torsion-free rank of $UF[G]$ when $F$ is not algebraic closed and $G$ is absolute arbitrary is requisite for the description of the torsion-free part in $UF[G]$, and thus for the isomorphism structure of this group. In this work we have established only a partial answer.

A final question is does $UF[G]$ being splitting imply that the same holds for $G$, i.e., in other words, if $UF[G]$ is splitting is then $G$ splitting? It seems to the author that this is not the case and even more that $G$ is not p-splitting.

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