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A STABILITY RESULT OF A PARAMETRIZED MINIMUM PROBLEM

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Abstract. This paper considers variational inequalities with pseudomonotone maps depending on a parameter and studies the behaviour of their solutions. The main result gives sufficient conditions for the stability of the initial minimum problem under small perturbation of the parameter.

1. **Introduction**

The parametrization is a welcome concept for almost every minimizing problem with solution and for the behaviour under perturbation.

The aim of this paper is to apply the result obtained in [5] for a particular type of parametric variational inequalities.

A lot of problems are reduced to looking for

$$(M) \qquad \inf\{\, I(u) : u \in C \,\},$$

where C is a nonempty subset of a real Banach space X and $I : C \to \mathbb{R}$ is given.

Some papers deal with the existence of the solution or with their regularity. Other papers study the "path" of the solution function provided by a family of parametrized problems, i.e. if it is single-valued, multivalued, continuous or not and so on.

For our purpose, let $\Omega \subset \mathbb{R}^n$ be a bounded domain and the minimizing problem in discussion

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$$(M)_0 \qquad \min\{\, I(u) = \int_\Omega f\big(\, t, \nabla u(t)\,\big) dt : u \in v_0 + X \,\},$$

where $X = H_0^{1,q}(\Omega), 1 < q < +\infty, v_0 \in X$ given with $I(v_0) < +\infty$, the integrand $f : \overline{\Omega} \times \mathbb{R}^n \to \mathbb{R}$ is continuous.

For I differentiable it is known that a (local) solution u_0 of $(M)_0$ has to satisfy the equilibrum equation $I'(u) = 0$.

For the real Banach space X, X^* denotes the dual space and $< x, u >$ the duality pairing between $x \in X$ and $u \in X^*$. If the admissible set C is a closed convex subset of X then u_0 has to satisfy the variational inequality

$$(VI) \qquad < I'(u), v - u > \geq 0, \quad \text{for each } v \in C.$$

The parametric form for the problem (VI) requires the following data. Let P be a topological space - the set of parameters, $K : P \to 2^X$ and $J : P \times X \to 2^{X^*}$ be given set-valued maps so that $K(p) \subseteq \operatorname{Dom} J(p, \cdot)$ for each $p \in P$, where $\operatorname{Dom} J(p, \cdot)$ denotes the domain of the map $J(p, \cdot) : X \to 2^{X^*}$, i.e. the set $\{\, u \in X \mid J(p, u) \neq \emptyset \,\}$.

For a given $p \in P$ we consider the following problem: find an element $u_p \in K(p)$ and $x \in J(p, u_p)$ so that

$$(VIP)_p \qquad < x, v - u_p > \geq 0, \quad \text{for each } u \in K(p).$$

For a fixed $p_0 \in P$ suppose that $u_0 \in K(p_0)$ is the unique solution for $(VIP)_{p_0}$.

Then, the problem $(VIP)_{p_0}$ is called *stable under perturbations* if there exist a neighborhood U_0 of p_0 and a mapping $\bar{u} : U_0 \to X$ so that:

i) $\bar{u}(p)$ is a solution for $(VIP)_p$, for any $p \in U_0$;

ii) $\bar{u}(p_0) = u_0$;

iii) $\bar{u}$ is continuous at p_0.

Section 3 deals with sufficient conditions for the stability under perturbations of the initial problem $(M)_{p_0}$.

2. **Definitions and auxiliary results**

Consider $\alpha : (0, +\infty) \to (0, +\infty)$ a nondecreasing function.

The map $I : P \times X \to \mathbb{R}$ is called *uniformly α-pseudoconvex* on $U \subseteq P$, if for each $p \in U$ and $u, v \in X, u \neq v$ and $0 \leq s \leq 1$ one has

$$< I'(p,u), v-u > \geq 0 \Rightarrow I(p,v) \leq I(p, v + s(u-v)) + s(1-s)\alpha(\|v-u\|)||v-u||,$$

where $I'(p,u)$ denotes the gradient of $I(p,\cdot)$ at the point u.

The map $J : P \times X \to 2^{X^*}$ is called *uniformly α-pseudomonotone* on $U \subseteq P$, if for each $p \in U$ and $u, v \in X, u \neq v, x \in J(p,u), y \in J(p,v)$ one has

$$< x, v-u > \geq 0 \ \Rightarrow < y, v-u > \geq \alpha(\|v-u\|) \cdot \|v-u\|.$$

An important notion for some parametric problems is consistency. For the sequential case one can consult Grave's Theorem [2, pg. 95] while for the continuous case see [1], [5].

Definition 1. *Let $p_0 \in P$, $u_0 \in K(p_0)$ and $\gamma > 1$ be fixed. The map $J : P \times X \to 2^{X^*}$ is called* consistent *in p at (p_0, u_0) if for each $0 < r \leq 1$, there exist a neighborhood U_r of p_0 and a function $\beta : U_r \to \mathbb{R}$ continuous at p_0 with $\beta(p_0) = 0$ so that, for every $p \in U_r$, there exist $u_p \in K(p)$ and $x \in J(p, u_p)$ such that*

$$\|u_p - u_0\| \leq \beta(p)$$

and

$$< x, v - u_p > + \beta(p) \cdot \|v - u_p\| \geq 0,$$

for all $v \in K(p)$ with $r < \|v - u_p\| \leq \gamma$.

Note that for $p = p_0, u_{p_0}$ is u_0.

The mapping $A : X \to 2^{X^*}$ is said to be *upper semicontinuous* (*usc*) at $u_0 \in X$ if, for any open set V containing $A(u_0)$, there exist a neighborhood Δ of u_0 so that $A(\Delta) \subset V$.

Theorem 1. ([5]) *Let P be a topological space, X be a real Banach space, $K : P \to 2^X$ be with values closed convex sets in X and $J : P \times X \to 2^{X^*}$ be a set valued map.*

Let $p_0 \in P$ and $u_0 \in K(p_0)$ be fixed. Suppose that:

i) *u_0 is a solution of $(VIP)_{p_0}$;*

ii) *J is consistent in p at (p_0, u_0);*

iii) *there exists a neighborhood U of p_0 so that the mappings $J(p, \cdot)$ are uniformly α-pseudomonotone and $J(p, \cdot)$ is usc from the line segments in X to X^* for each $p \in U$;*

iv) *for each p, u the set $J(p, u)$ is compact.*

Then, the problem $(VIP)_{p_0}$ is stable under perturbations.

3. **Main Result**

In this section we are going to apply Theorem 1 to the solutions of $(M)_p$ in particular

$$(M)_p \qquad\qquad \min\{\, I(p,u) : u \in K(p)\,\},$$

where the functionals involving the parameter are given by

$$I(p,u) = \int_\Omega f_p\big(t, \nabla u(t)\big)\, dt.$$

Now, for $p_0 \in P$ fixed suppose that $u_0 \in K(p_0)$ is the unique solution of $(M)_{p_0}$.

In this case, the problem $(M)_{p_0}$ is called *stable under perturbations* if there exist a neighborhood U_0 of p_0 and a mapping $\bar{u} : U_0 \to X$ so that:

i) $\bar{u}(p)$ is a solution for $(M)_p$, for any $p \in U_0$;

ii) $\bar{u}(p_0) = u_0$;

iii) $\bar{u}$ is continuous at p_0.

Let P be a topological space, let X be a reflexive Banach space and Y a normed space. Let $C \subseteq X$ and $D \subseteq Y$ be nonempty closed convex sets and consider the mappings $a : P \to Y$, $L : P \to (X,Y)^*$ continuous, where $(X,Y)^*$ denotes the space of all linear, continuous mappings defined on X with values in Y.

The admissible set of the problem $(M)_p$ is considered the set

$$K(p) = \{u \in C \mid a(p) + L(p)(u) \in D\}.$$

For a $p \in P$ the admissible set $K(p)$ is called *regular* if

$$0 \in \operatorname{int}\{a(p) + L(p)(u) - y : u \in C, y \in D\}.$$

Lemma 1. ([7]) *Suppose that $K(p)$ is regular and $u_0 \in K(p_0)$. Then, for each $d > 0$, there exists a neighborhood U_d of p_0 such that $K(p) \cap B(u_0; d) \neq \emptyset$ for each $p \in U_d$. Moreover, there exists a constant $c_d > 0$ such that, for every $p_1, p_2 \in U_d$ one has*

$$\operatorname{dist}\big(u, K(p_2) \cap B(u_0; d)\big) \leq c_d[\|L(p_1) - L(p_2)\| + \|a(p_1) - a(p_2)\|],$$

for each $u \in K(p_1) \cap B(u_0; d)$.

Now, considering an initial problem and a small displacement of the data we state the stability under perturbation.

Theorem 2. *Suppose that $K(p_0)$ is regular and that:*

i) *u_0 is a solution of $(M)_{p_0}$;*

ii) *the map $(p, u) \longmapsto I'(p, u)$ is weakly continuous at (p_0, u_0);*

iii) *there exists a neighborhood U of p_0 such that for each $p \in U, t \in \Omega$, $\dfrac{\partial f_p}{\partial \nabla u}(t, \cdot)$ is continuous from $X = H^{1,q}(\Omega)$ to the weak* topology of X^* and $f_p(t, \cdot)$ are strictly convex on U;*

iv) *for each $p \in U, t \in \Omega$, $\dfrac{\partial f_p}{\partial \nabla u}(t, \cdot)$ is locally bounded around u_0.*

Then, the problem $(M)_{p_0}$ is stable under perturbations.

Proof. Since u_0 is a minimum point of the functional $I(p_0, \cdot)$ on the set $K(p_0)$ we have

$$< I'(p_0, u_0), u - u_0 > \geq 0, \quad \text{for each} \quad u \in K(p_0).$$

Define $J : P \times X \to 2^{X^*}$ by $J(p, u) = \{I'(p, u)\}$, for each $p \in P$ and $u \in X$.

Let U_1 be the neighborhood of p_0, provided by Lemma 1. For each $p \in U_1$ let $u_p \in K(p) \cap B(u_0; 1)$ be the element such that

$$\|u_p - u_0\| \leq c_1[\|L(p) - L(p_0)\| + \|a(p) - a(p_0)\|].$$

Put $x = I'(p, u_p)$ (by Definition 1) and take the neighborhood U_γ and the constant c_γ given also by Lemma 1. Denote $c := \max\{c_1, c_\gamma\}$ and $U_0 := U_1 \cap U_\gamma$. For $v \in K(p)$ with $r < \|v - u_p\| \le \gamma$ define the control function

$$\beta(p) = \max\Big\{ -2\frac{1}{\|v-u_0\| + \|u_p - u_0\|} \cdot < I'(p,u_p) - I'(p_0,u_0), v - u_0 >,$$
$$\sqrt{c[\|L(p) - L(p_0)\| + \|a(p) - a(p_0)\|]}\ \Big\}.$$

From *iv)* $I'(p, u_p)$ is also locally bounded. Let $C_v > 0$ for which $\|I'(p, u_p)\| \le C_v$.

Choose $U_r \subset U_0$ a neighborhood of p_0 such that the restriction of the control function to U_r satisfies the following conditions:

$\beta(p) \le 1$, for each $p \in U_r$;
$\frac{1}{2}\|v - u_0\| - \beta(p)\left(C_v + 3\|I'(p_0, u_0)\| + \frac{3}{2}\beta(p)\right) \ge 0$, for each $p \in U_r$.

Observe that

$$\|u_p - u_0\| \le \beta^2(p) \le \beta(p).$$

By *ii)* β is continuous at p_0 and $\beta(p_0) = 0$.

Now, let $v \in K(p)$ for which $r < \|v - u_p\| \le \gamma$. We have

$$\|v - u_0\| \le \|v - u_p\| + \|u_p - u_0\| \le \gamma + 1.$$

Again by Lemma 1 there exists $v_0 \in K(p_0) \cap B(u_0; \gamma + 1)$ such that

$$\|v - v_0\| \le \beta^2(p).$$

The relationship we must verify is

$$< I'(p, u_p), v - u_p > +\beta(p) \cdot \|v - u_p\| \ge 0.$$

For simplicity denote by $I'_p := I'(p, u_p)$ and $I'_0 := I'(p_0, u_0)$. We will use

$$< I'_0, v_0 - u_0 >\ge 0,$$

due to the fact that $v_0 \in K(p_0)$.

So, we have

$$\begin{aligned}
& < I'_p, v - u_p > + \beta(p) \cdot \|v - u_p\| = \\
= \ & < I'_p - I'_0, v - u_p > + < I'_0, v - u_p > + \beta(p)\|v - u_p\| = \\
= \ & < I'_p - I'_0, v - u_0 > + < I'_p - I'_0, u_0 - u_p > + \\
& + < I'_0, v - v_0 > + < I'_0, v_0 - u_0 > + < I'_0, u_0 - u_p > + \beta(p)\|v - u_p\| \geq \\
\geq \ & -\frac{1}{2}(\|v - u_0\| + \|u_0 - u_p\|) \cdot \beta(p) - \|u_0 - u_p\| \cdot \|I'_p - I'_0\| + \\
& + < I'_0, v - v_0 > + < I'_0, u_0 - u_p > + \beta(p)(\|v - u_0\| - \|u_0 - u_p\|) \geq
\end{aligned}$$

$$\begin{aligned}
\geq \ & \frac{1}{2}\|v - u_0\| \cdot \beta(p) - \|u_0 - u_p\| \cdot \|I'_p - I'_0\| - \|v - v_0\| \cdot \|I'_0\| - \\
& - \|u_0 - u_p\| \cdot \|I'_0\| - \frac{3}{2}\beta(p)\|u_0 - u_p\| = \\
= \ & \frac{1}{2}\|v - u_0\| \cdot \beta(p) - \|u_0 - u_p\| \left(\|I'_p - I'_0\| + \|I'_0\| + \frac{3}{2}\beta(p) \right) - \\
& - \beta^2(p) \cdot \|I'_0\| \geq \\
\geq \ & \frac{1}{2}\|v - u_0\| \cdot \beta(p) - \beta^2(p) \left(C_v + 3\|I'_0\| + \frac{3}{2}\beta(p) \right) = \\
= \ & \beta(p) \left[\frac{1}{2}\|v - u_0\| - \beta(p) \left(C_v + 3\|I'_0\| + \frac{3}{2}\beta(p) \right) \right] \geq 0,
\end{aligned}$$

therefore $J \equiv I'$ is consistent in p at (p_0, u_0).

By *iii)* $I(p, \cdot)$ is strictly convex for each $p \in U$ so that $I(p, \cdot)$ is uniformly α-pseudoconvex, thus $J(p, \cdot) = I'(p, \cdot)$ are uniformly α-pseudomonotone (see [5], [3]). The conclusion follows by Theorem 1. □

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipshitz frontier, and $f_p(t, \nabla u(t)) = g(t, p) \cdot h(t, \nabla u(t))$, for each $p \in P$ and each $t \in \Omega$.

Proposition 1. *If $h \in C^1$ and $g(t, \cdot)$ is continuous at p_0 for each $t \in \Omega$, then the mapping $(p, u) \longmapsto I'(p, u)$ is weakly continuous at (p_0, u_0).*

Proof. We estimate $\left|I_p'(u)(v) - I_0'(u_0)(v)\right| \leq$

$$\leq \int_\Omega \left|g(t,p)\cdot \frac{\partial h}{\partial \nabla u}\big(t,\nabla u(t)\big) - g(t,p_0)\cdot \frac{\partial h}{\partial \nabla u}\big(t,\nabla u_0(t)\big)\right| \cdot |\nabla v| dt \leq$$

$$\leq \int_\Omega \left|g(t,p) - g(t,p_0)\right| \cdot \Big|\frac{\partial h}{\partial \nabla u}\big(t,\nabla u(t)\big)\Big| \cdot |\nabla v| dt +$$

$$+ \int_\Omega |g(t,p_0)| \cdot \Big|\frac{\partial h}{\partial \nabla u}\big(t,\nabla u(t)\big) - \frac{\partial h}{\partial \nabla u}\big(t,\nabla u_0(t)\big)\Big| \cdot |\nabla v| dt \leq$$

$$\leq \|v\|_X \cdot \left(\int_\Omega \left|g(t,p) - g(t,p_0)\right|^{q'} \cdot \Big|\frac{\partial h}{\partial \nabla u}\big(t,\nabla u_0(t)\big)\Big|^{q'} dt\right)^{1/q'} +$$

$$+ \|v\|_X \cdot \left(\int_\Omega |g(t,p_0)|^{q'} \cdot \Big|\frac{\partial h}{\partial \nabla u}\big(t,\nabla u(t)\big) - \frac{\partial h}{\partial \nabla u}\big(t,\nabla u_0(t)\big)\Big|^{q'} dt\right)^{1/q'} \to 0,$$

once that $p \to p_0$ and $u \to u_0$, for each $v \in X$. Here q' is the dual of q, i.e $\frac{1}{q}+\frac{1}{q'} = 1$. $\square$

For the existence and unicity of the solution problem $(M)_p$ we refer to [4, pg. 87].

Proposition 2. *If g and h satisfy the following conditions:*

i) *$g(t,p) > 0$ and $h(t,\cdot)$ is strictly convex for each $t \in \Omega$;*

ii) *there exists $c > 0$ such that $h(t,\xi) \geq c(|\xi|^q - 1)$ for each $(t,\xi) \in \Omega \times \mathbb{R}^n$,*

then the problem $(M)_p$ has a unique solution in $H^{1,q}(\Omega)$ for each $p \in P$.

Example 1. *Let $X = H^1(\Omega)$ with $\Omega = (0,1), 0 < p_0 < 1$ fixed, $P = [p_0, 1)$ the set of parameters, and the initial problem*

$$(M)_{p_0} \qquad \min\{I(p_0,u) : u \in C\},$$

where $I(p_0,u) = \displaystyle\int_0^1 (t+p_0)\cdot u'^2(t)\,dt$ and $C = \{u \in X : u'(p_0) = 1, u(1) = 0\}$. The solution u_0 is given by $u_0(t) = 2p_0 \ln(t+p_0)/\ln(1+p_0)$.

The parametrized problem is

$$(M)_p \qquad \min\{I(p,u) : u \in K(p)\},$$

where $I(p,u) = \displaystyle\int_0^1 (t+p)\cdot u'^2(t)\,dt$ and $K(p) = \{u \in X : u'(p) = 1, u(1) = 0\}$. $(M)_{p_0}$ is stable under perturbations and the solution function $\bar{u}$ can be obtained explicitly $\bar{u}(p)(t) = u_p(t) = 2p\ln(t+p)/\ln(1+p)$.

Remark 1. *The following problem*

$$\min\{\int_0^1 t\cdot u'^2(t)\,dt : u'(0)=1, u(1)=0\},$$

has no solution in $X = H^{1,1}(\Omega)$ *because* $h(t,\xi) = t\cdot\xi^2$ *does not satisfy the hypotheses of Proposition* 2 *namely the existence of* $c > 0$ (*the proof is similar to* [4, pg. 56].

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